Asymptotic stability of Ekman boundary layers in rotating stratified fluids

Hajime Koba

Tokyo university

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Geophysical fluid dynamics

- Fluids: gases and liquids as water and air.
- Geophysical fluids: large-scale fluids as the earth's atmosphere, ocean and climate.
- Features of geophysical fluids: Rotation, Stratification, and Heat effects.
- Rotation: the earth's rotation (Coriolis force).
- Stratification: Density and Temperature.

What is an Ekman boundary layer ?

What is an Ekman boundary layer ?

- Boundary layers appearing in the atmosphere and ocean
- A stationary solution of a rotating Navier-Stokes equations with a boundary condition (half space, infinite layer)

Our purpose

- Constructing a stationary solution of a geophysical system (our Ekman layer)
- Investigating the Ekman layer (stability and instability)

Known results (Ekman layers without heat and stratification effects)

- Giga-Inui-Mahalov-Matsui-Saal '07:
  Existence of a unique mild solution of an Ekman perturbed system with non-decaying initial data.
- Hess-Hieber-Mahalov-Saal '10:
  The nonlinear stability of Ekman boundary layers (weak solution).
- Hieber-Stannat '11:
  The stability of Ekman boundary layers in a stochastic sense.

Navier-Stokes equations

- Navier-Stokes equations:

$$(NS)\begin{cases}\partial_t u - \nu\Delta u + (u,\nabla)u + \nabla p = 0,\\ \nabla\cdot u = 0,\end{cases}$$

where $\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$, $\nabla = (\partial_1, \partial_2, \partial_3)$,
$u = (u^1, u^2, u^3)$: the velocity,
p: the pressure,
$\nu > 0$: the viscosity coefficient,
$\nabla \cdot u = 0$: incompressible condition.

Coriolis force

- Rotating Navier-Stokes equations:

$$(RNS)\begin{cases}\partial_t u - \nu\Delta u + (u,\nabla)u + \nabla p = -\Omega e_3 \times u,\\ \nabla\cdot u = 0,\end{cases}$$

where $\times$: exterior product,
$e_3 = (0,0,1)$: the rotating axis,
$\Omega \in \mathbb{R}$: the rotation rate (rotation parameter).

Stratification and heat effects

- Navier-Stokes Boussinesq equations with stratification effect:

$$(NSBS)\begin{cases}\partial_t u - \nu\Delta u + (u,\nabla)u + \nabla p = \mathcal{G} e_3\theta, \\ \partial_t\theta - \kappa\Delta\theta + (u,\nabla)\theta = -N^2 u^3, \end{cases}$$

where θ: the temperature (distribution),
$\kappa > 0$: the heat diffusion rate,
$\mathcal{G} \in \mathbb{R}\setminus\{0\}$: gravity,
$N \in \mathbb{R}$: Brunt-Väisälä frequency(stratification parameter).

Our system

- A rotating Navier-Stokes-Boussinesq equations with stratification effects:

$$
(S)\begin{cases}
\partial_t u - \nu\Delta u + (u,\nabla)u + \Omega d\times u + \nabla p = \mathcal{G}e_3\theta,\\
\partial_t\theta - \kappa\Delta\theta + (u,\nabla)\theta = -N^2u^3,\\
\nabla\cdot u = 0,\\
u|_{t=0} = (u_0^1, u_0^2, u_0^3),\\
\theta|_{t=0} = \theta_0,
\end{cases}
$$

where $\nu,\kappa > 0$, $\Omega, N \in \mathbb{R}$, $\mathcal{G} \in \mathbb{R}\setminus\{0\}$, $e_3 = (0,0,1)$,
$d \in \mathbb{S}^2 := \{d = (d_1, d_2, d_3) \in \mathbb{R}^3 : |d| = 1\}$ with $d_3 \neq 0$.

Construction of our Ekman layer

Assume that $\Omega d_3 > 0$. Let $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{R}$ and let us set $\delta = \sqrt{\frac{2\nu}{\Omega d_3}}$.

Form:

$$\begin{cases} u_E = u_E(x_3),\ \theta_E = \theta_E(x_3), \\ p_E = \widetilde{p}(x_3) - \Omega d_3 a_2 x_2 + \Omega b_2 d_3 x_1, \\ \widetilde{p}(0) = [\delta\Omega\{(a_2 - a_1)(d_1 + d_2) + (b_2 - b_1)(d_2 - d_1)\}]/2. \end{cases}$$

Boundary condition at $x_3 = 0, \infty$ and a slope condition:

$$\begin{cases} u_E|_{\{x_3=0\}} = (a_1, b_1, 0),\ \theta_E|_{\{x_3=0\}} = c_1, \\ \lim_{x_3\to\infty} u_E(x_3) = (a_2, b_2, 0), \\ d\theta_E/dx_3 = c_2. \end{cases}$$

Our Ekman layer

$$u_E = \begin{pmatrix} u_E^1 \\ u_E^2 \\ u_E^3 \end{pmatrix} = \begin{pmatrix} a_2 + (a_1 - a_2)e^{-\frac{x_3}{\delta}} \cos \frac{x_3}{\delta} + (b_1 - b_2)e^{-\frac{x_3}{\delta}} \sin \frac{x_3}{\delta} \\ b_2 + (b_1 - b_2)e^{-\frac{x_3}{\delta}} \cos \frac{x_3}{\delta} + (a_2 - a_1)e^{-\frac{x_3}{\delta}} \sin \frac{x_3}{\delta} \\ 0 \end{pmatrix},$$

$$\theta_E = c_2 x_3 + c_1,$$

$$\begin{aligned} p_E =& \frac{1}{2}\mathcal{G} c_2 x_3^2 + (\mathcal{G} c_1 + \Omega d_2 a_2 - \Omega d_1 b_2) x_3 \\ &+ \frac{\delta\Omega}{2} e^{-\frac{x_3}{\delta}} \cos \frac{x_3}{\delta} [(a_2 - a_1)(d_1 + d_2) + (b_2 - b_1)(d_2 - d_1)] \\ &+ \frac{\delta\Omega}{2} e^{-\frac{x_3}{\delta}} \sin \frac{x_3}{\delta} [(a_1 - a_2)(d_2 - d_1) + (b_2 - b_1)(d_1 + d_2)] \\ &- \Omega d_3 a_2 x_2 + \Omega b_2 d_3 x_1. \end{aligned}$$

Main theorem I(Existence of a weak solution)

Set $\mathcal{M} := \sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2}$. Assume that $(u_0 - u_E, \theta_0 - \theta_E) \in L^2_\sigma(\mathbb{R}^3_+) \times L^2(\mathbb{R}^3_+)$. If

$$(c_2 + N^2)\mathcal{G} > 0 \text{ and } \frac{2\mathcal{M}}{\sqrt{\nu\Omega d_3}} < 1 \cdots \text{(R)},$$

then there exists at least one global weak solution (u, θ, p) of (S) with the initial datum (u_0, θ_0), satisfying the strong energy inequality. Moreover, the solution (u, θ, p) satisfies

$$\lim_{T\to\infty} \int_T^{T+1} \left(\|u(t) - u_E\|^2_{L^2(\mathbb{R}^3_+)} + \|\theta(t) - \theta_E\|^2_{L^2(\mathbb{R}^3_+)} \right) dt = 0.$$

Furthermore, assume that there exists an another weak solution $(\widetilde{u}, \widetilde{\theta}, \widetilde{p})$ of (S) with the initial datum (u_0, θ_0). If $(\widetilde{u} - u_E, \widetilde{\theta} - \theta_E)$ belongs to the class $L^{p_1}(0, T; L^{p_2}(\mathbb{R}^3_+))$ with $2/p_1 + 3/p_2 = 1$ for some $p_2 > 3$ and $T > 0$, then $u = \widetilde{u}$ and $\theta = \widetilde{\theta}$ on $[0, T)$.

Main theorem II(Existence of a strong solution)

Assume that the initial datum $(u_0 - u_E, \theta_0 - \theta_E) \in H^1_{0,\sigma}(\mathbb{R}^3_+) \times H^1_0(\mathbb{R}^3_+)$. If

$$\|u_0 - u_E\|_{H^1} + \|\theta_0 - \theta_E\|_{H^1} << 1,$$

then there exists a unique global-in-time strong solution (u, θ, p) of (S) with initial datum (u_0, θ_0) such that

$$\lim_{t\to\infty} \left(\|u(t) - u_E\|_{L^2(\mathbb{R}^3_+)} + \|\theta(t) - \theta_E\|_{L^2(\mathbb{R}^3_+)} \right) = 0,$$

where p is a pressure associated with (u, θ).

Corollary (Smoothness of weak solutions)

If a weak solution of (S) satisfies the strong energy inequality, then the weak solution is smooth with respect to time when time is sufficiently large. Moreover, the weak solution satisfies

$$\lim_{t\to\infty}\left(\|u(t)-u_E\|_{L^2(\mathbb{R}^3_+)}+\|\theta(t)-\theta_E\|_{L^2(\mathbb{R}^3_+)}\right)=0.$$

Known results (NS)

- H. Fujita and T. Kato, '62 '64:
  Existence of a unique global-in-time strong solution of (NS) when the initial datum is sufficiently small.
- K. Masuda, 84: If a weak solution of (NS) satisfies the strong energy inequality, then the weak solution has the asymptotic stability. Moreover, he studied the uniqueness of weak solutions.
- T. Miyakawa and H. Sohr, 88:The existence of weak solution of (NS) satisfying the strong energy inequality. Moreover, they showed that the weak solution is smooth with respect to time when time is sufficiently large.

Notations and function spaces

$$\widetilde{\nabla} := (\partial_1, \partial_2, \partial_3, 0),$$

$$C^\infty_{0,\sigma} := C^\infty_{0,\sigma}(\mathbb{R}^3_+) := \{f = (f^1, f^2, f^3) \in [C^\infty_0(\mathbb{R}^3_+)]^3;\ \nabla \cdot f = 0\},$$

$$\widetilde{L}^p_\sigma := \widetilde{L}^p_\sigma(\mathbb{R}^3_+) := \overline{C^\infty_{0,\sigma} \times C^\infty_0(\mathbb{R}^3_+)}^{\|\cdot\|_{L^p}} \ (= L^p_\sigma \times L^p(\mathbb{R}^3_+)),$$

$$\widetilde{H}^1_{0,\sigma} := \widetilde{H}^1_{0,\sigma}(\mathbb{R}^3_+) := \overline{C^\infty_{0,\sigma} \times C^\infty_0(\mathbb{R}^3_+)}^{\|\cdot\|_{H^1}}, \widetilde{C}^\infty_{0,\sigma} := C^\infty_{0,\sigma} \times C^\infty_0(\mathbb{R}^3_+)$$

$$G_p := G_p(\mathbb{R}^3_+) := \{f \in [L^p(\mathbb{R}^3_+)]^3;\ f = \nabla g,\ g \in L^p_{loc}(\overline{\mathbb{R}^3_+})\},$$

$$G^{m,p}_0 := G^{m,p}_0(\mathbb{R}^3_+) := \{f \in [W^{m-1,p}_0(\mathbb{R}^3_+)]^3;\ f = \nabla g,\ g \in W^{m,p}_0(\mathbb{R}^3_+)\}$$

for $1 < p < \infty$ and $m \in \mathbb{N}$ with the norms

$$\|f\|_{W^{m,p}} := \sum_{0 \le |\alpha| \le m} \|D^\alpha f\|_{L^p} \text{ and } \|f\|_{H^1} := \|f\|_{W^{1,2}},$$

$$\|f\|_{L^\infty} := \operatorname{ess.sup}_{x \in \mathbb{R}^3_+}\{|f(x)|\},\ \langle \cdot, \cdot \rangle := (\cdot, \cdot)_{L^2}.$$

Formulation near the stationary solution

Set $w := (w^1, w^2, w^3, w^4) := (u - u_E, \sqrt{\frac{\mathcal{G}}{N^2+c_2}}(\theta - \theta_E))$ and $\widetilde{q} := p - p_E$. The function $(w, \widetilde{q})$ satisfies

$$(OS)\begin{cases}\partial_t w + \mathcal{A}w + \mathcal{S}w + \mathcal{B}_E w + (w, \widetilde{\nabla})w + \widetilde{\nabla}\widetilde{q} = 0,\\ \widetilde{\nabla}\cdot w = 0,\ w|_{t=0} = w_0,\ w|_{\partial\mathbb{R}^3_+} = 0,\end{cases}$$

with $w_0 := (u_0 - u_E, \sqrt{\frac{\mathcal{G}}{N^2+c_2}}(\theta_0 - \theta_E))$, $\widetilde{u}_E := (u_E^1, u_E^2, 0, 0)$,
$\mathcal{B}_E w := (\widetilde{u}_E, \widetilde{\nabla})w + w^3\partial_3\widetilde{u}_E$,
$\mathcal{A} :=$diag$\{-\nu\Delta, -\nu\Delta, -\nu\Delta, -\kappa\Delta\}$,

$$\mathcal{S} := \begin{pmatrix} 0 & -\Omega d_3 & \Omega d_2 & 0\\ \Omega d_3 & 0 & -\Omega d_1 & 0\\ -\Omega d_2 & \Omega d_1 & 0 & -\sqrt{(N^2+c_2)\mathcal{G}}\\ 0 & 0 & \sqrt{(N^2+c_2)\mathcal{G}} & 0\end{pmatrix}.$$

Constructing an energy inequality

- Let w be a solution of (OS). Since

$$\langle \mathcal{S}w, w\rangle = \langle (w, \widetilde{\nabla})w, w\rangle = \langle (\widetilde{u}_E, \widetilde{\nabla})w, w\rangle = \langle \widetilde{\nabla}\widetilde{q}, w\rangle = 0,$$

we test (OS) by w to obtain

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|_{L^2}^2 + \nu\|\nabla\overline{w}(t)\|_{L^2}^2 + \kappa\|\nabla w^4(t)\|_{L^2}^2 + \langle w^3\partial_3\widetilde{u}_E, w\rangle = 0,$$

where $\overline{w} := (w^1, w^2, w^3)$.

Lemma

Let $\alpha \geq 0$, $\beta > 0$ and $u \in W_0^{1,p}(\mathbb{R}_+)$ for $1 \leq p < \infty$. Then

$$\|(\cdot)^\alpha e^{-\beta(\cdot)}u(\cdot)\|_{L^p(\mathbb{R}_+)} \leq \frac{1}{(p\beta)^{\alpha+1}}\left[\widetilde{\Gamma}(\alpha, p)\right]^{1/p}\|u'\|_{L^p(\mathbb{R}_+)}$$

with $\widetilde{\Gamma}(\alpha, p) := \int_0^\infty z^{p\alpha+p-1}e^{-z}dz$.

Constructing an energy inequality

Using the above lemma and integrating with respect to time,

$$\|w(t)\|_{L^2}^2 + C_E \int_s^t \|\widetilde{\nabla} w(\tau)\|_{L^2}^2 d\tau \leq \|w(s)\|_{L^2}^2 \leq \|w_0\|_{L^2}^2$$

for $t \geq s \geq 0$ with $C_E := 2\min\{\nu - \sqrt{2}\delta\mathcal{M}, \kappa\}$.
Under the restrictions

$$(c_2 + N^2)\mathcal{G} > 0 \text{ and } \frac{2\mathcal{M}}{\sqrt{\nu\Omega d_3}} < 1 \quad \cdots (R),$$

we see that $C_E > 0$.

Define operators

P: the Helmholtz projection from $[L^p(\mathbb{R}^3_+)]^3$ to $L^p_\sigma(\mathbb{R}^3_+)$ for some $1 < p < \infty$. Define the extended Helmholtz projection $\widetilde{P}$ as follows:

$$\widetilde{P} := \begin{pmatrix} P & \\ & 1 \end{pmatrix}.$$

Applying $\widetilde{P}$ to (OS), we obtain the abstract system:

$$\begin{cases} \partial_t v + L_{E,p} v = -\widetilde{P}(v, \widetilde{\nabla})v, \\ v|_{t=0} = v_0 = \widetilde{P} w_0 \end{cases}$$

with the linear operator $L_{E,p}$ in $\widetilde{L}^p_\sigma$ defined by

$$\begin{cases} L_{E,p} v & := \widetilde{P}(\mathcal{A} + \mathcal{S} + \mathcal{B}_E)v, \\ D(L_{E,p}) & := [W^{2,p}(\mathbb{R}^3_+)]^4 \cap [W^{1,p}_0(\mathbb{R}^3_+)]^4 \cap \widetilde{L}^p_\sigma(\mathbb{R}^3_+). \end{cases}$$

Adjoint operator

In particular, we define $L_E := L_{E,2}$, and it is easy to see that its adjoint operator L_E^* is defined by

$$D(L_E^*) = D(L_E) \text{ and } L_E^* v = \widetilde{P}(\mathcal{A} + \mathcal{S}^* + \mathcal{B}_E^*)v$$

with $\mathcal{B}_E^* v := -(\widetilde{u}_E, \widetilde{\nabla})v + \mathcal{B}^E v$, $\mathcal{S}^* := \mathcal{S}^T$,

$$\mathcal{B}^E = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \partial_3 u_E^1 & \partial_3 u_E^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is easy to check that

$$\langle L_E v_1, v_2 \rangle = \langle v_1, L_E^* v_2 \rangle \ (v_1, v_2 \in D(L_E)).$$

Definition of Weak solutions

Let $w_0 \in \widetilde{L}^2_\sigma$. We call a vector-valued function $(w, \widetilde{q})(= (w^1, w^2, w^3, w^4, \widetilde{q}))$ a weak solution of (OS) with the initial datum w_0, if for all $T > 0$ and for all $s, t, \varepsilon \geq 0$ such that $0 \leq s < \varepsilon < t < T$ the following six properties hold:

(i)(function class)

$$w \in L^\infty(0, T; \widetilde{L}^2_\sigma) \cap L^2(0, T; \widetilde{H}^1_{0,\sigma}),$$

$$\widetilde{\nabla}\widetilde{q} \in [L^2(\varepsilon, T; [L^2(\mathbb{R}^3_+)]^3) \oplus L^{5/4}(\varepsilon, T; [L^{5/4}(\mathbb{R}^3_+)]^3)] \times \{0\},$$

Definition of Weak solutions

(ii) $w : [0, T] \to \widetilde{L}^2_\sigma$ is weakly continuous,
(iii) (an energy inequality)

$$\|w(t)\|^2_{L^2} + C_E \int_0^t \|\widetilde{\nabla} w(\tau)\|^2_{L^2} d\tau \le \|w_0\|^2_{L^2}$$

with $C_E := 2\min\{\nu - \sqrt{2}\delta\mathcal{M}, \kappa\}$, where $\delta = \sqrt{2\nu}/\sqrt{\Omega d_3}$ and $\mathcal{M} = \sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2}$,

Definition of Weak solutions

(iv)(weak form I)

$$\int_0^T \langle w, \phi \rangle \psi' d\tau - \nu \int_0^T \langle \nabla \overline{w}, \nabla \overline{\phi} \rangle \psi d\tau - \kappa \int_0^T \langle \nabla w^4, \nabla \phi^4 \rangle \psi d\tau$$
$$- \int_0^T \langle \mathcal{S} w, \phi \rangle \psi d\tau - \int_0^T \langle (\widetilde{u}_E, \widetilde{\nabla}) w, \phi \rangle \psi d\tau - \int_0^T \langle w^3 \partial_3 \widetilde{u}_E, \phi \rangle \psi d\tau$$
$$- \int_0^T \langle (w, \widetilde{\nabla}) w, \phi \rangle \psi d\tau = -\langle w_0, \phi \rangle \psi(0)$$

holds for all $\phi = (\phi^1, \phi^2, \phi^3, \phi^4) \in \widetilde{H}^1_{0,\sigma}$ and all $\psi \in C^1([0,T]; \mathbb{R})$ with $\psi(T) = 0$, where $\langle \cdot, \cdot \rangle = (\cdot, \cdot)_{L^2}$, $\psi' = d\psi/d\tau$, $\overline{w} = (w^1, w^2, w^3)(= u - u_E)$, and $\overline{\phi} = (\phi^1, \phi^2, \phi^3)$,

Definition of Weak solutions

(v)(weak form II)

$$\int_s^t \langle w, \Phi' \rangle d\tau - \nu \int_s^t \langle \nabla \overline{w}, \nabla \overline{\Phi} \rangle d\tau - \kappa \int_s^t \langle \nabla w^4, \nabla \Phi^4 \rangle d\tau$$
$$- \int_s^t \langle \mathcal{S} w, \Phi \rangle d\tau - \int_s^t \langle (\widetilde{u}_E, \widetilde{\nabla}) w, \Phi \rangle d\tau - \int_s^t \langle w^3 \partial_3 \widetilde{u}_E, \Phi \rangle d\tau$$
$$- \int_s^t \langle (w, \widetilde{\nabla}) w, \Phi \rangle d\tau = \langle w(t), \Phi(t) \rangle - \langle w(s), \Phi(s) \rangle$$

holds for all $\Phi = (\Phi^1, \Phi^2, \Phi^3, \Phi^4) \in C^1([s,t]; \widetilde{H}^1_{0,\sigma})$, where $\Phi' = \partial \Phi / \partial \tau$ and $w(0) = w_0$,

Definition of Weak solutions

(vi) the function $(w, \widetilde{q})$ satisfies the following identity:

$$\int_\varepsilon^t \langle \mathcal{S}w, \Psi\rangle d\tau + \int_\varepsilon^t \langle (\widetilde{u}_E, \widetilde{\nabla})w, \Psi\rangle d\tau + \int_\varepsilon^t \langle w^3 \partial_3 \widetilde{u}_E, \Psi\rangle d\tau + \int_\varepsilon^t \langle (w, \widetilde{\nabla})w, \Psi\rangle d\tau + \int_\varepsilon^t \langle \widetilde{\nabla}\widetilde{q}, \Psi\rangle d\tau = 0$$

for all $\Psi = (\Psi^1, \Psi^2, \Psi^3, \Psi^4) \in C([\varepsilon, t]; G_0^{2,2} \times \{0\})$, where
$G_0^{2,2} = \{f \in [W_0^{1,2}(\mathbb{R}^3_+)]^3;\ f = \nabla g,\ g \in W_0^{2,2}(\mathbb{R}^3_+)\}$.

Strong energy inequality

Let $w_0 \in \widetilde{L}^2_\sigma$ and $(w, \widetilde{q})(= (w^1, w^2, w^3, w^4, \widetilde{q}))$ be a weak solution of (OS) with the initial datum w_0. We say that $(w, \widetilde{q})$ satisfies the strong energy inequality if

$$
\|w(t)\|_{L^2}^2 + 2\nu \int_s^t \|\nabla \overline{w}\|_{L^2}^2 d\tau + 2\kappa \int_s^t \|\nabla w^4\|_{L^2}^2 d\tau + 2 \int_s^t \langle w^3 \partial_3 \widetilde{u}_E, w \rangle d\tau \le \|w(s)\|_{L^2}^2
$$

holds for a.e. $s \ge 0$, including $s = 0$, and all $t > s$, where $w(0) = w_0$.

Strong energy equality

Analogously, one can define the strong energy equality:

$$\|w(t)\|_{L^2}^2 + 2\nu \int_s^t \|\nabla \overline{w}\|_{L^2}^2 d\tau + 2\kappa \int_s^t \|\nabla w^4\|_{L^2}^2 d\tau + 2 \int_s^t \langle w^3 \partial_3 \widetilde{u}_E, w \rangle d\tau = \|w(s)\|_{L^2}^2$$

for all $s, t \geq 0 (s < t)$, where $w(0) = w_0$.

Main results

Let $\nu, \kappa > 0$, $\Omega, N, a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{R}$, $\mathcal{G} \in \mathbb{R} \setminus \{0\}$, and $d = (d_1, d_2, d_3) \in \mathbb{S}^2$ with the restrictions (R). Then the following seven assertions hold:
(1) For every $w_0 \in \widetilde{L}^2_\sigma$ there exists at least one weak solution of (OS) with the initial datum w_0, which satisfies the strong energy inequality.
(2) Let $w_0 \in \widetilde{L}^2_\sigma$ and $s_1, t_1 \geq 0$ such that $0 \leq s_1 < t_1$. Let $(w, \widetilde{q})$ be a weak solution of (OS) with the initial datum w_0. Assume that w belongs to the class $L^{p_1}(s_1, t_1; [L^{p_2}(\mathbb{R}^3_+)]^4)$ with $2/p_1 + 3/p_2 = 1$ for some $p_2 > 3$. Then the solution $(w, \widetilde{q})$ satisfies the strong energy equality for all s and t $(s_1 \leq s < t \leq t_1)$.

Main results

(3) Let $T > 0$ and $w_0 \in \widetilde{L}^2_\sigma$. Let $(w, \widetilde{q}_1)$ and $(v, \widetilde{q}_2)$ be two weak solutions of (OS) with the same initial datum w_0. Assume that the solution $(w, \widetilde{q})$ satisfies the strong energy inequality. If $v \in L^{p_1}(0, T; [L^{p_2}(\mathbb{R}^3_+)]^4)$ with $p_2 > 3$ and $2/p_1 + 3/p_2 = 1$, then $w = v$ on $[0, T)$.

(4) Let $w_0 \in \widetilde{L}^2_\sigma$ and $(w, \widetilde{q})$ be a weak solution of (OS) with the initial datum w_0. If the solution $(w, \widetilde{q})$ satisfies the strong energy inequality, then

$$\lim_{T\to\infty} \int_T^{T+1} \|w(t)\|^2_{H^1} dt = 0.$$

Main results

(5) Let $w_0 \in \widetilde{L}^2_\sigma$. Assume that

$$\|\mathrm{e}^{-tL_E} w_0\|_{L^2} = O(t^{-1/2}) \text{ as } t \to \infty,$$
$$\|\widetilde{\nabla}\mathrm{e}^{-tL_E^*}\|_{\mathcal{L}(\widetilde{L}^2_\sigma)} \leq Ct^{-1/2}, \ t > 0$$

with a positive constant C independent of t. Then there exists at least one weak solution $(w, \widetilde{q})$ of (OS) with the initial datum w_0 such that

$$\|w(t)\|_{L^2} = O(t^{-1/4}) \text{ as } t \to \infty.$$

Here e^{-tL_E} is a C_0-semigroup whose generator is the operator $-L_E$ and $\mathrm{e}^{-tL_E^*}$ is a C_0-semigroup with the generator $-L_E^*$.

Main results

(6) Let $w_0 \in \widetilde{H}^1_{0,\sigma}$. There is a positive number δ_0 independent of $\|w_0\|_{H^1}$ such that if $\|w_0\|_{H^1} < \delta_0$ then there exists a unique global-in-time strong solution $(w, \widetilde{q})$ of (OS) with the initial datum w_0, satisfying the strong energy equality and

$$\lim_{t \to \infty} \|w(t)\|_{L^2} = 0.$$

Here

$$w \in C([0,\infty); \widetilde{H}^1_{0,\sigma}) \cap C((0,\infty); D(L_E)) \cap C^1((0,\infty); \widetilde{L}^2_\sigma),$$
$$\widetilde{\nabla} \widetilde{q} \in C((0,\infty); G_2 \times \{0\}),$$

and $\widetilde{q}$ is a pressure associated with w.

Main results

(7) Let $w_0 \in \widetilde{L}^2_\sigma$ and $(w, \widetilde{q})$ be a weak solution of (OS) with the initial datum w_0. If the weak solution $(w, \widetilde{q})$ satisfies the strong energy inequality, then there exists a positive number T_0 such that

$$w \in C^1([T_0, \infty); \widetilde{L}^2_\sigma).$$

Moreover, the weak solution satisfies

$$\lim_{t \to \infty} \|w(t)\|_{L^2} = 0.$$

Outline of the proof

- Properties of operators
- Existence of a strong solution
- Existence of weak solutions
- Strong energy inequality and Uniqueness of weak solutions
- Asymptotic stability
- Smoothness on large time

Stokes-Laplace operators A_p

Definition (Stokes-Laplace operators)

Let $p > 1$. We define A_p in $\widetilde{L}^p_\sigma$ as follows:

$$\begin{cases} D(A_p) & := \widetilde{L}^p_\sigma \cap [W^{1,p}_0(\mathbb{R}^3_+)]^4 \cap [W^{2,p}(\mathbb{R}^3_+)]^4, \\ A_p w & := \widetilde{P}\mathcal{A}w, \ w \in D(A_p), \end{cases}$$

where $\mathcal{A} =$ diag $\{-\nu\Delta, -\nu\Delta, -\nu\Delta, -\kappa\Delta\}$ and $\widetilde{P}$ is the extended Helmholtz operator. In particular, we write $A := A_2$.

V. A. Solonnikov' 77, S. Ukai'87, Borchers-Miyakawa'88, Desch-Hieber-Prüss'01, Denk-Hieber-Prüss'03.

Properties of the Stokes-Laplace operators A_p

Lemma

Let $p > 1$. Then

(i) *The operator $-A_p$ generates a bounded analytic semigroup on $\widetilde{L}^p_\sigma$.*

(ii) *The operator A_p has maximal L^q-regularity for each $q > 1$.*

Lemma (Key facts)

Let $t > 0$. For all $f_0 \in \widetilde{L}^2_\sigma \cap [H^1(\mathbb{R}^3_+)]^4$

$$e^{-tA}\partial_1 f_0 = \partial_1 e^{-tA} f_0, \qquad e^{-tA}\partial_2 f_0 = \partial_2 e^{-tA} f_0,$$

$$\widetilde{P}\partial_1 f_0 = \partial_1 \widetilde{P} f_0, \qquad \widetilde{P}\partial_2 f_0 = \partial_2 \widetilde{P} f_0.$$

Maximal L^p-regularity

Let X be a Banach space and $T \in (0, \infty]$, $J = (0, T)$. A linear closed operator $\mathscr{A} : D(\mathscr{A}) \to X$ in X is said to have maximal L^p-regularity for $p > 1$ on $(0, T)$, if for each $(f, u_0) \in L^p(J; X) \times (X, D(\mathscr{A}))_{1-1/p,p}$ there exists a unique function u satisfying

$$\begin{cases} u' + \mathscr{A} u = f, \\ u(0) = u_0, \end{cases}$$

for a.e. $t \in J$ and

$$\|u'\|_{L^p(J;X)} + \|\mathscr{A} u\|_{L^p(J;X)} \le C\left(\|f\|_{L^p(J;X)} + \|u_0\|_{(X,D(\mathscr{A}))_{1-1/p,p}}\right),$$

where $C > 0$ is independent of f and u_0. Here $(X, D(\mathscr{A}))_{1-1/p,p}$ is a real interpolation space between function spaces X and $D(\mathscr{A})$, and $\|\cdot\|_{(X,D(\mathscr{A}))_{1-1/p,p}}$ is a norm of the real interpolation space. In particular, we say that $\mathscr{A}$ has maximal L^p-regularity for $p > 1$ when $T = \infty$.

Difficulties

Since our Ekman layer has a special form,

- The main operator $L_E (= L_{E,2})$ is not selfadjoint in L^2-space.
- It is difficult to obtain an explicit expression of a solution to the linearized system of (OS).
  In other words, it is not easy to derive the desired resolvent estimates for the Ekman operators $L_{E,p}$.

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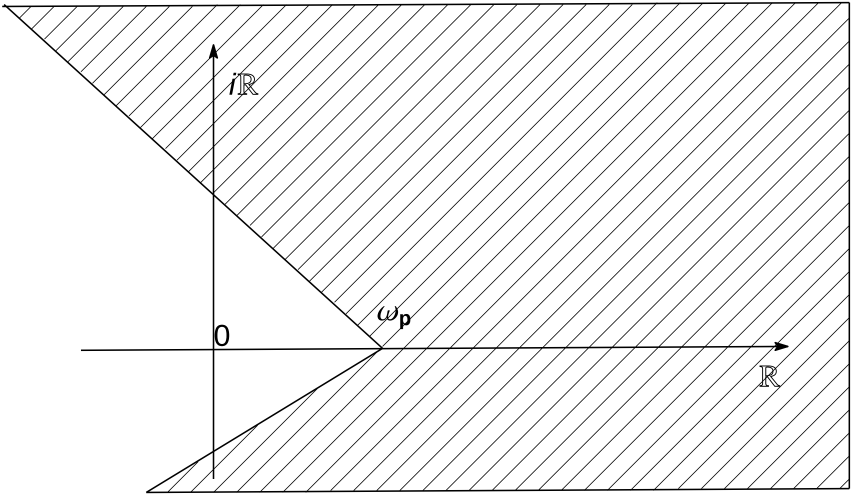
  In other words, it is not easy to derive the desired resolvent estimates for the Ekman operators $L_{E,p}$.

However,

- The operator $-L_{E,p}$ generates an analytic semigroup on $\widetilde{L}^p_\sigma$.
- There exists $\eta_p > 0$ such that $(L_{E,p} + \eta_p)$ has maximal L^q-regularity $1 < q < \infty$.

Resolvent set $\rho(-L_{E,p})$, $\|e^{-tL_{E,p}}\| \leq Me^{\omega_p t}$



Contraction C_0-semigroup

Let $v_0 \in \widetilde{L}^2_\sigma$. Set $v := (v^1, v^2, v^3, v^4) := \mathrm{e}^{-tL_E} v_0$.
Since $-L_E$ generates an analytic semigroup on $\widetilde{L}^2_\sigma$, we have

$$\begin{cases} v_t + L_E v = 0, \\ v(0) = v_0. \end{cases}$$

An easy computation gives

$$\begin{aligned} \frac{d}{dt}\|v(t)\|^2_{L^2} &= \int_{\mathbb{R}^3_+} v_t(t) \cdot \overline{v}(t) dx + \int_{\mathbb{R}^3_+} v(t) \cdot \overline{v}_t(t) dx \\ &= \langle -L_E v, v \rangle + \langle v, -L_E v \rangle. \end{aligned}$$

Contraction C_0-semigroup

By integration by parts, we have

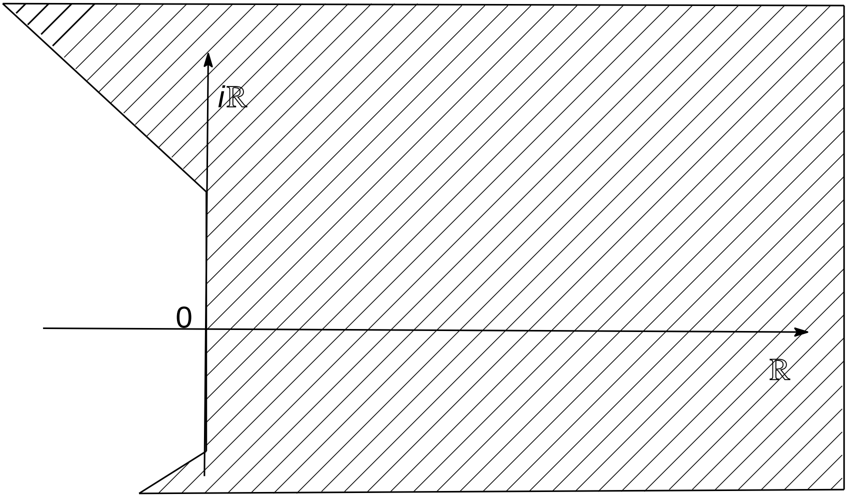
$$\frac{d}{dt}\|v(t)\|_{L^2}^2 + 2\nu\sum_{j=1}^{3}\|\nabla v^j\|_{L^2}^2 + 2\kappa\|\nabla v^4\|_{L^2}^2$$
$$= \int_{\mathbb{R}^3_+} \partial_3 u_E^1(\overline{v^1}v^3 + v^1\overline{v^3})dx + \int_{\mathbb{R}^3_+} \partial_3 u_E^2(\overline{v^2}v^3 + v^2\overline{v^3})dx.$$

Applying a previous argument, we have

$$\|v(t)\|_{L^2}^2 + C_E\int_s^t \|\widetilde{\nabla} v(\tau)\|_{L^2}^2 d\tau \le \|v(s)\|_{L^2}^2$$
$$\le \|v_0\|_{L^2}^2 \text{ for all } s, t \ge 0 (s < t).$$

Remark that $C_E > 0$ under the restrictions (R).

Resolvent set $\rho(-L_E)$ under the restrictions (R)



Properties of operators $L_{E,p}$

Lemma

Let $p > 1$. Then

(i) *The operator $-L_{E,p}$ generates an analytic semigroup on $\widetilde{L}^p_\sigma$.*

(ii) *There exists a positive number η_p such that the operator $(L_{E,p} + \eta_p)$ has maximal L^q-regularity for each $q > 1$.*

(iii) *The operator $-L_E$ generates a contraction C_0-semigroup on $\widetilde{L}^2_\sigma$.*

Key lemma

(i)Let $t > 0$. For all $f_0 \in \widetilde{L}^2_\sigma \cap [H^1(\mathbb{R}; L^2(\mathbb{R}^2_+))]^4$

$$e^{-tL_E}\partial_1 f_0 = \partial_1 e^{-tL_E} f_0, \; e^{-tL_E^*}\partial_1 f_0 = \partial_1 e^{-tL_E^*} f_0 \text{ (Commutativity).}$$

(ii)For all $f_0 \in \widetilde{L}^2_\sigma$

$$\lim_{t\to\infty} \|e^{-tL_E} f_0\|_{L^2} = 0 \text{ (Linear stability).}$$

Proof of Commutativity

Recall that the operator L_E can be expressed as follows:

$$L_E = A + B$$

with $B := \widetilde{P}\partial_1 B_1 + \widetilde{P}\partial_2 B_2 + \widetilde{P}B_3$,

$$B_1 := \begin{pmatrix} u_E^1 & 0 & 0 & 0 \\ 0 & u_E^1 & 0 & 0 \\ 0 & 0 & u_E^1 & 0 \\ 0 & 0 & 0 & u_E^1 \end{pmatrix}, \; B_2 := \begin{pmatrix} u_E^2 & 0 & 0 & 0 \\ 0 & u_E^2 & 0 & 0 \\ 0 & 0 & u_E^2 & 0 \\ 0 & 0 & 0 & u_E^2 \end{pmatrix},$$

$$B_3 := \begin{pmatrix} 0 & \Omega d_3 & -\Omega d_2 + \partial_3 u_E^1 & 0 \\ -\Omega d_3 & 0 & \Omega d_1 + \partial_3 u_E^2 & 0 \\ \Omega d_2 & -\Omega d_1 & 0 & \sqrt{(c_2 + N^2)\mathcal{G}} \\ 0 & 0 & -\sqrt{(c_2 + N^2)\mathcal{G}} & 0 \end{pmatrix},$$

where $A = \widetilde{P}\mathrm{diag}\{-\nu\Delta, -\nu\Delta, -\nu\Delta, -\kappa\Delta\}$.

Proof of Commutativity

Let $\phi_0 \in \widetilde{C}^\infty_{0,\sigma}$ and $T > 0$. We consider the following two systems:

$$\begin{cases} u_t + (L_E + \eta)u = 0, \ t \in (0, T], \\ u|_{t=0} = \phi_0, \end{cases}$$

$$\begin{cases} v_t + (L_E + \eta)v = 0, \ t \in (0, T], \\ v|_{t=0} = \partial_1 \phi_0 \end{cases}$$

for some $\eta > 0$.
Since $-(L_E + \eta)$ generates an analytic semigroup on $\widetilde{L}^2_\sigma$, we write $u(t) = \mathrm{e}^{-t(L_E+\eta)}\phi_0$ and $v(t) = \mathrm{e}^{-t(L_E+\eta)}\partial_1\phi_0$.
Now we show $v(t) = \partial_1 u(t)$ for each $t \in (0, T]$.

Proof of Commutativity

Set

$$X_T := \{f \in C([0,T]; \widetilde{H}^1_{0,\sigma});\ \|f\|_{X_T} < \infty\}$$

with

$$\|f\|_{X_T} := \sup_{0\le t\le T}\{\|f(t)\|_{L^2} + \|A^{1/2}f(t)\|_{L^2}\}.$$

We now consider the following approximate solutions:

$$u_1(t) := \mathrm{e}^{-t(A+\eta)}\phi_0,$$

$$u_{m+1}(t) := \mathrm{e}^{-t(A+\eta)}\phi_0 - \int_0^t \mathrm{e}^{-(t-s)(A+\eta)}Bu_m(s)ds\ (m = 1,2,3,\dots),$$

$$v_1(t) := \mathrm{e}^{-t(A+\eta)}\partial_1\phi_0,$$

$$v_{m+1}(t) := \mathrm{e}^{-t(A+\eta)}\partial_1\phi_0 - \int_0^t \mathrm{e}^{-(t-s)(A+\eta)}Bv_m(s)ds\ (m = 1,2,3,\dots).$$

Proof of Commutativity

It is easy to check that

$$v_1(t) = \mathrm{e}^{-t(A+\eta)}\partial_1\phi_0 = \partial_1\mathrm{e}^{-t(A+\eta)}\phi_0 = \partial_1 u_1(t)$$

and

$$\begin{aligned} v_2(t) =& \mathrm{e}^{-t(A+\eta)}\partial_1\phi_0 - \int_0^t \mathrm{e}^{-(t-s)(A+\eta)} B\partial_1 u_1 ds \\ =& \partial_1\left(\mathrm{e}^{-t(A+\eta)}\phi_0 - \int_0^t \mathrm{e}^{-(t-s)(A+\eta)} B u_1 ds\right) \\ =& \partial_1 u_2(t). \end{aligned}$$

By induction, we see that $v_m = \partial_1 u_m$ for each $m \in \mathbb{N}$.

Proof of Commutativity

We first choose η sufficiently large, then we use a fixed point theorem and semigroup theory to obtain
a unique strong solution $u \in C([0,T];\widetilde{H}^1_{0,\sigma}) \cap C((0,T];D(L_E))$ of $[u_t + (L_E+\eta)u = 0,\ u(0) = \phi_0]$ and
a unique strong solution $v \in C([0,T];\widetilde{H}^1_{0,\sigma}) \cap C((0,T];D(L_E))$ of $[v_t + (L_E+\eta)v = 0,\ v(0) = \partial_1\phi_0]$ such that

$$\|u_m - u\|_{X_T} = 0 \text{ as } m \to \infty,$$
$$\|v_m - v\|_{X_T} = 0 \text{ as } m \to \infty.$$

Since $v_m = \partial_1 u_m$, we conclude that $v(t) = \partial_1 u(t)$ for each $t \in (0,T]$, that is, $\mathrm{e}^{-tL_E}\partial_1\phi_0 = \partial_1 \mathrm{e}^{-tL_E}\phi_0$.

Proof of Linear stability

Define the tangential operator $\widetilde{\partial_1}$ as follows:

$$\widetilde{\partial_1} := \partial_1 I \equiv \begin{pmatrix} \partial_1 & 0 & 0 & 0 \\ 0 & \partial_1 & 0 & 0 \\ 0 & 0 & \partial_1 & 0 \\ 0 & 0 & 0 & \partial_1 \end{pmatrix}$$

and

$$D(\widetilde{\partial_1}) := \widetilde{L}^2_\sigma \cap [H^1(\mathbb{R}; L^2(\mathbb{R}^2_+))]^4.$$

Lemma

$$(i)\widetilde{\partial_1} : D(\widetilde{\partial_1}) \to \widetilde{L}^2_\sigma,$$

$$(ii) R(\widetilde{\partial_1}) \text{ is dense in } \widetilde{L}^2_\sigma(\mathbb{R}^3_+).$$

Proof of Linear stability

We first assume that $a \in R(\widetilde{\partial}_1)$. Since e^{-tL_E} is a contraction C_0-semigroup, we use the Cauchy-Schwarz inequality to obtain

$$\|e^{-tL_E}a\|_{L^2} \leq \frac{1}{t}\int_0^t \|e^{-sL_E}a\|_{L^2} ds \leq \frac{1}{\sqrt{t}}\left(\int_0^t \|e^{-sL_E}a\|_{L^2}^2 ds\right)^{1/2}.$$

Since $a \in R(\widetilde{\partial}_1)$, $\exists b \in D(\widetilde{\partial}_1)$ s.t. $a = \widetilde{\partial}_1 b$. The energy inequality $\Rightarrow$

$$\int_0^t \|e^{-sL_E}a\|_{L^2}^2 ds = \int_0^t \|\partial_1 e^{-sL_E}b\|_{L^2}^2 ds \leq C_E^{-1}\|b\|_{L^2}^2.$$

Thus,

$$\|e^{-tL_E}a\|_{L^2} \leq t^{-1/2} C_E^{-1/2}\|b\|_{L^2} \to 0 \ (t \to \infty).$$

Since $R(\widetilde{\partial}_1)$ is dense in $\widetilde{L}^2_\sigma$, we see the linear stability.

Construction of a unique strong solution

Let $T > 1$. We consider the next system:

$$\begin{cases} dw/dt + (L_E + \eta)w = -\widetilde{P}(w, \widetilde{\nabla})w + \eta w, \\ w|_{t=0} = w_0 \in \widetilde{H}^1_{0,\sigma} \end{cases}$$

for some $\eta > 0$.
We consider the following approximate solutions ($j = 1, 2, \cdots$) by

$$\begin{aligned} w_1(t) =& \mathrm{e}^{-t(L_E+\eta)} w_0, \\ w_{j+1}(t) =& \mathrm{e}^{-t(L_E+\eta)} w_0 - \int_0^t \mathrm{e}^{-(t-\tau)(L_E+\eta)} \widetilde{P}(w_j, \widetilde{\nabla}) w_j(\tau) d\tau \\ &+ \eta \int_0^t \mathrm{e}^{-(t-\tau)(L_E+\eta)} w_j(\tau) d\tau. \end{aligned}$$

Construction of a unique strong solution

Set

$$Y_T := \{f \in C([0,T]; \widetilde{H}^1_{0,\sigma});\ \|f\|_{X_T} < \infty\}$$

with

$$\|f\|_{Y_T} := \sup_{0\le\tau\le T} \|(L_E+\eta)^{1/2} f(\tau)\|_{L^2} + \sup_{0<\tau\le T} [\tau^{1/4} \|(L_E+\eta)^{3/4} f(\tau)\|_{L^2}].$$

We first choose η sufficiently small, then we take $w_0 \in \widetilde{H}^1_{0,\sigma}$ with $\|w_0\|_{H^1} << 1$ to obtain a unique strong solution $w \in C([0,T]; \widetilde{H}^1_{0,\sigma}) \cap C((0,T]; D(L_E)) \cap C^1((0,T]; \widetilde{L}^2_\sigma)$.
The strong energy equality $\Rightarrow$ a unique global-in-time strong solution $w \in C([0,\infty); \widetilde{H}^1_{0,\sigma}) \cap C((0,\infty); D(L_E)) \cap C^1((0,\infty); \widetilde{L}^2_\sigma)$.

Yosida approximation

Lemma (Yosida approximation)

For $m \in \mathbb{N}$, we set

$$\mathcal{J}_m = (1 + m^{-1} L_E)^{-1}.$$

Then $\mathcal{J}_m$ has the following properties:
(i) *Let $p \in \mathbb{N}$. For each $w \in \widetilde{L}^p_\sigma$*

$$\lim_{m\to\infty} \|\mathcal{J}_m w - w\|_{L^p} = 0.$$

(ii) *For all $w \in \widetilde{L}^2_\sigma$*

$$\|\mathcal{J}_m w\|_{L^2} \le \|w\|_{L^2}.$$

(iii) *There exists $C(m) > 0$ such that for all $w \in \widetilde{L}^2_\sigma$*

$$\|\mathcal{J}_m w\|_{L^\infty} \le C(m)\|w\|_{L^2}.$$

Construction of weak solutions

Fix $m \in \mathbb{N}$. We consider the next system:

$$\begin{cases} w_m' + L_E w_m = -\widetilde{P}(\mathcal{J}_m w_m, \widetilde{\nabla}) w_m, \\ w_m(0) = \mathcal{J}_m w_0, \end{cases}$$

where $\mathcal{J}_m = (1 + m^{-1} L_E)^{-1}$.
We solve the following integral form:

$$w_m(t) = \mathrm{e}^{-tL_E} \mathcal{J}_m w_0 + \int_0^t \mathrm{e}^{-(t-s)L_E} F_m w_m(s) ds$$

with $F_m w := -\widetilde{P}(\mathcal{J}_m w, \widetilde{\nabla}) w$, where $w_m = (w_m^1, w_m^2, w_m^3, w_m^4)$.

Construction of weak solutions

Let $T > 0$. Set

$$Z_T := \{f \in C([0,T]; \widetilde{H}^1_{0,\sigma}); \|f\|_{Z_T} < \infty\}$$

with

$$\|f\|_{Z_T} := \sup_{0 \le t \le T} \{\|f(t)\|_{L^2} + \|L_E^{1/2} f(t)\|_{L^2}\}.$$

Fixed point theorem $\Rightarrow$ a unique local-in-time strong solution $w_m \in C([0,T_*]; \widetilde{H}^1_{0,\sigma}) \cap C((0,T_*]; D(L_E)) \cap C^1((0,T_*]; \widetilde{L}^2_\sigma)$ for some $T_* > 0$.
Energy inequality $\Rightarrow$
$w_m \in C([0,T]; \widetilde{H}^1_{0,\sigma}) \cap C((0,T]; D(L_E)) \cap C^1((0,T]; \widetilde{L}^2_\sigma)$ for each fixed $T > 0$.

Construction of weak solutions

Energy inequality, Maximal L^p-regularity, Real interpolation theory, Cut-off functions
$\Rightarrow$ a weak solution satisfying the strong energy inequality.

K. Masuda, 84 $\Rightarrow$ the uniqueness of weak solutions.

More precisely, let $(w, \widetilde{q}_1)$ and $(v, \widetilde{q}_2)$ be two weak solutions. Assume that the solution $(w, \widetilde{q})$ satisfies the strong energy inequality. If $v \in L^{p_1}(0, T; [L^{p_2}(\mathbb{R}^3_+)]^4)$ with $p_2 > 3$ and $2/p_1 + 3/p_2 = 1$, then $w = v$ on $[0, T)$.

Asymptotic stability

We follow the arguments of Masuda'84 and Miyakawa-Sohr'88.

$$\int_T^{T+1} \|w(t)\|_{L^2}^2 dt$$
$$\leq \left(\int_T^{T+1} \|w(t)\|_{H^1}^2 dt\right)^{1/3} \left(\int_T^{T+1} \|(L_E + 1)^{-1/4} w(t)\|_{L^2}^2 dt\right)^{2/3}$$
$$\leq C(\|w_0\|_{L^2}) \left(\int_T^{T+1} \|(L_E + 1)^{-1/4} w(t)\|_{L^2}^2 dt\right)^{2/3}.$$

$$\|(1 + L_E)^{-1/4} w(t)\|_{L^2}$$
$$\leq \|e^{-tL_E}(1 + L_E)^{-1/4} w(s)\|_{L^2} + C \int_s^t \|\widetilde{\nabla} w(\tau)\|_{L^2}^2 d\tau.$$

By the linear stability and the strong energy inequality,

$$\|(1 + L_E)^{-1/4} w(t)\|_{L^2} \to 0 (t >> s >> 1).$$

Smoothness of weak solutions

As a result, if w satisfies the strong energy inequality, then

$$\lim_{T\to\infty}\int_T^{T+1}\|w(t)\|_{H^1}^2 dt = 0.$$

For an arbitrary number $\varepsilon > 0$ we can take $\tau_0 > 0$ such that

$$w(\tau_0) \in \widetilde{H}^1_{0,\sigma}, \ \|w(\tau_0)\|_{H^1} < \varepsilon.$$

The existence of a unique strong solution
and the uniqueness of weak solutions, $\Rightarrow$
the weak solution is smooth with respect to time when time is sufficiently large.