Global and almost global solutions for the Navier-Stokes equations in Besov spaces and Triebel-Lizorkin spaces

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Navier-Stokes equations

$$(\mathrm{NS})\quad \begin{cases} \partial_t u - \Delta u + (u\cdot\nabla)u + \nabla p = 0 & \text{for } t>0, x\in\mathbb{R}^n, \\ \mathrm{div}\ u = 0 & \text{for } t>0, x\in\mathbb{R}^n, \\ u(0,x) = u_0(x) & \text{for } x\in\mathbb{R}^n. \end{cases}$$

Known Results. **(Global solutions for small initial data)**

Well-posedness			Ill-posedness
T. Kato (1984)	**Kozono, Yamazaki (1994)**	**Koch, Tataru (2001)**	**Bourgain, Pavlović (2008)**
$L^n(\mathbb{R}^n)$	$\dot{B}_{p,\infty}^{-1+\frac{n}{p}}(\mathbb{R}^n)(n<p<\infty)$	BMO^{-1}	$\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^n)$

Remark. $L^n(\mathbb{R}^n) \subset \dot{B}_{p,\infty}^{-1+\frac{n}{p}}(\mathbb{R}^n) \subset BMO^{-1} \subset \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^n)$.

$|x|^{-1} \in \dot{B}_{p,\infty}^{-1+\frac{n}{p}}(\mathbb{R}^n) \setminus L^n(\mathbb{R}^n)$.

Definition. Let $\{\phi_j\}_{j\in\mathbb{Z}}$ be Littlewood-Paley's dyadic decomposition with $\operatorname{supp}\widehat{\phi_j} \subset \{\,\xi \in \mathbb{R}^n \,|\, 2^{j-1} \le |\xi| \le 2^{j+1}\,\}$.
Let $s \in \mathbb{R}, 1 \le p, q \le \infty$.
(i) Besov spaces $\dot{B}^s_{p,q}(\mathbb{R}^n)$.

$$\|u\|_{\dot{B}^s_{p,q}} := \left\{ \sum_{j\in\mathbb{Z}} \Big(2^{sj}\|\phi_j * u\|_{L^p}\Big)^q \right\}^{\frac{1}{q}}.$$

(ii) Triebel-Lizorkin spaces $\dot{F}^s_{p,q}(\mathbb{R}^n)$

$$\|u\|_{\dot{F}^s_{p,q}} := \left\| \left\{ \sum_{j\in\mathbb{Z}} \Big(2^{sj}|\phi_j * u|\Big)^q \right\}^{\frac{1}{q}} \right\|_{L^p}.$$

Remark. $L^p(\mathbb{R}^n) = \dot{F}^0_{p,2}(\mathbb{R}^n)$ for $1 < p < \infty$

$$L^\infty(\mathbb{R}^n) \subset BMO = \dot{F}^0_{\infty,2}(\mathbb{R}^n) \subset \dot{B}^0_{\infty,\infty}(\mathbb{R}^n)$$

Well-posedness		Ill-posedness
$\dot{B}^{-1+\frac{n}{p}}_{p,\infty}(\mathbb{R}^n)$ $(p<\infty)$	$BMO^{-1} = \dot{F}^{-1}_{\infty,2}(\mathbb{R}^n)$	$\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^n)$

Let ϕ_L **and** ϕ_H **satisfy** $\widehat{\phi_L} + \widehat{\phi_H} \equiv 1$,

$$\text{supp } \widehat{\phi_L} \subset \{|\xi| \le 2\}, \quad \text{supp } \widehat{\phi_H} \subset \{|\xi| \ge 1\}.$$

Main Theorem. Let $n \ge 2$, $n < p < \infty$.

(1) If $\|u_0\|_{\dot{B}^{-1}_{\infty,\infty}} \ll 1$ **and** $\|u_0\|_{\dot{B}^{-1+\frac{n}{p}}_{p,\infty}} \le C_1 \exp\left\{C_2 \|u_0\|^{-1}_{\dot{B}^{-1}_{\infty,\infty}}\right\}$, **there exists a global solution to** (NS).

(2) If $\varepsilon := \|\phi_L * u_0\|_{\dot{B}^{-1}_{\infty,\infty}} + \|\phi_H * u_0\|_{\dot{F}^{-1}_{\infty,2}} \ll 1$, **there exists a solution to** (NS) **on the time interval** $\left[0, C_1 e^{C_2 \varepsilon^{-1}}\right]$.

Remark. $\|u_0\|_{\dot{F}^{-1}_{\infty,2}} \ll 1$: **Well-posedness.**

$\|u_0\|_{\dot{B}^{-1}_{\infty,\infty}} \ll 1$: **Ill-posedness.**

Proof of Main Theorem (1).

$$\|u\|_Y := \sup_{t>0} t^{\frac{1}{2}-\frac{n}{2p}}\|u(t)\|_{L^p}, \qquad \|u\|_Z := \sup_{t>0} t^{\frac{1}{2}}\|u(t)\|_{L^\infty}.$$

Remark. $\|e^{t\Delta}u_0\|_Y = \|u_0\|_{\dot{B}^{-1+\frac{n}{p}}_{p,\infty}}$, $\|e^{t\Delta}u_0\|_Z = \|u_0\|_{\dot{B}^{-1}_{\infty,\infty}}$.

$$\Psi(u)(t) := e^{t\Delta}u_0 - \int_0^t e^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot(u\otimes u)d\tau,$$

$$X := \left\{ u \,\middle|\, \|u\|_Y \le C\|u_0\|_{\dot{B}^{-1+\frac{n}{p}}_{p,\infty}},\ \|u\|_Z \le C\|u_0\|_{\dot{B}^{-1}_{\infty,\infty}} \right\},$$

$$d(u,v) := \|u-v\|_Y.$$

$$\|u\|_Y := \sup_{t>0} t^{\frac{1}{2}-\frac{n}{p}} \|u(t)\|_{L^p}, \qquad \|u\|_Z := \sup_{t>0} t^{\frac{1}{2}} \|u(t)\|_{L^\infty}.$$

Proposition. **(1)** $\exists C > 0$ **such that**

$$\left\| \int_0^t e^{(t-\tau)\Delta} \mathbb{P} \nabla \cdot (u \otimes u) d\tau \right\|_Y \le C \|u\|_Z \|u\|_Y.$$

(2) $\exists C, \theta > 0$ **such that for any** $0 < \delta \le \dfrac{1}{2}$

$$\left\| \int_0^t e^{(t-\tau)\Delta} \mathbb{P} \nabla \cdot (u \otimes u) d\tau \right\|_Z \le C \Big\{ \delta^\theta \|u\|_Y + \log(e + \delta^{-1}) \|u\|_Z \Big\} \|u\|_Z.$$

$$\delta^\theta \|u_0\|_{\dot{B}^{-1+\frac{n}{p}}_{p,\infty}} + \log(e + \delta^{-1}) \|u_0\|_{\dot{B}^{-1}_{\infty,\infty}} \ll 1$$

$$\Longrightarrow \|u_0\|_{\dot{B}^{-1}_{\infty,\infty}} \ll 1 \text{ and } \|u_0\|_{\dot{B}^{-1+\frac{n}{p}}_{p,\infty}} \le C_1 \exp\Big\{ C_2 \|u_0\|^{-1}_{\dot{B}^{-1}_{\infty,\infty}} \Big\}$$

Proof of Proposition, (1).

$$
\begin{aligned}
&t^{\frac{1}{2}-\frac{n}{2p}}\left\|\int_0^t e^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot(u\otimes u)d\tau\right\|_{L^p}\\
&\le Ct^{\frac{1}{2}-\frac{n}{2p}}\int_0^t (t-\tau)^{-\frac{1}{2}}\|u\otimes u\|_{L^p}d\tau\\
&\le Ct^{\frac{1}{2}-\frac{n}{2p}}\int_0^t (t-\tau)^{-\frac{1}{2}}\|u\|_{L^\infty}\|u\|_{L^p}d\tau\\
&\le C\|u\|_Z\|u\|_Y t^{\frac{1}{2}-\frac{n}{2p}}\int_0^t (t-\tau)^{-\frac{1}{2}}\tau^{-\frac{1}{2}+\frac{n}{2p}}\tau^{-\frac{1}{2}}d\tau
\end{aligned}
$$

By changing the variable $\tau \mapsto t\tau$,

$$
t^{\frac{1}{2}-\frac{n}{2p}}\int_0^t (t-\tau)^{-\frac{1}{2}}\tau^{-\frac{1}{2}+\frac{n}{2p}}\tau^{-\frac{1}{2}}d\tau = \int_0^1 (1-\tau)^{-\frac{1}{2}}\tau^{-1+\frac{n}{2p}}d\tau.
$$

Proof of Proposition, (2). $[0,t] = [0,\delta t] \cup [\delta t, t]$.

$$t^{\frac{1}{2}} \Big\| \int_0^{\delta t} e^{(t-\tau)\Delta} \mathbb{P}\nabla \cdot (u \otimes u) d\tau \Big\|_{L^\infty}$$
$$\leq C t^{\frac{1}{2}} \int_0^{\delta t} (t-\tau)^{-\frac{1}{2}-\frac{n}{2p}} \|u \otimes u\|_{L^p} d\tau$$
$$\leq C t^{\frac{1}{2}} \int_0^{\delta t} (t-\tau)^{-\frac{1}{2}-\frac{n}{2p}} \|u\|_{L^p} \|u\|_{L^\infty} d\tau$$
$$\leq C \|u\|_Y \|u\|_Z t^{\frac{1}{2}} \int_0^{\delta t} (t-\tau)^{-\frac{1}{2}-\frac{n}{2p}} \tau^{-\frac{1}{2}+\frac{n}{2p}} \tau^{-\frac{1}{2}} d\tau.$$

By changing the variable $\tau \mapsto t\tau$,

$$t^{\frac{1}{2}} \int_0^{\delta t} (t-\tau)^{-\frac{1}{2}-\frac{n}{2p}} \tau^{-\frac{1}{2}+\frac{n}{2p}} \tau^{-\frac{1}{2}} d\tau = \int_0^{\delta} (1-\tau)^{-\frac{1}{2}-\frac{n}{2p}} \tau^{-1+\frac{n}{2p}} d\tau$$
$$\leq C \delta^{\frac{n}{2p}}.$$

For the integral in $[\delta t, t]$,

$$
\begin{aligned}
t^{\frac{1}{2}}\Big\| \int_{\delta t}^{t} e^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot(u\otimes u)d\tau\Big\|_{L^\infty} &\le Ct^{\frac{1}{2}}\int_{\delta t}^{t}(t-\tau)^{-\frac{1}{2}}\|u\otimes u\|_{L^\infty}d\tau \\
&\le Ct^{\frac{1}{2}}\int_{\delta t}^{t}(t-\tau)^{-\frac{1}{2}}\|u\|_{L^\infty}^2 d\tau \\
&\le C\|u\|_Z^2 t^{\frac{1}{2}}\int_{\delta t}^{t}(t-\tau)^{-\frac{1}{2}}\tau^{-1}d\tau.
\end{aligned}
$$

By changing the variable $\tau \mapsto t\tau$,

$$
t^{\frac{1}{2}}\int_{\delta t}^{t}(t-\tau)^{-\frac{1}{2}}\tau^{-1}d\tau = \int_{\delta}^{1}(1-\tau)^{-\frac{1}{2}}\tau^{-1}d\tau \le C\log(e+\delta^{-1}).
$$