On uniqueness of stationary solutions to the Navier-Stokes equations in exterior domains

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December 1

Ω: Exterior domain in $\mathbb{R}^3$

$$
(NS) \begin{cases} -\Delta u + u \cdot \nabla u + \nabla p = \text{div } F & \text{in } \Omega, \\ \text{div } u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u(x) \to 0 & \text{as } |x| \to \infty. \end{cases}
$$

$u = (u_1, u_2, u_3)$: unknown velocity vector

p: unknown pressure

div F: given external force

Kozono-Yamazaki (1998)

(1)$F \in L_{3/2,\infty}(\Omega)$, $\|F\|_{L_{3/2,\infty}(\Omega)} \le \exists\delta$

$\Rightarrow$ $\exists$ solution $\{u,p\} \in \dot{H}^1_{3/2,\infty}(\Omega) \times L_{3/2,\infty}(\Omega)$ of (NS)

(2)$\{v,q\} \in \dot{H}^1_{3/2,\infty}(\Omega) \times L_{3/2,\infty}(\Omega)$: another solution

$\|u\|_{L_{3,\infty}(\Omega)}, \|v\|_{L_{3,\infty}(\Omega)} \le \exists\widetilde{\delta} \Rightarrow \{u,p\} = \{v,q\}$

$(\dot{H}^1_{3/2,\infty}(\Omega) \subset L_{3,\infty}(\Omega))$

Question

Either u or v is small in $L_{3,\infty}(\Omega)$ $\Rightarrow$ Uniqueness?

Function spaces: $1 < p < \infty$ **and** $1 \leq q \leq \infty$

- $L_{p,q}(\Omega)$: **Lorentz space**
- $\dot{H}^1_p(\Omega)$: **the completion of** $C_0^\infty(\Omega)$ **in the norm** $\|\nabla \cdot\|_{L_p(\Omega)}$
- $\dot{H}^1_{p,q}(\Omega) := (\dot{H}^1_{p_0}(\Omega), \dot{H}^1_{p_1}(\Omega))_{\theta,q} \qquad (1/p = (1-\theta)/p_0 + \theta/p_1)$

L_p **theory**

- **Nonlinear structure of (NS)** $\Rightarrow$ **Seek a solution** $u \in \dot{H}^1_{3/2}(\Omega)$
- $u \in \dot{H}^1_{3/2}(\Omega)$ **is a solution**

$$\Rightarrow \int_{\partial\Omega} (T[u,p] + F) \cdot \nu \, dS = 0$$

$(T[u,p] = (\partial_i u_j + \partial_j u_i - \delta_{ij} p)_{i,j=1}^3)$

- **The best decay rates:** $|u(x)| = O(|x|^{-1})$, $|\nabla u(x)| = O(|x|^{-2})$

Known results

- **Kozono-Yamazaki (1999)**

  $u, v \in \dot{H}^1_2(\Omega)$: **solutions of (NS)**

  $\|\nabla u\|^2_{L_2(\Omega)} \le -(F, \nabla u)$ **and** $\|v\|_{L_{3,\infty}(\Omega)} < \exists\delta \Rightarrow u = v$

- **Taniuchi (2009), Farwig-Taniuchi (2011)**
  (restricted to the stationary problem)

  $u, v \in L_{3,\infty}(\Omega)$: **solutions of (NS)**

  $\|u\|_{L_{3,\infty}(\Omega)} < \exists\widetilde{\delta}$ **and** $u, v \in L_{6,2}(\Omega) \Rightarrow u = v$

Main result

Theorem. *Suppose that* $\{u, p\}, \{v, q\} \in \dot{H}^1_{3/2,\infty}(\Omega) \times L_{3/2,\infty}(\Omega)$ *are solutions of (NS). There exists an absolute constant* $\delta > 0$ *such that if*

$$\|u\|_{L_{3,\infty}(\Omega)} \leq \delta$$

and

$$u, v \in L_r(\Omega)$$

for some $r > 3$*, then* $\{u, p\} = \{v, q\}$*.*

For the proof, we consider

$$(W)\begin{cases} -\Delta w + w\cdot\nabla u + v\cdot\nabla w + \nabla\pi = 0 & \text{in } \Omega,\\ \text{div } w = 0 & \text{in } \Omega,\\ w = 0 & \text{on } \partial\Omega, \end{cases}$$

$(w := u - v, \pi := p - q)$ and its dual equation

$$(D)\begin{cases} -\Delta\psi - \sum_{i=1}^{3} u_i\nabla\psi_i - v\cdot\nabla\psi + \nabla\chi = f & \text{in } \Omega,\\ \text{div } \psi = 0 & \text{in } \Omega,\\ \psi = 0 & \text{on } \partial\Omega. \end{cases}$$

- The relation

Solvability of (D) $\longleftrightarrow$ Uniqueness of (W)

Basic idea

- **Solvability of (D) in the class** $\psi \in \dot{H}^1_{p',q'}(\Omega)$
- **Take** $w \in \dot{H}^1_{p,q}(\Omega)$ **as a test function in the weak form of (D):**

$$(\nabla \psi, \nabla \varphi) - (\sum_{i=1}^{3} u_i \nabla \psi_i, \varphi) - (v \cdot \nabla \psi, \varphi) - (\chi, \mathrm{div}\ \varphi) = (f, \varphi)$$

and $\psi \in \dot{H}^1_{p',q'}(\Omega)$ **in the weak form of (W):**

$$(\nabla w, \nabla \widetilde{\varphi}) - (\sum_{i=1}^{3} u_i \nabla \widetilde{\varphi}_i, w) - (v \cdot \nabla \widetilde{\varphi}, w) - (\pi, \mathrm{div}\ \widetilde{\varphi}) = 0$$

$\Rightarrow$ **We obtain**

$$(f, w) = 0 \quad \text{for all } f \in \dot{H}^1_{p,q}(\Omega)^*$$

Difficulty

$C_0^\infty(\Omega)$ is not dense in $\dot{H}^1_{3/2,\infty}(\Omega)$

Outline of the proof

- (W): Additional regularity $u, v \in L_r(\Omega)$ $(r > 3)$ and the regularity criterion for the Stokes equation $\Rightarrow$ Show $w \in \dot{H}^1_{p,q}(\Omega)$ where $C_0^\infty(\Omega)$ is dense

- (D): L_2 theory and the bootstrap argument $\Rightarrow$ Desired regularity $\psi \in \dot{H}^1_{p',q'}(\Omega)$

- The duality argument (previous page)

Regularity criterion for the Stokes equation

Let $1 < p_0 < 3$, $3/2 < p_1 < 3$ and $1 < q_0, q_1 \leq \infty$.

For $f \in \dot{H}^1_{p_0', q_0'}(\Omega)^* \cap \dot{H}^1_{p_1', q_1'}(\Omega)^*$, $\{u, p\} \in \dot{H}^1_{p_0, q_0}(\Omega) \times L_{p_0, q_0}(\Omega)$ is a solution of

$$\begin{cases} -\Delta u + \nabla p = f & \text{in } \Omega, \\ \text{div } u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

$\Rightarrow \{u, p\} \in \dot{H}^1_{p_1, q_1}(\Omega) \times L_{p_1, q_1}(\Omega)$