L_p-theory for a generalized viscoelastic fluid model on a domain and with a free surface

Manuel Nesensohn
IRTG 1529
Fachbereich Mathematik
TU Darmstadt

Joint work with Matthias Geißert and Dario Götz

Introduction

Fixed domains

Free boundary value problem

Oldroyd-B and generalized viscoelastic fluid model

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Oldroyd-B model:

$$\partial_t u - \Delta u + \nabla \pi + u.\nabla u = \operatorname{div} \tau, \quad \operatorname{div} u = 0, \quad u|_{\partial\Omega} = 0 \qquad \text{in } (0, T_0) \times \Omega$$

$$\partial_t \tau + u.\nabla \tau + \tau = \nabla u + \nabla u^T + \tau \nabla u^T + \nabla u \tau \qquad \text{in } (0, T_0) \times \Omega$$

$$u(0) = u_0, \quad \tau(0) = \tau_0 \qquad \text{in } \Omega$$

u: velocity filed, π: pressure, τ: elastic part of the stress

Oldroyd-B and generalized viscoelastic fluid model

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Generalized nonlinear viscoelastic fluid model:

- $-\Delta u \curvearrowright A(u)u$ with $A(u)u = -\operatorname{Div} 2\alpha(|Eu|^2)Eu$
  (α viscosity function, $Eu = \frac{1}{2}(\nabla u + (\nabla u)^T)$ symmetric part of the gradient)
- $\operatorname{div} \tau \curvearrowright \operatorname{div} \mu(\tau)$ $\quad \nabla u + \nabla u^T + \tau \nabla u^T + \nabla u \tau \curvearrowright g(\nabla u, \tau)$

Oldroyd-B and generalized viscoelastic fluid model

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- $\operatorname{div}\tau \curvearrowright \operatorname{div}\mu(\tau) \qquad \nabla u + \nabla u^T + \tau\nabla u^T + \nabla u\tau \curvearrowright g(\nabla u, \tau)$

Boundary conditions:

- $u|_{\partial\Omega} = 0 \curvearrowright u = 0$ on Γ_D, $\quad (u \cdot \nu, [2\alpha(|Eu|^2)Eu\nu + \mu(\tau)\nu]_{\tan}) = 0$ on Γ_S

Oldroyd-B and generalized viscoelastic fluid model

Oldroyd-B model:

$$\partial_t u - \Delta u + \nabla\pi + u.\nabla u = \operatorname{div}\tau, \quad \operatorname{div} u = 0, \quad u|_{\partial\Omega} = 0 \qquad \text{in } (0, T_0) \times \Omega$$

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Generalized nonlinear viscoelastic fluid model:

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Free boundary $\Gamma_F(t)$:

- $u|_{\partial\Omega} = 0 \curvearrowright 2\alpha(|Eu|^2)Eu\nu - \pi\nu + \mu(\tau)\nu = 0, \quad V = u \cdot \nu$ on $\Gamma_F(t)$

Part A: Fixed Domains

Generalized nonlinear viscoelastic fluid model on a fix domain Ω with Dirichlet and perfect slip boundary conditions

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System:

$$
(1) \quad \begin{cases}
\partial_t u + A(u)u + \nabla\pi + u.\nabla u = \operatorname{div}\mu(\tau) & \text{in } (0,T_0)\times\Omega \\
\operatorname{div} u = 0 & \text{in } (0,T_0)\times\Omega \\
\partial_t \tau + u.\nabla\tau + \tau = g(\nabla u,\tau) & \text{in } (0,T_0)\times\Omega \\
u = 0 & \text{in } (0,T_0)\times\Gamma_D \\
(u\cdot\nu, [2\alpha(|Eu|^2)Eu\nu + \mu(\tau)\nu]_{\tan}) = 0 & \text{in } (0,T_0)\times\Gamma_S \\
u(0) = u_0 & \text{in } \Omega \\
\tau(0) = \tau_0 & \text{in } \Omega
\end{cases}
$$

u: velocity filed, π: pressure, τ: elastic part of the stress
$A(u)v$ is defined by:

$$[A(u)v]_j = -\sum_{k,l,m} A^{l,m}_{j,k}(Eu)\partial_l\partial_m v_k = -\alpha(|Eu|^2)\Delta v_j - \sum_{k,l,m} 4\alpha'(|Eu|^2)(Eu)_{jl}(Eu)_{km}\partial_l\partial_m v_k$$

Main result: Arbitrary data

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Theorem (Götz, N.)

Assume:

- $T_0 > 0$, $n + 2 < p < \infty$, α, μ, g *and* $\partial\Omega$ *sufficiently smooth*
- $\alpha(s) > 0$, $\alpha(s) + 2s\alpha'(s) > 0$ *for* $s \geq 0$
- Ω *bounded domain*
- $\partial\Omega = \Gamma_D \cup \Gamma_S$ *with* Γ_D *and* Γ_S *disjoint, open and close*
- $u_0 \in W_p^{2-2/p}$, $\tau_0 \in H_p^1$ *with* $\operatorname{div} u_0 = 0$,

$$u_0 = 0 \text{ on } \Gamma_D \quad \text{and} \quad (u_0 \cdot \nu, [2\alpha(|Eu_0|^2)Eu_0\nu + \mu(\tau_0)\nu]_{tan}) = 0 \text{ on } \Gamma_S$$

Then: $\exists!\ T < T_0$ *and a unique strong solution of* (1) *on* $J = (0, T)$ *in*

$$u \in L_p(J; H_p^2) \cap H_p^1(J; L_p), \quad \pi \in L_p(J; \hat{H}_p^1), \quad \tau \in L_\infty(J; H_p^1) \cap W_\infty^1(J; L_p)$$

Main result: Small data

Theorem (Geißert, Götz, N.)

Assume:

- $T_0 > 0$, $n + 2 < p < \infty$, α, μ, g *and* $\partial\Omega$ *sufficiently smooth*
- $\alpha(0) > 0$ *and* $g(0, 0) = 0$
- Ω *domain s.t.* $\lambda_0 + A_{Stokes}$ *admits* $\mathcal{BIP}$ *(e.g. bounded, exterior domain, layer)*
- $\Gamma_S = \emptyset$
- $u_0 \in W_p^{2-2/p}$, $\tau_0 \in H_p^1$ *with* div $u_0 = 0$, $u_0|_{\partial\Omega} = 0$

Then: $\exists\, \kappa(T_0) > 0$ *and a unique strong solution of* (1) *on* $J_0 = (0, T_0)$ *in*

$$u \in L_p(J_0; H_p^2) \cap H_p^1(J_0; L_p), \quad \pi \in L_p(J_0; \hat{H}_p^1), \quad \tau \in L_\infty(J_0; H_p^1) \cap W_\infty^1(J_0; L_p)$$

provided

$$\|u_0\|_{W_p^{2-2/p}} + \|\tau_0\|_{H_p^1} + |\nabla g(0, 0)| \leq \kappa(T_0)$$

Known and related results

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Generalized Navier-Stokes equations:

- Amann (1994): Generalized Navier-Stokes, small initial data, L_p-theory
- Bothe, Prüß (2007): Generalized Navier-Stokes, arbitrary initial data, L_p-theory

Viscoelastic fluid model:

- Guillopé, Saut (1990): Oldroyd-B, bounded domain, L_2-theory
- Ferández-Cara, Guillén, Ortega (1998): Oldroyd-B, bounded domain, L_p-theory
- Hakim (1999): White-Metzner, bounded domain, L_2-theory
- Molinet, Talhouk (2004), White-Metzner, bounded domain, L_2-theory
- Vorotnikov, Zvyagion (2004): generalized nonlinear viscoelastic fluid, whole space, L_2-theory

Sketch of the proof

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- Linearize / write in form of a fixed point equation
- Solve linearized problem
- Apply Fixed point theorem
- Show uniqueness of solution

Fixed point formulation

- Choose suitable (u_*, π_*)
- Ansatz: $u = u_* + w$ and $\pi = \pi_* + \psi$
- System (1) is equivalent to fixed point equation $\Phi : (\bar{w}, \bar{\tau}) \to (w, \tau)$

$$\begin{cases} \partial_t w + A(u_*)w + \nabla\psi = F(\bar{\tau}, \bar{w}), \quad \operatorname{div} w = 0 & \text{in } (0, T_0) \times \Omega \\ \partial_t \tau + (u_* + \bar{w}).\nabla\tau + \tau = G(\bar{\tau}, \bar{w}) & \text{in } (0, T_0) \times \Omega \\ w = 0 & \text{in } (0, T_0) \times \Gamma_D \\ (w \cdot \nu, B_S(u_*)w) = (0, H(\bar{w}) + T(\bar{\tau})) & \text{in } (0, T_0) \times \Gamma_S \\ w(0) = u_0 - u_*(0), \quad \tau(0) = \tau_0 & \text{in } \Omega \end{cases}$$

with $B_S(u_*)w = \left[\sum_{k,l=1}^{n} A^{k,l}(Eu_*)\nu_k(-i\partial_l)w \right]_{\tan}$

-

Case: Arbitrary data	Case: Small data
u_* sol. of Stokes with $u_*(0) = u_0$	$u_* \equiv 0$
$A(u_*)$: generalized Stokes operator	$A(u_*) = A(0) = -\alpha(0)\Delta$

Contraction mapping principle?

The transport equation

$$(2) \qquad \partial_t \tau + \bar{u}.\nabla\tau + \tau = g \quad \text{in } (0,T)\times\Omega, \qquad \tau(0) = \tau_0$$

is not regularizing in space, i.e.

Proposition

Assume:

- $\bar{u} \in L_p(J; H_p^2)$, $\tau_0 \in H_p^1$ *with* $\bar{u}\cdot\nu = 0$ *and*
- $g \in L_1(J; H_p^1)$

$\Rightarrow \exists!$ *a unique strong solution* $\tau \in L_\infty(J; H_p^1) \cap H_p^1(J; L_p)$ *of* (2)

Two solutions τ_1, τ_2 to the data $\bar{u}_1, g_1$ resp. $\bar{u}_2, g_2$ fulfills the equation

$$\partial_t(\tau_2 - \tau_1) + \bar{u}_1.\nabla(\tau_2 - \tau_1) + (\tau_2 - \tau_1) = g_2 - g_1 - (\bar{u}_2 - \bar{u}_1).\underbrace{\nabla\tau_2}_{L_\infty(L_p)}$$

Case: Arbitrary data

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Existence: Apply Schauder's fixed point theorem. Define the sets:

$$X_w = L_p(J; H_p^2) \cap H_p^1(J; L_p), \qquad X_\tau := L_\infty(J; H_p^1) \cap W_\infty^1(J; L_p),$$

$$K_w = \{w \in {}_0X_w : \|w\|_{X_w} \le R_1\}, \quad K_\tau := \{\tau \in X_\tau : \tau(0) = \tau_0, \|\tau\|_{L_\infty(H_p^1)} \le R_2$$
$$\|\partial_t \tau\|_{L_\infty(L_p)} \le R_3\}$$

- Choose R_1, R_2, R_3 such that $\Phi(K_w \times K_\tau) \subset K_w \times K_\tau$:
  Key: Maximal regularity of generalized Stokes (Bothe, Prüß)
  Estimate of the transport equation (last slide)
- $K_w \times K_\tau$ is compact in $C(J; C^1) \times C(J; C)$
- $\Phi|_{K_w \times K_\tau}$ is continuous in the topology of $C(J; C^1) \times C(J; C)$:
  Key in step 2 and 3: Compact embeddings

$$X_w \overset{c}{\hookrightarrow} C(J; C^1) \quad \text{and} \quad X_\tau \overset{c}{\hookrightarrow} C(J; C).$$

Case: Arbitrary data

Uniqueness:

- Let u_j, τ_j be two solutions. Since $p > n+2$ and Ω bounded
  $\Rightarrow u_j \in L_2(J; H_2^2) \cap H_2^1(J; L_2)$, $\tau_j \in L_2(J; H_2^1) \cap H_2^1(J; L_2)$.
- Apply energy methods
- Boundary integrals vanish due to boundary conditions
- Define $S(Eu) = 2\alpha(|Eu|^2)Eu$
  Note that $\operatorname{div} S(Eu) = A(u)u$
  Strong monotonicity of S:

$$\alpha(s) > 0, \quad \alpha(s) + 2s\alpha'(s) > 0, \quad s \geq 0$$
$$\Rightarrow \exists \alpha_0 > 0 : (S(Eu_2) - S(Eu_1)) : (Eu_2 - Eu_1) \geq \alpha_0 |Eu_2 - Eu_1|$$

Idea: Use the following version of the contraction mapping principle:

Proposition

Assume

- *X Banach space with a separable predual, Y Banach space with $X \hookrightarrow Y$*
- *$K \subset X$ convex, cosed and bounded*
- *$\Phi\colon X \to X$ with $\Phi(K) \subset K$ and for $\eta < 1$*

$$\|\Phi(x) - \Phi(y)\|_Y \leq \eta \|x - y\|_Y, \quad x, y \in K$$

Then: $\exists!$ a unique fixed point of Φ in K

Our setting:

$$X_u = L_p(J_0; H_p^2 \cap H_{p,0}^1) \cap H_p^1(J_0; L_p), \quad X_\tau = L_\infty(J_0; H_p^1) \cap W_\infty^1(J_0; L_p),$$
$$Y_u = L_p(J_0; H_{p,0}^1) \cap H_p^{1/2}(J_0; L_p), \qquad Y_\tau = L_\infty(J_0; L_p)$$

Case: Small Data

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We can write $F(\tau, w) = \operatorname{div} \mathbf{F}(\tau, w)$. Consider

$$(3) \quad \partial_t u - \Delta u + \nabla \pi = \operatorname{div} \mathbf{f}, \quad \operatorname{div} u = 0, \quad u|_{\partial\Omega} = 0 \qquad \text{in } (0, T_0) \times \Omega, \quad u(0) = 0$$

$$(4) \quad \partial_t \tau + \bar{u}.\nabla \tau + \tau = g \qquad \text{in } (0, T_0) \times \Omega, \quad \tau(0) = 0$$

Proposition

- *For $\mathbf{f} \in L_p(J; H_p^1)$ we can estimate the unique solution of equation* (3) *by*

$$\|u\|_{Y_u} = \|u\|_{L_p(J;H_p^1) \cap H_p^{\frac{1}{2}}(J;L_p)} \leq C \|\mathbf{f}\|_{L_p(J;L_p)}$$

- *For $g \in L_1(H_p^1)$ we can estimate the unique solution of equation* (4) *by*

$$\|\tau\|_{Y_\tau} = \|\tau\|_{L_\infty(J;L_p)} \leq C \|g\|_{L_1(J;L_p)}$$

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Part B: Free boundary value problem

Generalized viscoelastic fluid model with a free boundary and without surface tension

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System:

$$\left\{\begin{array}{rcll} \partial_t u + A(u)u + \nabla\pi + u.\nabla u & = & \operatorname{div}\mu(\tau) & \text{in } (0,T_0)\times\Omega(t) \\ \operatorname{div} u & = & 0 & \text{in } (0,T_0)\times\Omega(t) \\ \partial_t\tau + u.\nabla\tau + \tau & = & g(\nabla u,\tau) & \text{in } (0,T_0)\times\Omega(t) \\ (2\alpha(|Eu|^2)Eu - \pi + \mu(\tau))\nu & = & 0 & \text{in } (0,T_0)\times\Gamma_F(t) \\ V & = & u\cdot\nu & \text{in } (0,T_0)\times\Gamma_F(t) \\ u & = & 0 & \text{in } (0,T_0)\times\Gamma_D \\ u(0) & = & u_0 & \text{in } \Omega_0 \\ \tau(0) & = & \tau_0 & \text{in } \Omega_0 \\ \Gamma_F(0) & = & \Gamma_{F,0} & \end{array}\right.$$

u: velocity filed, π: pressure, τ: elastic part of the stress, Γ_F: free surface, V: normal velocity of Γ_F

Formulation in Lagrange coordinates

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Set $(v, \theta, \eta)(t, \xi) = (u, \pi, \zeta)(t, X_u(t, \xi))$ with $X_u(t, \xi) = \xi + \int_0^t u(s, X_u(s, \xi))ds$
System in Lagrangian coordinates:

$$
(5) \qquad \begin{cases}
\partial_t v + A(v)v + \nabla\theta = \bar{F}(v, \theta, \eta) & \text{in } (0, T_0) \times \Omega_0 \\
\operatorname{div} u = \bar{F}_d(v) & \text{in } (0, T_0) \times \Omega_0 \\
\partial_t \eta + \eta = \bar{G}(v, \eta) & \text{in } (0, T_0) \times \Omega_0 \\
(2\alpha(|Eu|^2)Eu - \theta)\nu_0 = \bar{H}(v, \theta, \eta) & \text{in } (0, T_0) \times \Gamma_{F,0} \\
v = 0 & \text{in } (0, T_0) \times \Gamma_D \\
v(0) = u_0 & \text{in } \Omega_0 \\
\eta(0) = \tau_0 & \text{in } \Omega_0
\end{cases}
$$

Note: Transport term vanishes, i.e. $(\partial_t \tau + u.\nabla\tau)(t, X_u(t, \xi)) = \partial_t \eta(t, \xi)$

Main result

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Theorem (N.)

Assume:

- $T_0 > 0$, $n + 2 < p < \infty$, α, μ, g *and* $\partial\Omega_0$ *sufficiently smooth*
- $\alpha(s) > 0$, $\alpha(s) + 2s\alpha'(s) > 0$ *for* $s \geq 0$, $\mu(0) = g(0,0) = 0$
- Ω_0 *domain with compact boundary*
- $\partial\Omega_0 = \Gamma_D \cup \Gamma_{F,0}$ *with* Γ_D *and* $\Gamma_{F,0}$ *disjoint, open and close*
- $u_0 \in W_p^{2-2/p}$, $\tau_0 \in H_p^1$ *with* div $u_0 = 0$,

$$u_0 = 0 \text{ on } \Gamma_D \quad \text{and} \quad [2\alpha(|Eu_0|^2)Eu_0\nu + \mu(\tau_0)\nu]_{tan} = 0 \text{ on } \Gamma_{F,0}$$

Then: $\exists!\ T < T_0$ *and a unique strong solution of* (5) *on* $J = (0, T)$ *in*

$$v \in L_p(J; H_p^2) \cap H_p^1(J; L_p), \quad \theta \in L_p(J; \hat{H}_p^1), \quad \eta \in L_\infty(J; H_p^1) \cap W_\infty^1(J; L_p)$$

$$\gamma_{\Gamma_{F,0}}\theta \in L_p(J, W_p^{1-\frac{1}{p}}) \cap W_p^{\frac{1}{2}-\frac{1}{2p}}(J; L_p)$$

Known and related results

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Navier-Stokes equations with a free boundary in Euler coordinates

- Beale, Nishida (1985): Layer-like domain, L_2-theory
- Prüß, Simonett (2009, 2010): Two-phase flow with surface tension, L_p-theory
- Denk, Geißert, Hieber, Saal, Sawada (2011): Spin coating process with surface tension, L_p-theory

Navier-Stokes equations with a free boundary in Lagrange coordinates

- Solonnikov (1977, . . .): with and without surface tension, Hölder spaces, L_2-theory, L_p-theory
- Shibata, Shimizu (2007, 2011): with and without surface tension, L_p-L_q-theory
- Beale, Tanaka, Tani . . .

Generalized Navier-Stokes equation with a free boundary

- Plotnikov (1993): Two-Phase flow, monotone operator theory and energy estimates
- Abels (2007): Two-Phase flow, monotone operator theory and energy estimates

Sketch of the proof

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- Linearize around suitable (v_*, θ_*, η_*)
- Ansatz: $v = w + v_*$, $\theta = \theta_* + \zeta$ and $\eta = \tau_0 + \zeta$
- System (5) is equivalent to the fixed point equation $\Phi : (\bar{w}, \bar{\psi}, \bar{\zeta}) \to (w, \psi, \zeta)$

$$\begin{cases} \partial_t w + A(v_*)w + \nabla\psi = F(\bar{w}, \bar{\psi}, \bar{\zeta}), \quad \operatorname{div} w = F_d(\bar{w}) & \text{in } (0, T_0) \times \Omega_0 \\ \partial_t \zeta + \zeta = G(\bar{w}, \bar{\zeta}) & \text{in } (0, T_0) \times \Omega_0 \\ B_N(w, \psi) = H(\bar{w}, \bar{\psi}, \bar{\zeta}) & \text{in } (0, T_0) \times \Gamma_{F,0} \\ w = 0 & \text{in } (0, T_0) \times \Gamma_D \\ w(0) = 0, \quad \zeta(0) = 0 & \text{in } \Omega_0 \end{cases}$$

  with $B_N(v_*)(w, \psi) = \sum_{k,l=1}^{n} A^{k,l}(Ev_*)\nu_{0,k}(-i\partial_l)w + i\psi\nu_0$
- Apply standard contraction mapping principle:
  - Use Bothe and Prüß's result about maximal regularity of the generalized Stokes system
  - Integrate the transformed transport equation
  - Analyse the nonlinearities F, F_d, G, H

THANK YOU FOR YOUR ATTENTION