Blow-up solutions and its singularities of the heat equations with a nonlinear boundary condition

Junichi Harada (Waseda University)

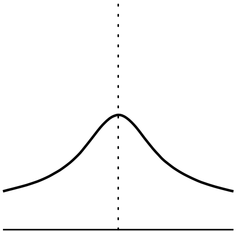
The 4th Japanese-German International Workshop on Mathematical Fluid Dynamics

December 2, 2011

Introduction

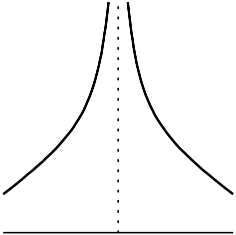
Typical blow-up phenomenon for semilinear parabolic equations

$$\partial_t u = \Delta u + f(u), \quad \text{with } B.C. + I.C.$$



classical solution $(0 < t < T)$

T: blow-up time



blow-up profile $(t = T)$

$$U(x) = \lim_{t \to T} u(x,t)$$

Our equation and assumption

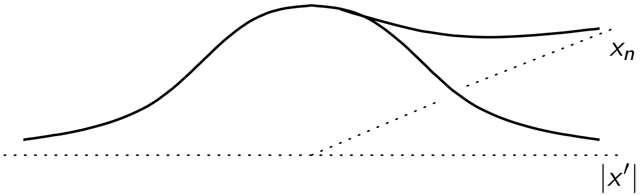
We study blow-up phenomenon of positive solutions to

$$\begin{cases} \partial_t u = \Delta u & \text{in } \mathbb{R}^n_+ \times (0, T), \\ \partial_\nu u = u^q & \text{on } \partial\mathbb{R}^n_+ \times (0, T) \quad + \text{I.C.} \end{cases} \tag{P}$$

where $q \in (1, n/(n-2))$.

Definition. We call $u(x)$ x_n-axial symmetric, if $u(x) = u(|x'|, x_n)$.

Assumption (monotonicity). $x' \cdot \nabla' u_0(x) \le 0, \quad \partial_n u_0(x) \le 0.$



$$\begin{cases} \partial_t u = \Delta u & \text{in } \mathbb{R}^n_+ \times (0, T), \\ \partial_\nu u = u^q & \text{on } \partial\mathbb{R}^n_+ \times (0, T) \quad + \text{I.C.} \end{cases} \tag{P}$$

Properties of equation (P)

- A comparison argument is available.
- Since solutions are defined on a half space $\mathbb{R}^n_+$, x_n-direction is a special one.
- Blow-up phenomenon occurs only on the boundary $\partial\mathbb{R}^n_+$ under some conditions.
- Scaling invariance. $u_\lambda(x, t) = \lambda^\gamma u(\lambda x, \lambda^2 t)$.
- Energy functional.

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^n_+} |\nabla u|^2 dx - \frac{1}{q+1} \int_{\partial\mathbb{R}^n_+} |u|^{q+1} dx'.$$

$$\partial_t u = \Delta u + u^p \qquad \text{(F)}$$

Trivial blow-up solutions for equation (F) "$\Delta u = 0$"

$$\partial_t u = u^p \quad \Rightarrow \quad u_{\text{triv}}(t) = c_p(T-t)^{-1/(p-1)}$$

Backward self-similar (blow-up) solutions

$$u(x,t) = (T-t)^{-\gamma}\varphi(x/\sqrt{T-t}) \quad (\gamma > 0).$$

$$\Delta\varphi - \frac{y}{2}\cdot\nabla\varphi - \frac{1}{2(q-1)}\varphi = 0, \quad \partial_\nu\varphi = \varphi^q. \qquad \text{(BP)}$$

$$\Delta\varphi - \frac{y}{2}\cdot\nabla\varphi - \frac{1}{(p-1)}\varphi + \varphi^p = 0. \qquad \text{(BF)}$$

From Fila-Quittner (91), Chlebik-Fila (00), if $q \in (1, n/(n-2))$, (BP) has a unique positive bounded solution $\varphi_0(y) = \varphi_0(y_n)$:

$$\varphi_0'' - \frac{\xi}{2}\varphi_0' - \frac{1}{2(q-1)}\varphi_0 = 0 \text{ in } \mathbb{R}_+, \qquad \partial_\nu \varphi_0 = \varphi_0^q \text{ on } \{0\}.$$

Therefore our equation (P) has a trivial blow-up solution:

$$u_{\text{triv}}(x,t) = (T-t)^{-1/2(q-1)}\varphi_0((T-t)^{-1/2}x_n).$$

Above arguments, we just constructed a special blow-up solution. How about general blow-up solutions ?

Self-similar valuables $x = (T-t)^{1/2}y,\ T - t = e^{-s}$

$$\varphi(y,s) = (T-t)^{\gamma} u(x,t).$$

blow-up time $t = T \quad \rightarrow \quad s = \infty$.

$$\begin{cases} \partial_s \varphi = \Delta\varphi - \frac{y}{2}\cdot\nabla\varphi - \frac{1}{2(q-1)}\varphi & \text{in } \mathbb{R}^n_+ \times (s_T, \infty), \\ \partial_\nu \varphi = \varphi^q & \text{on } \partial\mathbb{R}^n_+ \times (s_T,\infty). \end{cases} \quad \text{(RP)}$$

Chlebik-Fila (00) ($\gamma = 1/2(q-1),\ q \in (1, n/(n-2))$)

- $\varphi(y,s)$ is uniformly bounded.
- $\varphi(y,s) \to \varphi_0(y_n)$ in $C_{\mathrm{loc}}(\overline{\mathbb{R}^n_+})$ $(s \to \infty)$.
  $\varphi_0(y_n)$ is a solution of (BP), that is the steady state of (RP).

Trivial blow-up profile for equation (P)

Asymptotic behavior near blow-up time

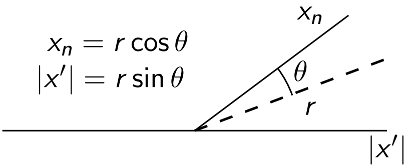
$$u(x,t) = (T-t)^{-1/2(q-1)}\varphi((T-t)^{-1/2}x, log(T-t))$$
$$\sim (T-t)^{-1/2(q-1)}\varphi_0((T-t)^{-1/2}x_n) = u_{\mathrm{triv}}(x,t).$$

Blow-up profile

$$\begin{aligned} U(x) &= \lim_{t\to T} u(x,t) \sim \lim_{t\to T} u_{\mathrm{triv}}(x,t) \\ &= c_q x_n^{-1/(q-1)} = c_q(\cos\theta)^{-1/(q-1)} r^{-1/(q-1)} \\ &=: U_{\mathrm{triv}}(x). \end{aligned}$$

Polar coordinate

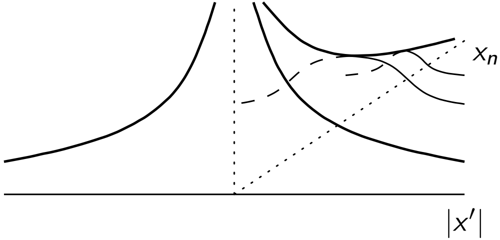
$$x_n = r\cos\theta$$
$$|x'| = r\sin\theta$$



Goal of this talk.

- What conditions on the initial data assure a single point blow-up ?
- When a single point blow-up occurs, how is the singularity of the blow-up profile on $\partial\mathbb{R}^n_+$ (along $|x'|$-axis) ?

Single pint blow-up.



Main results

Assumption

$$x' \cdot \nabla' u_0(x) \le 0, \qquad \partial_n u_0(x) \le 0. \tag{A}$$

Theorem 1 (single point blow-up)

Let $u_0(x)$ be $u_0 \in BC^1(\overline{\mathbb{R}^n_+})$, x_n-axial symmetric and satisfy (A). Then if $u(x,t)$ blows up in a finite time, the blow-up occurs only on the boundary.

Theorem 2 (spacial singularity of blow-up profile)

Let $u_0(x)$ be as in Theorem 1. Then if $u(x,t)$ blows up in a finite time, $u(x,t)$ has a blow-up profile $U(|x'|, x_n)$ and satisfies

$$c_1 \left(\frac{|x'|^2}{|\log |x'||} \right)^{-1/2(q-1)} \le U(|x'|, 0) \le c_2 \left(\frac{|x'|^2}{|\log |x'||} \right)^{-1/2(q-1)}.$$

For equation (F), Herrero-Velazquez (92-93) studied singularities of blow-up profile. We apply their strategy.

Strategy by Herrero-Velazquez (92-93)

1) Dynamical system on Hilbert space $L^2_\rho(\mathbb{R}^n_+)$.

2) Improvement of convergence range.

3) Singularity of blow-up profile.

4) Selection of eigen-mode.

Our main analysis is a large time behavior of $\varphi(y,s)$.

$$\varphi(y,s) = \varphi_0(y_n) + (\text{2'nd approximation}) + (\text{h.o.t})$$

◇ 1'st approximation $\varphi_0(y_n)$ $\to$ singularity of x_n-axis.

◇ 2'nd approximation $\to$ singularity of $|x'|$-axis.

Set $v(y,s) = \varphi(y,s) - \varphi_0(y_n)$.

$$\partial_s v = Av, \qquad \partial_\nu v = q\varphi_0^{q-1} v + f(v). \qquad \text{(Dy)}$$

$$Av = \Delta v - \frac{y}{2}\cdot\nabla v - \frac{1}{2(q-1)}v,$$

$$D(A) = \{v \in H^2_\rho(\mathbb{R}^n_+); \partial_\nu v = q\varphi_0^{q-1} v \text{ on } \partial\mathbb{R}^n_+\}.$$

◇ Weighted Sobolev space (weight function $\rho(y) = e^{-|y|^2/4}$)

$$L^p_\rho(\mathbb{R}^n_+) = \left\{ v \in L^1_{\text{loc}}(\mathbb{R}^n_+); \int_{\mathbb{R}^n_+} |v|^p \rho\, dy < \infty \right\},$$

$$H^k_\rho(\mathbb{R}^n_+) = \{v \in L^2_\rho(\mathbb{R}^n_+); D^\alpha v \in L^2_\rho(\mathbb{R}^n_+), \ \forall |\alpha| \le k\}.$$

Analysis of linearized equations

Let $K = q\varphi_0(0)^{q-1} > 0$.

$$\partial_s v = \Delta v - \frac{y}{2} \cdot \nabla v \;\text{ in } \mathbb{R}^n_+, \quad \partial_\nu v = Kv \;\text{ on } \partial\mathbb{R}^n_+. \tag{L1}$$

Set $w(y,s) = e^{\mu_K s} v(y,s)/b_K(y_n)$, where $\mu_K < 0$, $b_K(y_n)$ are the first eigenvalue and eigenfunction of

$$-\left(b'' + \frac{\xi}{2} b'\right) = \mu b \;\text{ in } \mathbb{R}_+, \qquad \partial_\nu b = Kb \;\text{ on } \{0\}.$$

$$\partial_s w = \Delta w - \frac{y}{2} \cdot \nabla w + \left(\frac{2b_K'(y_n)}{b_K(y_n)}\right) \partial_n w, \quad \partial_\nu w = 0. \tag{L2}$$

A goal is to derive global heat kernel estimates of (L2)

$$\Gamma_K(y,\xi,s) = \Gamma_{\mathbb{R}^{n-1}}(y',\xi',s)\gamma_K(y_n,\xi_n,s).$$

$\gamma_K(\xi,\xi',s)$ is a heat kernel of

$$\partial_s W = W_{\xi\xi} - \frac{\xi}{2} W_\xi + \left(\frac{2b_K'(\xi)}{b_K(\xi)}\right) W_\xi, \quad \partial_\nu W = 0. \qquad \text{(L3)}$$

In the original valuable, $U(x,t) = W((1-t)^{-1/2}x, -\log(1-t))$.

$$\partial_s U = U_{xx} + (1-t)^{-1/2} \left(\frac{2b_K'}{b_K}\right) U_x, \quad \partial_\nu U = 0.$$

Proposition. Let $\gamma_K(\xi,\xi',s)$ be the heat kernel of (L3). Then there exits $\theta \in (0,1)$ such that for $\xi \in (0, (\theta\xi' - 2)e^{s/2})$, $s \geq 1$

$$\gamma_K(\xi,\xi',s) \leq c\gamma_{\mathcal{N}}(\xi, \theta\xi', s),$$

where $\gamma_{\mathcal{N}}$ is the heat kernel of $\partial_s W = W_{\xi\xi} - \frac{\xi}{2} W_\xi$, $\partial_\nu W = 0$.