Stationary solutions of a three species population model with a protection zone

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Prey-Predator model (P0)

$$\begin{cases} u_t = \Delta u + u(\lambda - u - b\chi_{\Omega\setminus\Omega_0}(x)v) & \text{in } \Omega \times (0,\infty), \\ v_t = \Delta v + v(\mu + cu - v) & \text{in } \Omega\setminus\bar{\Omega}_0 \times (0,\infty), \\ \partial_n u = 0 & \text{on } \partial\Omega \times (0,\infty), \\ \partial_n v = 0 & \text{on } \partial(\Omega\setminus\bar{\Omega}_0) \times (0,\infty). \end{cases}$$

Ω, Ω_0 : bounded domains in $\mathbb{R}^N (\bar{\Omega}_0 \subset \Omega,\ N \geq 1)$.

$\partial\Omega, \partial\Omega_0$: smooth boundaries of Ω, Ω_0.

$u(x,t)$: population density of a prey species.

$v(x,t)$: population density of a predator species.

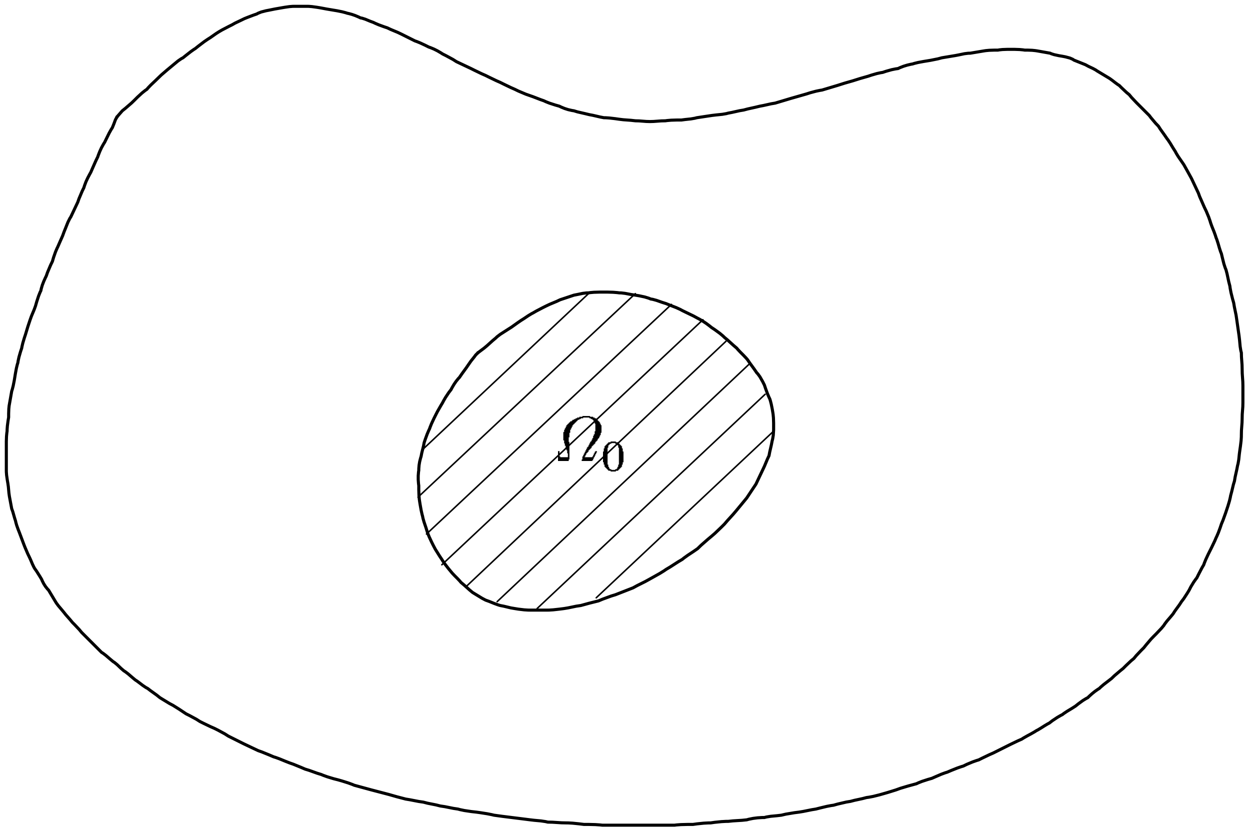


Fig.1. Protection zone Ω_0.

The prey species u can enter and leave Ω_0 freely.

The predator species v can not enter Ω_0.

Prey-Predator model (P0)

$$\begin{cases} u_t = \Delta u + u(\lambda - u - b\chi_{\Omega\setminus\Omega_0}(x)v) & \text{in } \Omega \times (0,\infty), \\ v_t = \Delta v + v(\mu + cu - v) & \text{in } \Omega\setminus\bar{\Omega}_0 \times (0,\infty), \\ \partial_n u = 0 & \text{on } \partial\Omega \times (0,\infty), \\ \partial_n v = 0 & \text{on } \partial(\Omega\setminus\bar{\Omega}_0) \times (0,\infty). \end{cases}$$

$\lambda > 0$: growth rate of the prey species.

$\mu \in \mathbb{R}$: growth rate of the predator species.

$b > 0, c > 0$: interaction coefficients.

$$\chi_{\Omega\setminus\Omega_0}(x) = \begin{cases} 1 & (x \in \Omega\setminus\Omega_0), \\ 0 & (x \in \Omega_0), \end{cases} \qquad \partial_n = \partial/\partial n.$$

Stationary problem (SP0)

$$\begin{cases} \Delta u + u(\lambda - u - b\chi_{\Omega\setminus\Omega_0}(x)v) = 0 & \text{in } \Omega, \\ \Delta v + v(\mu + cu - v) = 0 & \text{in } \Omega\setminus\bar{\Omega}_0, \\ \partial_n u = 0 & \text{on } \partial\Omega, \\ \partial_n v = 0 & \text{on } \partial(\Omega\setminus\bar{\Omega}_0). \end{cases}$$

positive solution of (SP0)

($u > 0$ in Ω, $v > 0$ in $\Omega\setminus\bar{\Omega}_0$)

$\Rightarrow$ coexistence state of two species.

Known results of (SP0)

Def.

$\lambda_1^D(\Omega_0)$: 1st eigenvalue of $-\Delta$ in Ω_0 (Dirichlet).

Proposition (Du-Shi '06, O. '11)

(i) Let $\mu \geq 0$.

(SP0) has a positive solution $\Leftrightarrow \lambda > {}^{\exists}\lambda^*(\mu, \Omega_0)$,

where $\lim_{\mu\to\infty} \lambda^*(\mu, \Omega_0) = \lambda_1^D(\Omega_0)$.

(ii) Let $\mu < 0$.

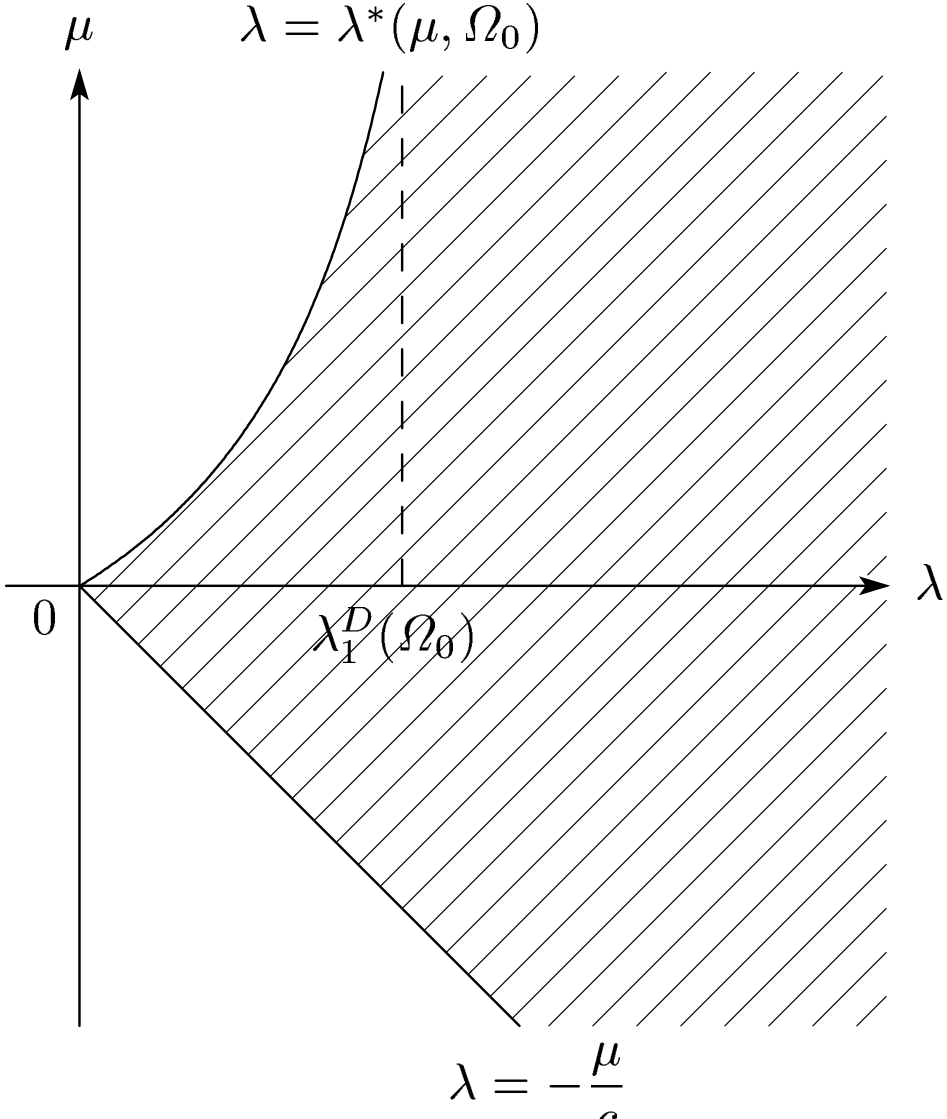
(SP0) has a positive solution $\Leftrightarrow \lambda > -\mu/c$.

$\lambda > 0$: growth rate of the prey species.

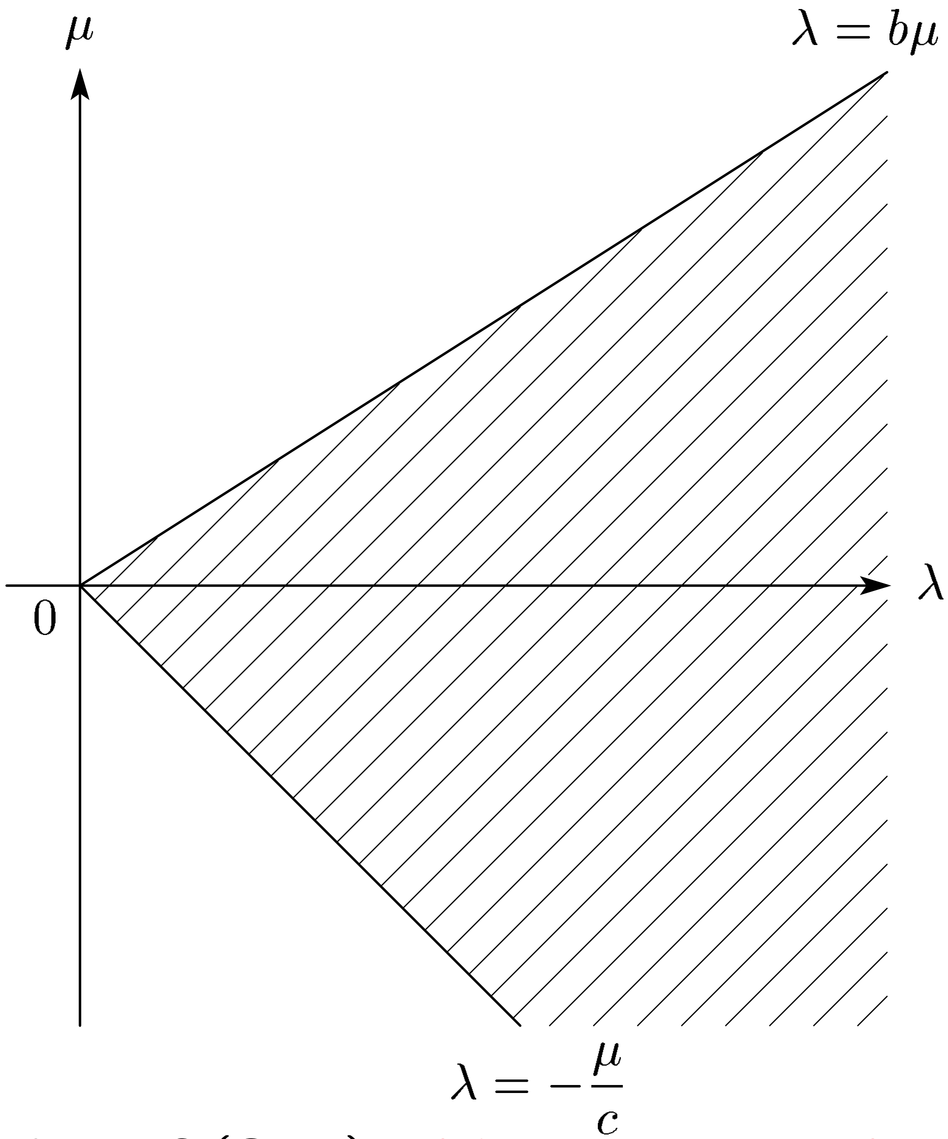
$\mu \in \mathbb{R}$: growth rate of the predator species.

Remark

$\lambda_1^D(\Omega_0)$: threshold prey growth rate for survival.



Coexistence region of (SP0)



Coexistence region of (SP0) without a protection zone ($\Omega_0 = \emptyset$)

Known results of (SP0)

Def.

$\lambda_1^D(\Omega_0)$: 1st eigenvalue of $-\Delta$ in Ω_0 (Dirichlet).

Proposition (Du-Shi '06, O. '11)

(i) Let $\mu \geq 0$.

(SP0) has a positive solution $\Leftrightarrow \lambda > {}^{\exists}\lambda^*(\mu, \Omega_0)$,

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(ii) Let $\mu < 0$.

(SP0) has a positive solution $\Leftrightarrow \lambda > -\mu/c$.

$\lambda > 0$: growth rate of the prey species.

$\mu \in \mathbb{R}$: growth rate of the predator species.

A three species prey-predator model (P)

$$
\begin{cases}
u_t = \Delta u + u(\lambda - u - b\chi_{\Omega\setminus\Omega_0}(x)v - d\chi_{\Omega_0}(x)w) & \text{in } \Omega \times (0,\infty), \\
v_t = \Delta v + v(\mu + cu - v) & \text{in } \Omega\setminus\bar{\Omega}_0 \times (0,\infty), \\
w_t = \Delta w + w(\nu + eu - w) & \text{in } \Omega_0 \times (0,\infty), \\
\partial_n u = 0 & \text{on } \partial\Omega \times (0,\infty), \\
\partial_n v = 0 & \text{on } \partial(\Omega\setminus\bar{\Omega}_0) \times (0,\infty), \\
\partial_n w = 0 & \text{on } \partial\Omega_0 \times (0,\infty).
\end{cases}
$$

$w(x,t)$: population density of another predator species.

$d > 0,\ \nu > 0,\ e > 0$

Stationary problem (SP)

$$\begin{cases} \Delta u + u(\lambda - u - b\chi_{\Omega\setminus\Omega_0}(x)v - d\chi_{\Omega_0}(x)w) = 0 & \text{in } \Omega, \\ \Delta v + v(\mu + cu - v) = 0 & \text{in } \Omega\setminus\bar{\Omega}_0, \\ \Delta w + w(\nu + eu - w) = 0 & \text{in } \Omega_0, \\ \partial_n u = 0 & \text{on } \partial\Omega, \\ \partial_n v = 0 & \text{on } \partial(\Omega\setminus\bar{\Omega}_0), \\ \partial_n w = 0 & \text{on } \partial\Omega_0. \end{cases}$$

positive solution of (SP)

($u > 0$ in Ω, $v > 0$ in $\Omega\setminus\bar{\Omega}_0$, $w > 0$ in Ω_0)

$\Rightarrow$ coexistence state of three species.

Existence of positive solutions of (SP)

Def.

$\lambda_1^N(q,\Omega)$: 1st eigenvalue of $-\Delta+q$ in Ω (Neumann).

Theorem 1

(SP) has a positive solution

$$\Leftrightarrow \lambda > \lambda_1^N\left(b\chi_{\Omega\setminus\Omega_0}(x)\mu + d\chi_{\Omega_0}(x)\nu, \Omega\right),$$

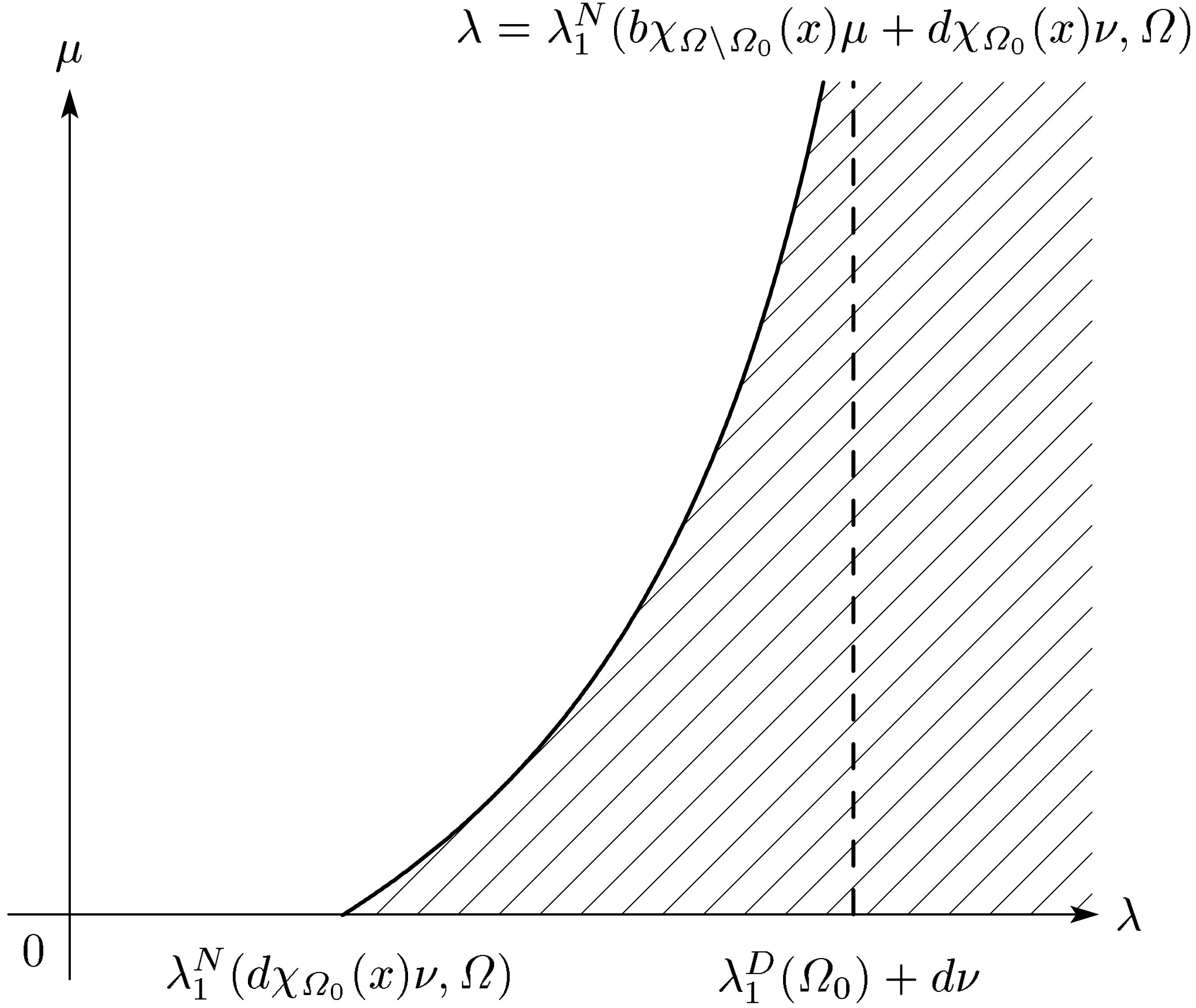
where $\lambda_1^N\left(b\chi_{\Omega\setminus\Omega_0}(x)\mu + d\chi_{\Omega_0}(x)\nu, \Omega\right)$ is continuous and strictly increasing with respect to μ satisfying

$$\lim_{\mu\to\infty} \lambda_1^N\left(b\chi_{\Omega\setminus\Omega_0}(x)\mu + d\chi_{\Omega_0}(x)\nu, \Omega\right) = \lambda_1^D(\Omega_0) + d\nu.$$

$\lambda_1^D(\Omega_0) + d\nu$: threshold prey growth rate.

λ : bifurcation parameter.

A branch of positive solutions of (SP) bifurcates from $(u, v, w) = (0, \mu, \nu)$ at $\lambda = \lambda_1^N(b\chi_{\Omega\setminus\Omega_0}(x)\mu + d\chi_{\Omega_0}(x)\nu, \Omega)$.



Proof of $\lim_{\mu\to\infty} \lambda_1^N \left(b\chi_{\Omega\setminus\Omega_0}(x)\mu + d\chi_{\Omega_0}(x)\nu, \Omega\right) = \lambda_1^D(\Omega_0) + d\nu$

Let ϕ_* satisfy

$$-\Delta\phi_* = \lambda_1^D(\Omega_0)\phi_* \text{ in } \Omega_0, \quad \phi_* = 0 \text{ on } \partial\Omega_0, \quad \int_{\Omega_0} \phi_*^2 dx = 1$$

and define $\tilde{\phi}_* \in H^1(\Omega)$ by

$$\tilde{\phi}_* \equiv \phi_* \text{ in } \Omega_0, \quad \tilde{\phi}_* \equiv 0 \text{ in } \Omega\setminus\Omega_0.$$

Then

$$\begin{aligned}
&\lambda_1^N \left(b\chi_{\Omega\setminus\Omega_0}(x)\mu + d\chi_{\Omega_0}(x)\nu, \Omega\right) \\
=& \inf_{\{\phi\in H^1(\Omega):\,\|\phi\|_2=1\}} \left(\int_\Omega |\nabla\phi|^2 dx + b\mu \int_{\Omega\setminus\Omega_0} \phi^2 dx + d\nu \int_{\Omega_0} \phi^2 dx\right) \\
\le& \int_\Omega |\nabla\tilde{\phi}_*|^2 dx + b\mu \int_{\Omega\setminus\Omega_0} \tilde{\phi}_*^2 dx + d\nu \int_{\Omega_0} \tilde{\phi}_*^2 dx \\
=& \lambda_1^D(\Omega_0) + d\nu.
\end{aligned}$$

For any sequence $\{\mu_i\}_{i=1}^{\infty}$ with $\lim_{i\to\infty}\mu_i = \infty$, let $\phi_i > 0$ satisfy

$$\begin{cases} -\Delta\phi_i + \left(b\chi_{\Omega\setminus\Omega_0}(x)\mu_i + d\chi_{\Omega_0}(x)\nu\right)\phi_i = \lambda_1^N\left(b\chi_{\Omega\setminus\Omega_0}(x)\mu_i + d\chi_{\Omega_0}(x)\nu, \Omega\right)\phi_i \text{ in } \Omega, \\ \partial_n\phi_i = 0 \text{ on } \partial\Omega, \quad \int_\Omega \phi_i^2 dx = 1. \end{cases}$$

Then

$$\int_\Omega |\nabla\phi_i|^2 dx \le \lambda_1^N\left(b\chi_{\Omega\setminus\Omega_0}(x)\mu_i + d\chi_{\Omega_0}(x)\nu, \Omega\right)\int_\Omega \phi_i^2 dx \le \lambda_1^D(\Omega_0) + d\nu.$$

So there exist a subsequence $\{\mu_i\}$ and $\phi_\infty \in H^1(\Omega)$ such that

$$\lim_{i\to\infty}\phi_i = \phi_\infty \ge 0 \text{ weakly in } H^1(\Omega), \text{ strongly in } L^2(\Omega), \quad \int_\Omega \phi_\infty^2 dx = 1.$$

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$$\lim_{i\to\infty}\phi_i = \phi_\infty \ge 0 \text{ weakly in } H^1(\Omega), \text{ strongly in } L^2(\Omega), \quad \int_\Omega \phi_\infty^2 dx = 1.$$

Moreover,

$$\int_\Omega |\nabla\phi_i|^2 dx + b\mu_i\int_{\Omega\setminus\Omega_0}\phi_i^2 dx + d\nu\int_{\Omega_0}\phi_i^2 dx = \lambda_1^N\left(b\chi_{\Omega\setminus\Omega_0}(x)\mu_i + d\chi_{\Omega_0}(x)\nu, \Omega\right).$$

By letting $i\to\infty$ ($\mu_i\to\infty$), $\int_{\Omega\setminus\Omega_0}\phi_\infty^2 dx = 0$.

Thus $\phi_\infty = 0$ a.e. in $\Omega\setminus\Omega_0$ and $\phi_\infty|_{\Omega_0} \in H_0^1(\Omega_0)$.

For any sequence $\{\mu_i\}_{i=1}^{\infty}$ with $\lim_{i\to\infty}\mu_i = \infty$, let $\phi_i > 0$ satisfy

$$\begin{cases} -\Delta\phi_i + \left(b\chi_{\Omega\setminus\Omega_0}(x)\mu_i + d\chi_{\Omega_0}(x)\nu\right)\phi_i = \lambda_1^N\left(b\chi_{\Omega\setminus\Omega_0}(x)\mu_i + d\chi_{\Omega_0}(x)\nu, \Omega\right)\phi_i \text{ in } \Omega, \\ \partial_n\phi_i = 0 \text{ on } \partial\Omega, \quad \int_\Omega \phi_i^2 dx = 1. \end{cases}$$

Then

$$\int_\Omega |\nabla\phi_i|^2 dx \le \lambda_1^N\left(b\chi_{\Omega\setminus\Omega_0}(x)\mu_i + d\chi_{\Omega_0}(x)\nu, \Omega\right)\int_\Omega \phi_i^2 dx \le \lambda_1^D(\Omega_0) + d\nu.$$

So there exist a subsequence $\{\mu_i\}$ and $\phi_\infty \in H^1(\Omega)$ such that

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By letting $i \to \infty$ $(\mu_i \to \infty)$, $\int_{\Omega\setminus\Omega_0}\phi_\infty^2 dx = 0$.

Thus $\phi_\infty = 0$ a.e. in $\Omega\setminus\Omega_0$ and $\phi_\infty|_{\Omega_0} \in H_0^1(\Omega_0)$. This yields

$$-\Delta\phi_\infty + d\nu\phi_\infty = \lim_{\mu\to\infty}\lambda_1^N\left(b\chi_{\Omega\setminus\Omega_0}(x)\mu + d\chi_{\Omega_0}(x)\nu, \Omega\right)\phi_\infty \text{ in } \Omega_0, \quad \phi_\infty = 0 \text{ on } \partial\Omega_0.$$

Therefore, $\lim_{\mu\to\infty}\lambda_1^N\left(b\chi_{\Omega\setminus\Omega_0}(x)\mu + d\chi_{\Omega_0}(x)\nu, \Omega\right) = \lambda_1^D(\Omega_0) + d\nu$.