On the norm-inflation solutions to the Navier-Stokes equations in the critical space

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Discussion with
Tsukasa Iwabuchi, Yasunori Maekawa,
Hideyuki Miura, Tsuyoshi Yoneda,
and members of IRTG1529.

§1. Introduction Navier-Stokes equations in $\mathbb{R}^3$:

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\text{(NS)} \quad \begin{cases} u_t - \Delta u + (u,\nabla)u + \nabla p = 0 & \text{in } \mathbb{R}^3 \times (0,T), \\ \nabla \cdot u = 0 & \text{in } \mathbb{R}^3 \times (0,T), \\ u\big|_{t=0} = u_0 & \text{in } \mathbb{R}^3. \end{cases}
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Besides, the heat equation is well-posed in $\dot{B}^{-1}_{\infty,\infty}$.

Known Results

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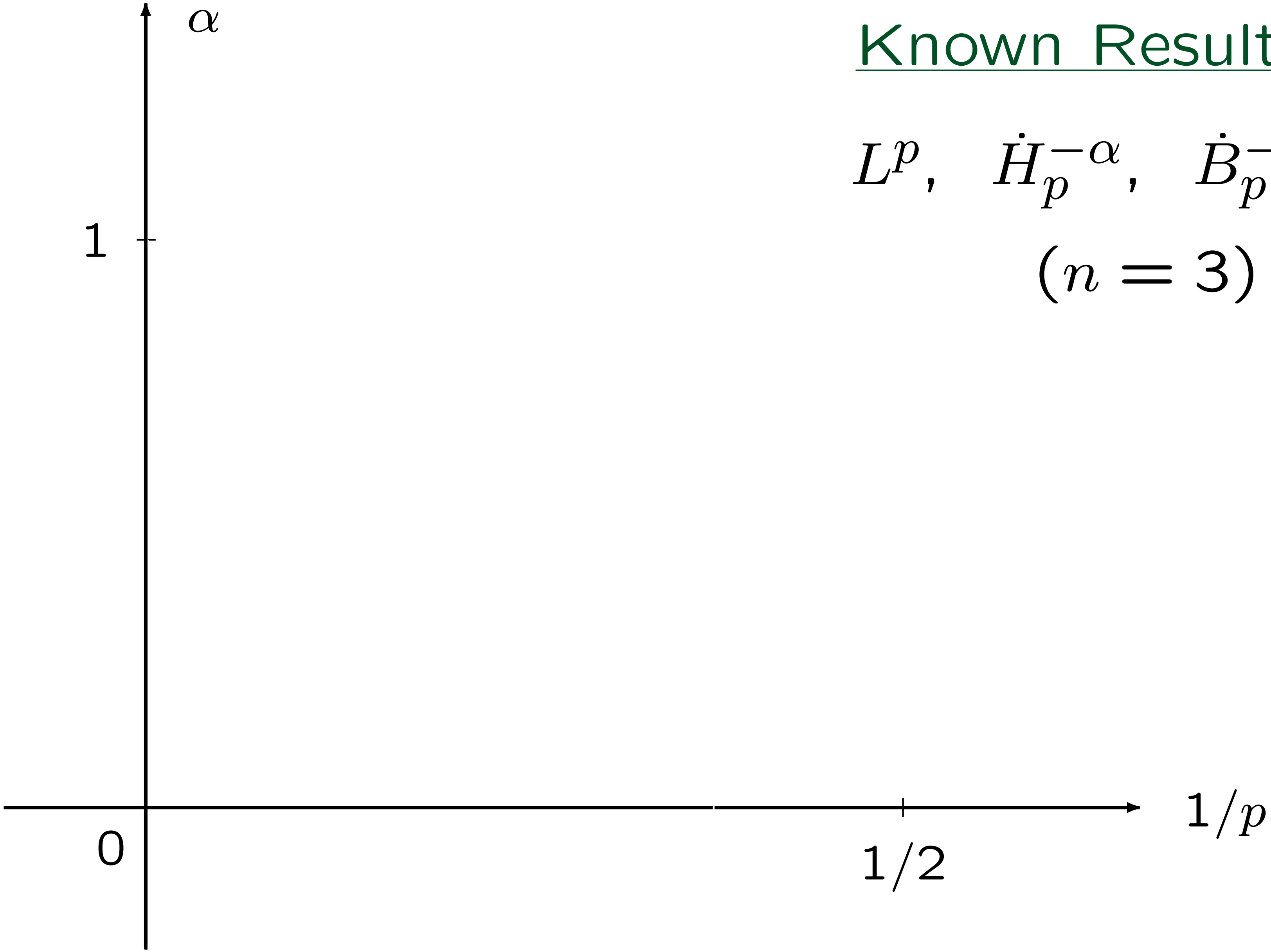
$$(n = 3)$$



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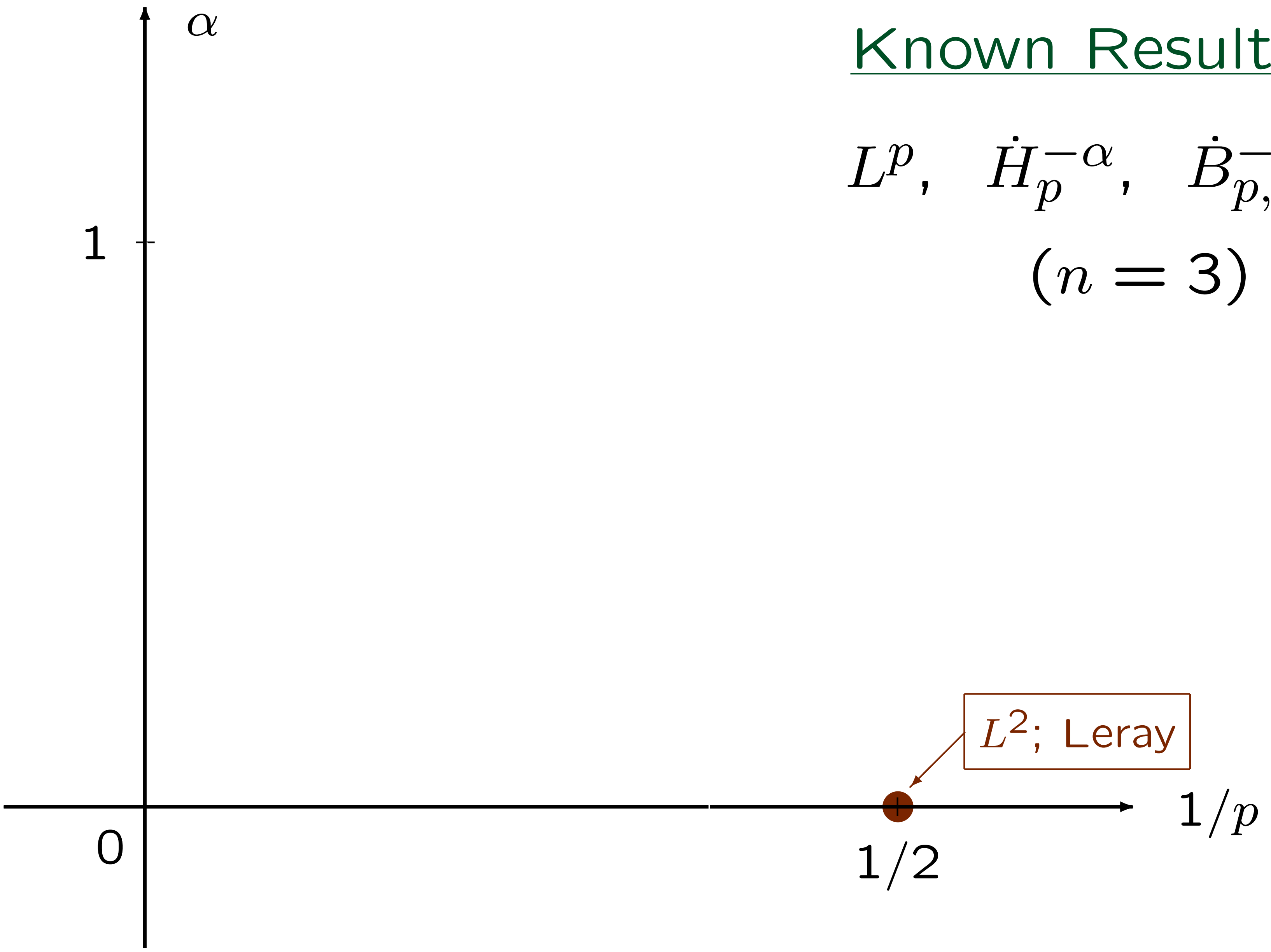

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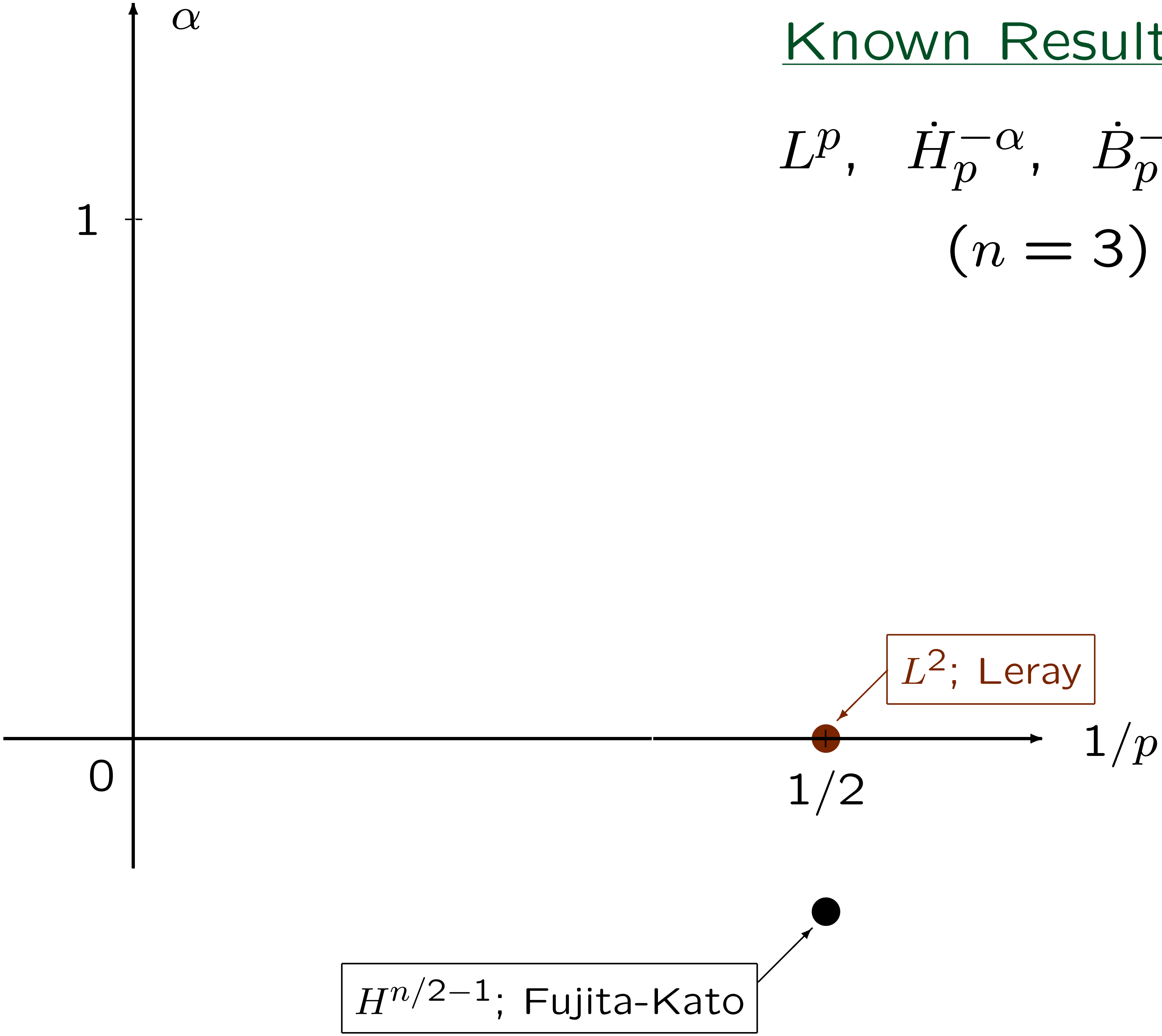

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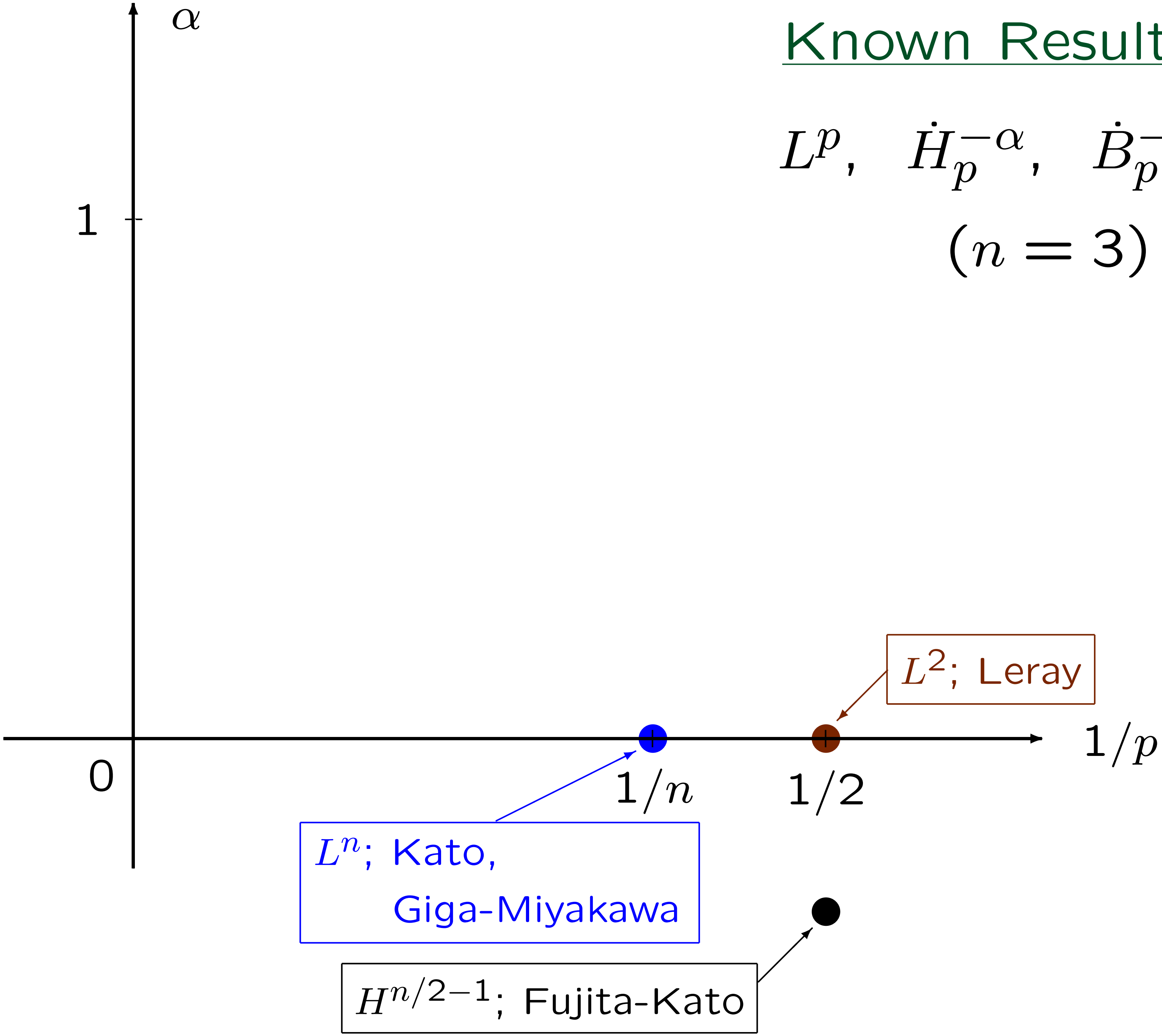

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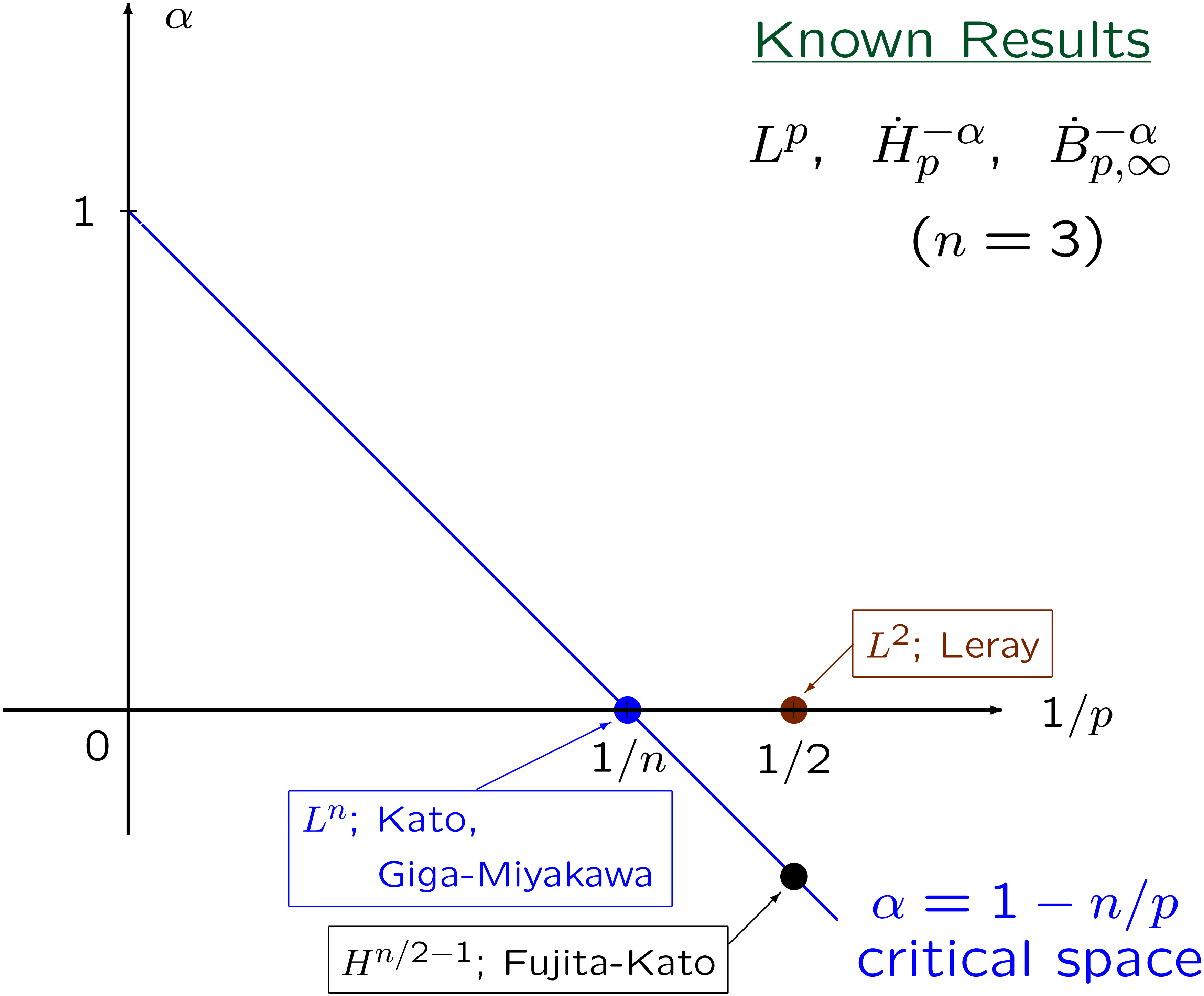

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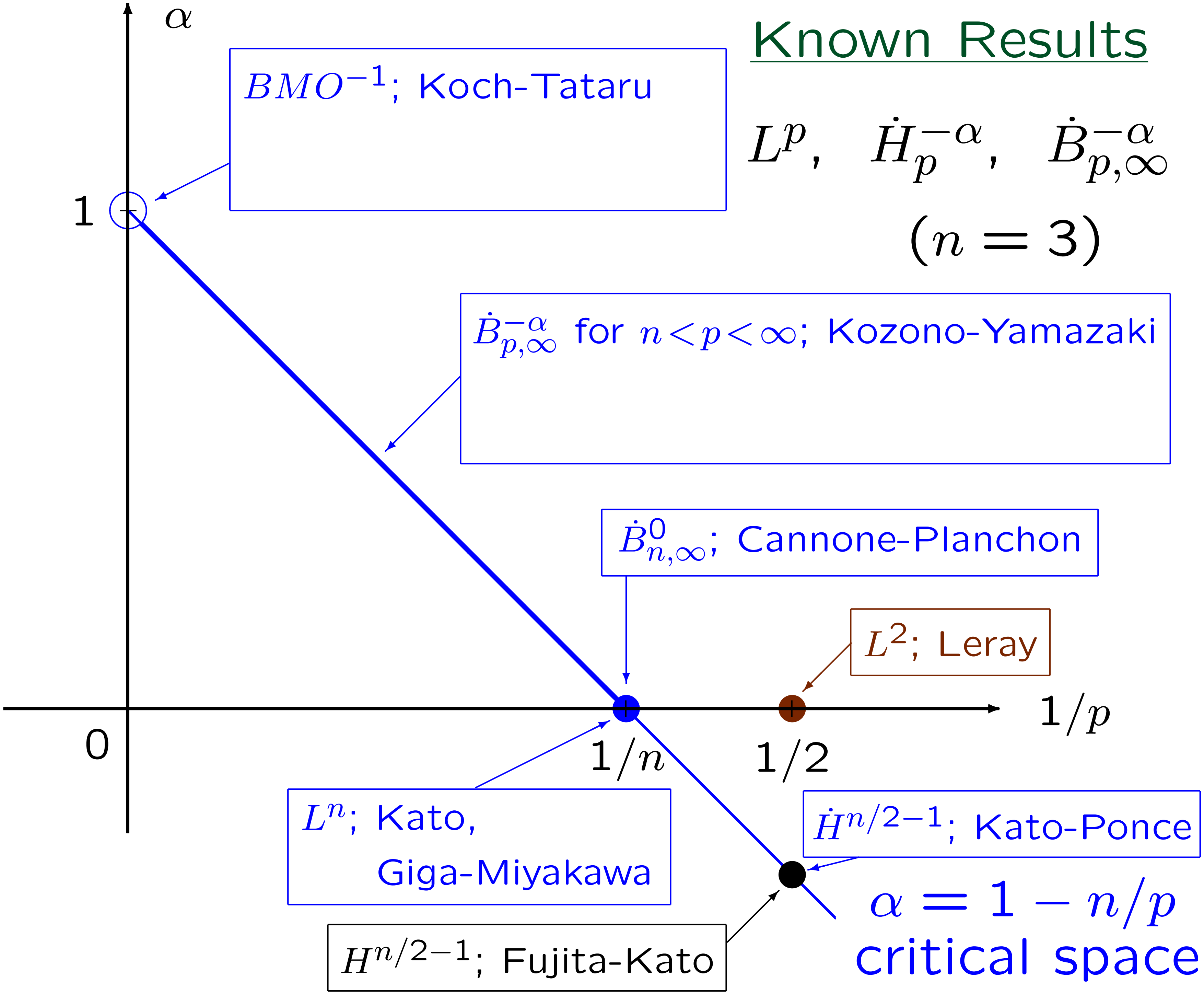

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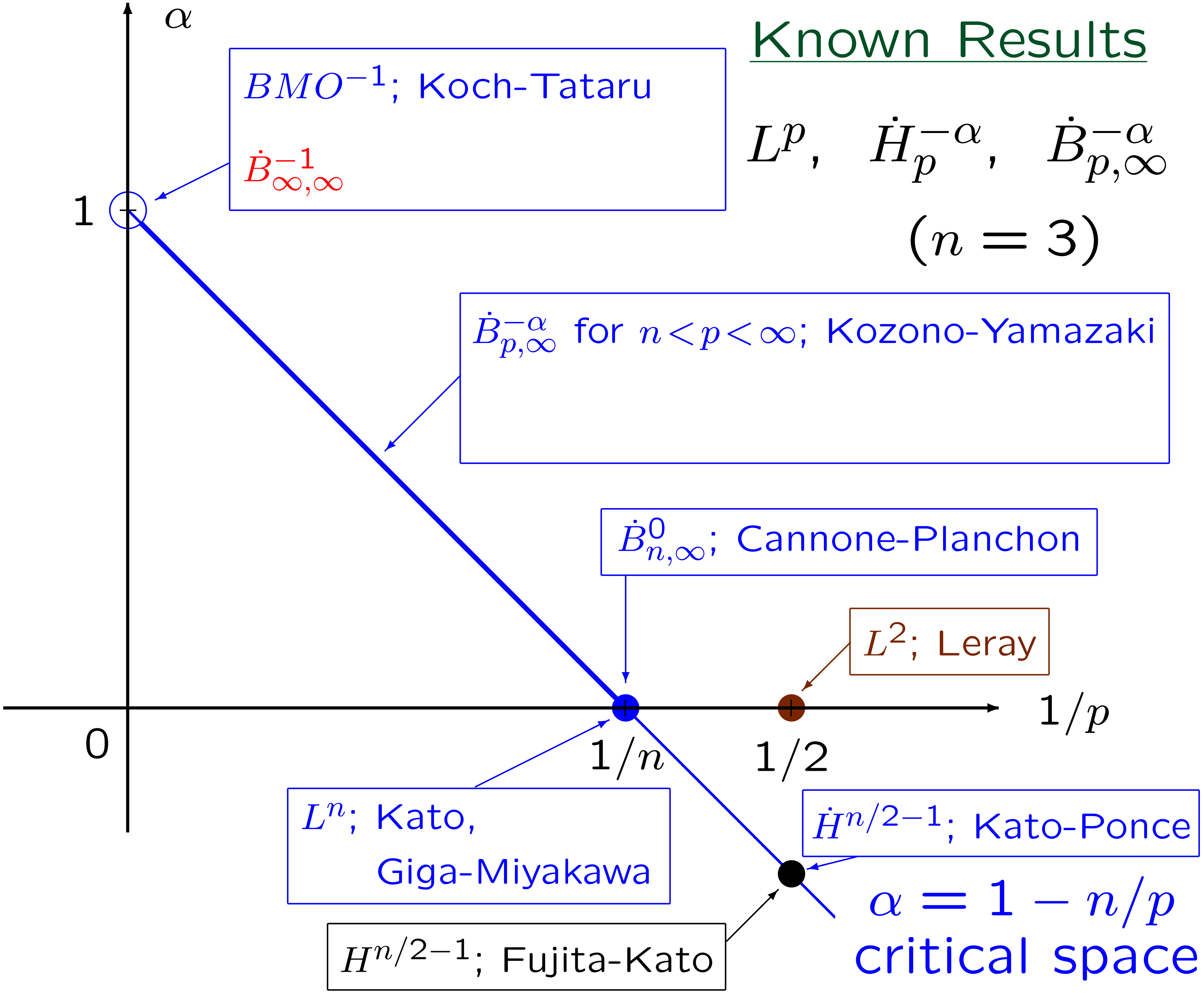

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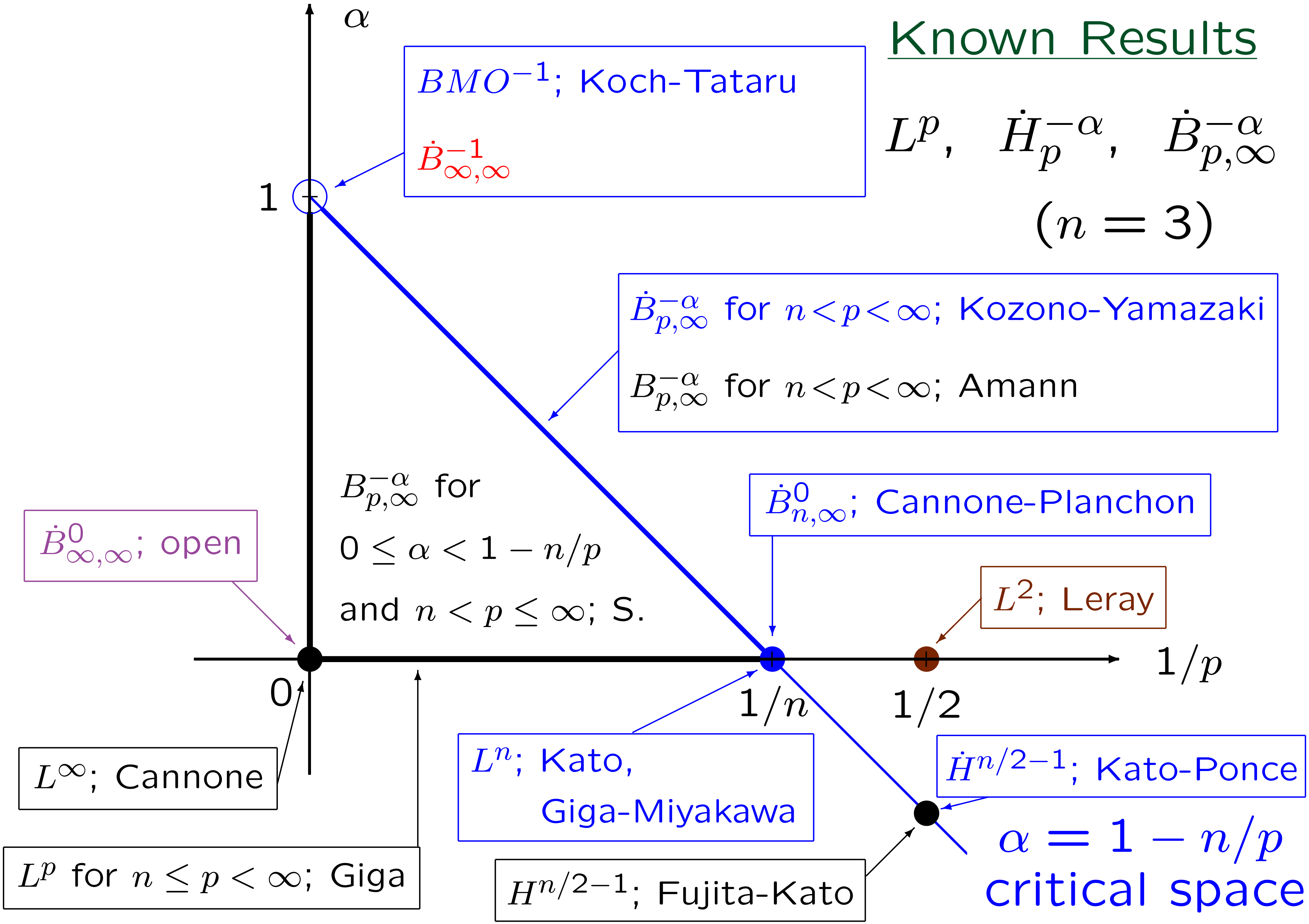

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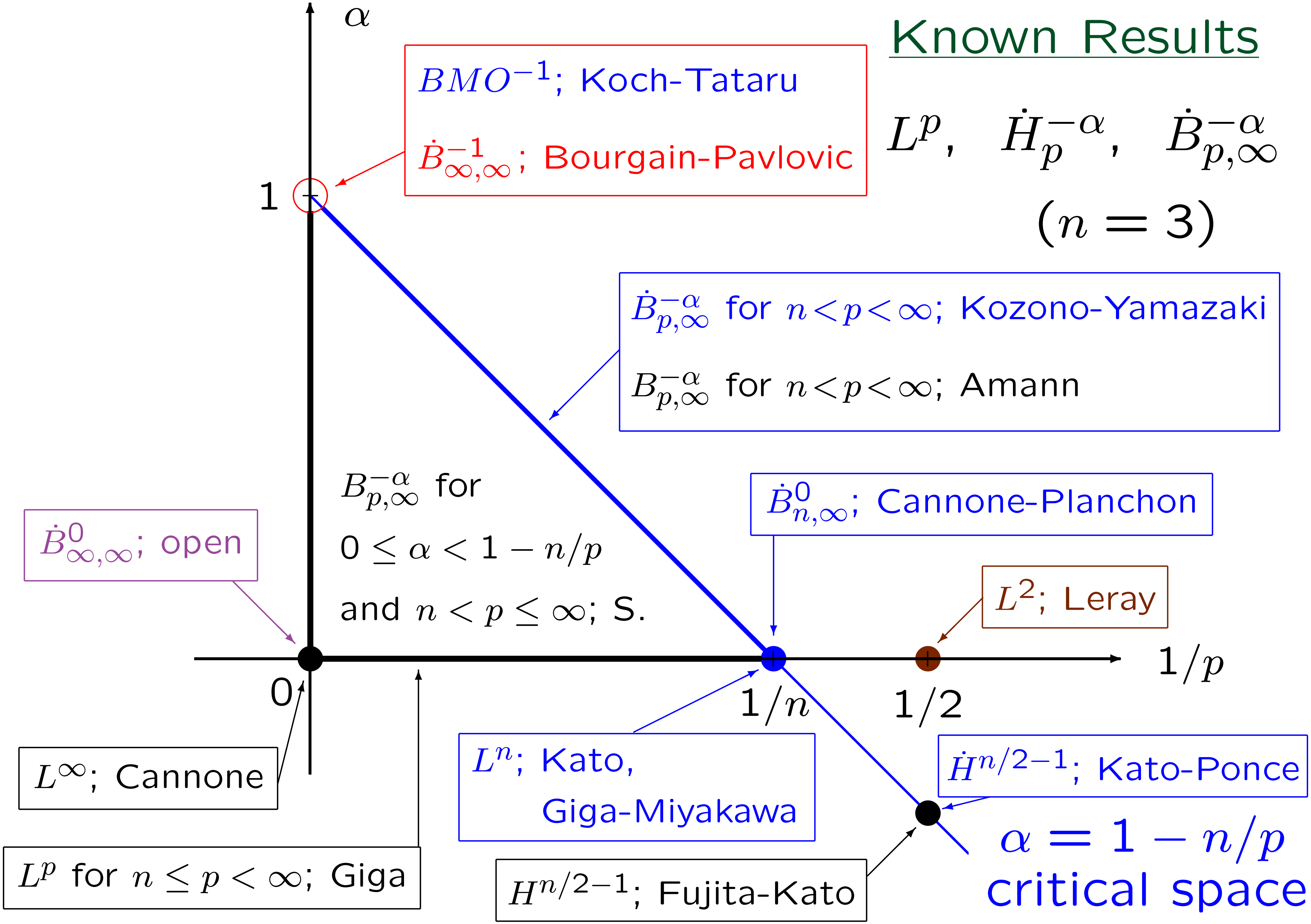

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$$\|f\| \simeq \sup_{j\in\mathbb{Z}} j^{-1}\|\phi_j * f\|_{L^\infty}, \ \dot{B}^{-1}_{\infty,\infty} \approx \nabla L^\infty, \ \|f\| \approx \|\nabla^{-1} f\|_{L^\infty}.$$

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§2. Main Results

Th.1 [Bourgain-Pavlovic] $^{\forall}\delta, {}^{\forall}T \in (0,1)$, $^{\exists}u_0 \in \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)$ s.t. $\|u_0\| < \delta$, $\nabla \cdot u_0 = 0$, $^{\exists}u$: a mild solution in $C([0,T]; \dot{B}^{-1}_{\infty,\infty})$ and $\|u(T)\| > 1/\delta$.

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Th.2 [S.] $^\forall T > 0$, $^\exists u_0$ s.t. $\{\|u_j(T)\|\}_{j=1}^{\infty}$ diverges.

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Initial Velocity Fix $u_0 \in \dot{B}^{-1}_{\infty,\infty}$ with $\nabla \cdot u_0 = 0$ by

$$u_0(x) = \left(0, u_0^2(x_1), u_0^3(x_1, x_2)\right)$$

$$= \left(0, \frac{Q}{\sqrt{r}} \sum_{s=1}^{r} h_s \cos(h_s x_1), \frac{Q}{\sqrt{r}} \sum_{s=1}^{r} h_s \cos(h_s x_1 - x_2)\right)$$

with parameters $Q > 0$ and r, γ, $\eta \in \mathbb{N}$, where

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$$\|v_1(t)\| \leq \|G_t\|_{L^1} \|u_0\| < \delta \ll 1 \quad \text{for} \quad {}^{\forall} t > 0.$$

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$$\|v_2(T)\| \sim \frac{Q^2}{4} e^{-T} \|\nabla^{-1} \sin x_2\|_{L^\infty} = \frac{Q^2}{4} e^{-T} > \frac{2}{\delta}.$$

Estimates for v_3, v_4, ... Since $v_2 = (0, 0, v_2^3)$ and

$$u_2 = v_1 + v_2 = (0, v_1^2(x_1, t), v_1^3(x_1, x_2, t) + v_2^3(x_1, x_2, t)),$$

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Thank you for your kind attention.