Time-Periodic Navier-Stokes Equations

Mads Kyed

International Research Training Group (IGK 1529)
Fachbereich Mathematik
Technische Universität Darmstadt (Germany)
Email: kyed@mathematik.tu-darmstadt.de

TECHNISCHE UNIVERSITÄT DARMSTADT

Tokyo, November 28, 2011
The 4th Japanese-German International Workshop on Mathematical Fluid Dynamics

- **Time-periodic Navier-Stokes equations:** Given a time-periodic force

$$f : \Omega \times \mathbb{R} \to \mathbb{R}^3, \quad \Omega \subset \mathbb{R}^3, \quad f(x,t) = f(x, t + \mathcal{T})$$

find a time-periodic solution

$$\begin{aligned} u &: \Omega \times \mathbb{R} \to \mathbb{R}^3 \qquad && u(x,t) = u(x, t+\mathcal{T}) \\ \mathfrak{p} &: \Omega \times \mathbb{R} \to \mathbb{R} && \mathfrak{p}(x,t) = \mathfrak{p}(x, t+\mathcal{T}) \end{aligned}$$

to

$$\begin{cases} \partial_t u - \Delta u + \nabla \mathfrak{p} + u \cdot \nabla u = f & \text{in } \Omega \times \mathbb{R} \\ \operatorname{div} u = 0 & \text{in } \Omega \times \mathbb{R} \end{cases}$$

subject to boundary conditions

$$u = u_* \quad \text{on } \partial\Omega \times \mathbb{R}$$

$$\lim_{|x| \to \infty} u(x,t) = u_\infty$$

- **Serrin 59'**:

$$
(\text{IVP}) \begin{cases} \partial_t u - \Delta u + \nabla \mathfrak{p} + u \cdot \nabla u = f & \text{in } \Omega \times (0, \infty) \\ \text{div } u = 0 & \text{in } \Omega \times (0, \infty) \\ u(x, 0) = u_0(x) & \text{in } \Omega \end{cases}
$$

  1. For <u>some</u> u_0 let $(u, \mathfrak{p})$ be a solution to IVP
  2. Put $\tilde{u}_0(x) := \lim_{t \to \infty} u(x, t)$.
  3. The solution $(\tilde{u}, \tilde{\mathfrak{p}})$ to IVP with initial value $\tilde{u}_0$ is time-periodic.

- Contributions based on Serrins method:
  - Miyakawa&Teramoto 82'
  - Maremonti 90'-08'
  - Galdi&Sohr 04'

- **Prouse 63'/Yudovich 60'**:

$$
\text{(IVP)} \begin{cases} \partial_t u - \Delta u + \nabla \mathfrak{p} + u \cdot \nabla u = f & \text{in } \Omega \times (0, \mathcal{T}) \\ \operatorname{div} u = 0 & \text{in } \Omega \times (0, \mathcal{T}) \\ u(x, 0) = u_0(x) & \text{in } \Omega \end{cases}
$$

  1. Poincaré map:

$$
u_0 \ \rightarrow \ \text{Solution to IVP} \ \rightarrow \ u(\cdot, \mathcal{T})
$$

  2. Find a fixed point $\tilde{u}_0$ of the Poincaré map.
  3. A solution $(\tilde{u}, \tilde{\mathfrak{p}})$ to IVP with inivial value $\tilde{u}_0$ is time-periodic.

- Contributions based on Prouse/Yudovich method:
  - Kaniel&Shinbrot 82'
  - Galdi&Silvestre 06'

- **Kozono&Nakao 96'**:

$$\begin{cases} \partial_t u - \Delta u + \nabla \mathfrak{p} + u \cdot \nabla u = f & \text{in } \mathbb{R}^3 \times \mathbb{R} \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^3 \times \mathbb{R} \\ \lim_{|x| \to \infty} u(x,t) = 0 \end{cases}$$

  Modifiy the method to Fujita&Kato within the framework of the Stokes semigroup.
- Contributions based on Kozono&Nakao method:
  - Yamazaki 00'

- **Time-periodic flow *past* a body ($u_\infty \neq 0$):**

$$\begin{cases} \partial_t u - \Delta u + \nabla \mathfrak{p} + u \cdot \nabla u = f & \text{in } \mathbb{R}^3 \times \mathbb{R} \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^3 \times \mathbb{R} \\ \lim_{|x| \to \infty} u(x,t) = u_\infty \end{cases}$$

  - Existence: Given time-periodic f find time-periodic solution $(u, \mathfrak{p})$.
  - Spatial asymptotics:

$$u(x,t) = \Gamma(x,t) + R(x,t)$$

where Gamma is explicitly known and

$$R(x,t) = o(|\Gamma(x,t)|) \text{ as } |x| \to \infty$$

- **Time-periodic flow *past* a body ($u_\infty \neq 0$):**

$$\begin{cases} \partial_t u - \Delta u + \nabla \mathfrak{p} - u_\infty \cdot \nabla u + u \cdot \nabla u = f & \text{in } \mathbb{R}^3 \times \mathbb{R} \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^3 \times \mathbb{R} \\ \lim_{|x| \to \infty} u(x,t) = 0 \end{cases}$$

  - Existence: Given time-periodic f find time-periodic solution $(u, \mathfrak{p})$.
  - Spatial asymptotics:

$$u(x,t) = \Gamma(x,t) + R(x,t)$$

  where Gamma is explicitly known and

$$R(x,t) = o(|\Gamma(x,t)|) \text{ as } |x| \to \infty$$

- **Reformulation on group:**
  - Identify $\mathbb{R}^3 \times (0, \mathcal{T})$ with the *group* $G := \mathbb{R}^3 \times \mathbb{R}/\mathcal{T}\mathbb{Z}$.
  - G inherits topology and differentiable structure from $\mathbb{R}^3 \times \mathbb{R}$.
  - G is a locally compact abelian group (LCA).
  - Schwartz-Bruhat space $\mathscr{S}(G)$.
  - Tempered distributions $\mathscr{S}'(G)$.
  - Reformulation: Given

$$f : G \to \mathbb{R}^3$$

find a solution

$$u : G \to \mathbb{R}^3, \quad \mathfrak{p} : G \to \mathbb{R}$$

to

$$\begin{cases} \partial_t u - \Delta u + \nabla \mathfrak{p} - u_\infty \cdot \nabla u + u \cdot \nabla u = f & \text{in } G \\ \operatorname{div} u = 0 & \text{in } G \\ \lim_{|x| \to \infty} u(x,t) = 0 & \end{cases}$$

- **Linear theory:** (Assume WLOG $u_\infty = \mathrm{e}_1$)

$$\begin{cases} \partial_t u - \Delta u + \nabla \mathfrak{p} - \partial_1 u = f & \text{in } G \\ \operatorname{div} u = 0 & \text{in } G \end{cases}$$

  - Dual group: $\hat{G} = \mathbb{R}^3 \times \mathbb{Z}$
  - Fourier transform: $\mathcal{F} : \mathscr{S}(G) \to \mathscr{S}(\hat{G})$ and $\mathcal{F} : \mathscr{S}'(G) \to \mathscr{S}'(\hat{G})$
  - Employ the Fourier transform:

$$\widehat{u}(\xi, k) = \frac{1}{|\xi|^2 + i(k - \xi_1)} \left(I - \frac{\xi \otimes \xi}{|\xi|^2} \right) \widehat{f}$$

  - Multiplier representation:

$$u = \mathcal{F}^{-1}\left[m(\xi, k) \widehat{\mathcal{P}_H f} \right]$$

with $\mathcal{P}_H =$Helmholtz projection and

$$m : \hat{G} \to \mathbb{C}, \quad m(\xi, k) := \frac{1}{|\xi^2| + i(k - \xi_1)}$$

- **Time averaging:**
  - $\mathcal{P}: L_q(\mathbb{R}^3 \times \mathbb{R}/\mathcal{T}\mathbb{Z}) \to L_q(\mathbb{R}^3), \quad \mathcal{P}u(x) := \int_{\mathbb{R}/\mathcal{T}\mathbb{Z}} u(x,t)\,\mathrm{d}t$
  - $\mathcal{P}: L_q(G) \to L_q(G)$ is a projection. Put $\mathcal{P}_\perp := \mathrm{Id} - \mathcal{P}$.
  - If $u : G \to \mathbb{R}^3$ is a solution to

$$\begin{cases} \partial_t u - \Delta u + \nabla \mathfrak{p} - \partial_1 u = f & \text{in } \mathbb{R}^3 \times \mathbb{R}/\mathcal{T}\mathbb{Z} \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^3 \times \mathbb{R}/\mathcal{T}\mathbb{Z} \end{cases}$$

  then $v := \mathcal{P}u$ is a solution to

$$\begin{cases} -\Delta v + \nabla p - \partial_1 v = \mathcal{P}f & \text{in } \mathbb{R}^3 \\ \operatorname{div} v = 0 & \text{in } \mathbb{R}^3 \end{cases}$$

- **Maximal regularity for the steady-state Oseen system:**

$$\mathrm{A}_{\mathrm{Oseen}} v := -\Delta v - \partial_1 v$$

$\mathrm{A}_{\mathrm{Oseen}} : \mathrm{X}_{q,\sigma}(\mathbb{R}^3) \to L_{q,\sigma}(\mathbb{R}^3)$ homeomorphism when $q \in (1,2)$ and

$$\mathrm{X}_{q,\sigma} := \left\{ v \mid \operatorname{div} v = 0,\ \nabla^2 v \in L_q,\ \partial_1 v \in L_q,\ \nabla v \in L_{\frac{4q}{4-q}},\ v \in L_{\frac{2q}{2-q}} \right\}$$

- **Symbol of projection:**

$$\mathcal{F}(\mathcal{P}u)(\xi,k) = \beta(k)\widehat{u}(\xi,k), \qquad \text{with } \beta(k) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k \neq 0 \end{cases}$$

- **Splitting of the linear problem:**

$$\begin{aligned} u &= \mathcal{F}^{-1}\left[\frac{1}{|\xi|^2 + i(k-\xi_1)}\widehat{\mathcal{P}_H f}\right] \\ &= \mathcal{F}^{-1}\left[\frac{\beta(k)}{|\xi|^2 + i(k-\xi_1)}\widehat{\mathcal{P}_H f}\right] + \mathcal{F}^{-1}\left[\frac{1-\beta(k)}{|\xi|^2 + i(k-\xi_1)}\widehat{\mathcal{P}_H f}\right] \\ &= \mathcal{P}u + \mathcal{F}^{-1}\left[M(\xi,k)\widehat{\mathcal{P}_H f}\right] \\ &= \mathcal{P}u + \mathcal{P}_\perp u \end{aligned}$$

with

$$M : \hat{G} \to \mathbb{C}, \quad M(\xi,k) := \frac{1-\beta(k)}{|\xi|^2 + i(k-\xi_1)}$$

- **Transference of multipliers:**

  Theorem (de Leeuw, Edwards&Gaudry)

  Let G, H be LCA groups and $\Phi : \hat{G} \to \hat{H}$ a continuous homomorphism. If $m \in L_\infty(\hat{H})$ is a $L_q(H)$-multiplier, then $m \circ \Phi$ is a $L_q(G)$-multiplier.

- **Estimate of $\mathcal{P}_\perp u$:**
  1. Put $H := \mathbb{R}^4$. Then $\hat{H} := \mathbb{R}^4$. Put $\Phi : \hat{G} \to \hat{H}$, $\Phi(\xi, k) := (\xi, k)$.
  2. Let $(\xi, \eta) \in \mathbb{R}^3 \times \mathbb{R} = \mathbb{R}^4$ and choose

$$\chi \in C_0^\infty(\mathbb{R}^4), \quad \chi(\xi, \eta) = 1 \text{ for } |\eta| < \frac{1}{2}, \ \chi(\xi, \eta) = 0 \text{ for } |\eta| \geq 1.$$

  3. Put

$$\tilde{M} : \hat{H} \to \mathbb{C}, \quad \tilde{M}(\xi, \eta) := \frac{1 - \chi(\eta)}{|\xi^2| + i(\eta - \xi_1)}$$

  4. By Marcinkiewicz's multiplier theorem $\tilde{M}$ is an $L_q(\mathbb{R}^4)$-multiplier.
  5. By de Leeuw's theorem $\tilde{M} \circ \Phi = M$ is an $L_q(G)$-multiplier and thus

$$\|\mathcal{P}_\perp u\|_q = \|\mathcal{F}^{-1}\left[M(\xi, k)\widehat{\mathcal{P}_H f}\right]\|_q \leq C\|\mathcal{P}_H f\|_q \leq C\|f\|_q$$

- **Maximal regularity for linear problem:**

$$\begin{cases} \partial_t u - \Delta u + \nabla \mathfrak{p} - \partial_1 u = f & \text{in } G \\ \operatorname{div} u = 0 & \text{in } G \end{cases}$$

Definition

$$\mathrm{X}_{q,\sigma}(\mathbb{R}^3) := \left\{ v \mid \operatorname{div} v = 0,\ \nabla^2 v \in L_q,\ \partial_1 v \in L_q,\ \nabla v \in L_{\frac{4q}{4-q}},\ v \in L_{\frac{2q}{2-q}} \right\}$$
$$W^{2,1}_{q,\sigma}(G) := \left\{ u \mid \operatorname{div} v = 0,\ \nabla^2 u \in L_q,\ \partial_t u \in L_q,\ u \in L_q \right\}$$

Theorem (K.11')

For every $f \in L_{q,\sigma}(G)$ there is a unique

$$(v, w) \in \mathrm{X}_{q,\sigma}(\mathbb{R}^3) \times W^{2,1}_{q,\sigma}(G)$$

such that

$$u(x,t) := v(x) + w(x,t) \quad (= \mathcal{P}u + \mathcal{P}_\perp u)$$

is a solution. Moreover

$$\|v\|_{\mathrm{X}_{q,\sigma}} + \|w\|_{W^{2,1}_{q,\sigma}} \leq C \|f\|_q$$

- **Existence of strong solutions:**

$$\begin{cases} \partial_t u - \Delta u + \nabla \mathfrak{p} - \partial_1 u + u \cdot \nabla u = f & \text{in } G \\ \operatorname{div} u = 0 & \text{in } G \\ \lim_{|x| \to \infty} u(x,t) = 0 \end{cases}$$

Theorem (K.11')

Let $q \in (1,2)$, $r \in (4, \infty)$. For every $f \in L_q(G) \cap L_r(G)$ with $\|f\|$ sufficiently small there is a locally unique $(v, w) \in \mathcal{K}_{q,\sigma}$ such that

$$u(x,t) := v(x) + w(x,t)$$

is a solution.

Proof.

Fixed-point argument based on the maximal regularity of the linear problem. □

- **Spacial asymptotics:**

$$\begin{cases} \partial_t u - \Delta u + \nabla \mathfrak{p} - \partial_1 u + u \cdot \nabla u = f & \text{in } G \\ \operatorname{div} u = 0 & \text{in } G \\ \lim_{|x| \to \infty} u(x,t) = 0 \end{cases}$$

Theorem (K.11')

A strong solution satisfies

$$u(x,t) := \Gamma_{\mathrm{Oseen}}(x) \cdot \left(\int_G f \right) + O(|x|^{-4/3+\varepsilon})$$

Proof.

- Recall $u = \mathcal{P}u + \mathcal{P}_\perp u$ and $\mathcal{P}u$ satisfies the steady-state problem.
- Show $\mathcal{P}_\perp u \in O(|x|^{-4/3+\varepsilon})$

□

Thank you for your attention!