Spatial Behaviour of the Navier Stokes Flow in a Rotating Frame

(work in progress)

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The Rotating Navier-Stokes Flow in $\mathbb{R}^3$

We will study the 3-dimensional rotating Navier-Stokes Equations

$$(RNS)\begin{cases} u_t - \Delta u + u \cdot \nabla u + \Omega e_3 \times u + \nabla p = f & \text{in } \mathbb{R}^3 \times [0,T), \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^3 \times [0,T), \\ u(0) = u_0 & \text{in } \mathbb{R}^3, \end{cases}$$

with a constant Coriolis parameter $\Omega \neq 0$ and the data $u_0, f \in L^\infty_\nu(\mathbb{R}^3)$, i.e.

$$\|v\|_{L^\infty_\nu} := \operatorname{ess\,sup}(1+|x|)^\nu |v(x)| < \infty.$$

Here e_3 denotes the vertical unit vector $(0,0,1)^T$ and the term

$$Ju := e_3 \times u$$

restricted to divergence free vector fields is called *Coriolis operator*.

A mild solution solves the integral equation:

$$u(t) = e^{-tA_\Omega}u_0 - \int_0^t e^{-(t-s)A_\Omega}\mathbb{P}(u \cdot \nabla u - f)(s)ds$$

with the *Stokes-Coriolis operator* $A_\Omega := -\mathbb{P}\Delta + \Omega\mathbb{P}J\mathbb{P}$.

The semigroup e^{-tA_Ω} is bounded in $L^p(\mathbb{R}^3)$, $1 < p < \infty$, and in $\mathrm{BMO}(\mathbb{R}^3)$. The norm estimate is even uniform in t for $p = 2$.

But in contrast to the non-rotating case, i.e. $\Omega = 0$, the operator e^{-tA_Ω} is not bounded in $L^\infty(\mathbb{R}^3)$.

Theorem (Existence and Uniqueness of Mild Solutions)

Let $\varepsilon > 0$. For every initial velocity $u_0 \in L^\infty_{\mu+\varepsilon}(\mathbb{R}^3)^3$ with $\operatorname{div} u_0 = 0$, $\mu \in (0,3]$, and external force $f \in C\left([0,T]; L^\infty_{\mu+\varepsilon}(\mathbb{R}^3)^3\right)$ there exists a constant $T > 0$ and a unique solution

$$u \in C_\omega\left([0,T]; L^\infty_\mu(\mathbb{R}^3)\right)$$

to the rotating Navier-Stokes equations.
In particular, with the constant C_0 there holds

$$\|u_0\|_{L^\infty_{\mu+\varepsilon}} + \sup_{0\leq t\leq T} \|f(t)\|_{L^\infty_{\mu+\varepsilon}} < \frac{1}{4C_0^2(\sqrt{T}+T)^2}.$$

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Sketch of Proof:

- The operator e^{-tA_Ω} is a convolution operator and the corresponding kernel K_t satisfies

$$\partial^\alpha K_t(x) \lesssim \left(\sqrt{t} + |x|\right)^{-3-|\alpha|}.$$

- Proving the boundedness of the operators

$$\mathcal{A} : L^\infty_{\mu+\varepsilon} \to C_\omega\left([0,T]; L^\infty_\mu\right),$$
$$\mathcal{A}(u)(t) := e^{-tA_t\Omega} u$$
$$\mathcal{B}(.,.) : C_\omega\left([0,T]; L^\infty_\mu\right) \to C_\omega\left([0,T]; L^\infty_\mu\right),$$
$$\mathcal{B}\left(u,u\right)\left(t\right) := -\int_0^t e^{(t-s)A_\Omega}\mathbb{P}\left(u\cdot\nabla u\right)\left(s\right)ds,$$
$$\mathcal{C} : C_\omega\left([0,T]; L^\infty_{\mu+\varepsilon}\right) \to C_\omega\left([0,T]; L^\infty_\mu\right),$$
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$$\mathcal{C}\left(f\right)\left(t\right) := \int_0^t e^{(t-s)A_\Omega}\mathbb{P}f(s)\, ds.$$

- Verifying the existence of a fixed point of the equation

$$u(t) = \mathcal{A}(u_0)(t) + \mathcal{B}\left(u,u\right)\left(t\right) + \mathcal{C}\left(f\right)\left(t\right)$$

in the Banach space $C_\omega\left([0,T]; L^\infty_\mu\right)$ by using the Kato iteration method.

Spatial Asymptotics of Non-Rotating Flow

Brandolese and Bae, see [1], proved the following profile solutions in a non-rotating frame, i.e. $\Omega = 0$:

Theorem (Spatial Asymptotic Behaviour of Non-Rotating Flow)

Let $\varepsilon > 0$ and $\Omega = 0$. For every initial velocity $u_0 \in L^\infty_\mu(\mathbb{R}^3)^3$ with $\mathrm{div} u_0 = 0$, $\mu > 2$, and external force $f \in C\left([0,T]; L^\infty_{3+\varepsilon}(\mathbb{R}^3)^3\right)$ let u be the solution of the previous Theorem. Then the following profile holds for almost all $|x| \geq \sqrt{t}$:

$$u(x,t) = e^{t\Delta} u_0(x) - \gamma \nabla \left[\frac{x}{|x|^3} \cdot \int_0^t \int_{\mathbb{R}^3} f \, dy \, ds \right] + \mathcal{O}_t(|x|^{-4}).$$

Spatial Asymptotics of Rotating Flow

The solution of the rotating Navier-Stokes equations (RNS) without external forces, solves the integral equation:

$$u(t) = e^{-tA_\Omega} u_0 - \int_0^t e^{-(t-s)A_\Omega} \mathbb{P}(u \cdot \nabla u)(s) ds.$$

The symbol of the operator $\exp(-tA_\Omega)$ which belongs to the linearised problem of (RNS) is given by

$$e^{-|\xi|^2 t} \left[\cos\left(\frac{\xi_3}{|\xi|} \Omega t \right) - \sin\left(\frac{\xi_3}{|\xi|} \Omega t \right) \mathbf{R}(\xi) \right],$$

with the *Riesz symbol*

$$\mathbf{R}(\xi) = \frac{1}{|\xi|} \begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix},$$

see Giga, Inui, Mahalov and Matsui, [3].

Due to this symbol let us use the following componentwise notations for the kernel of the convolution operator e^{-tA_Ω}:

$$D_{i,j}(x,t) := \mathcal{F}^{-1}\Big(\frac{1}{8\pi^3}e^{-4\pi^2 t|\xi|^2}\big[\cos(\frac{\xi_3}{|\xi|}\Omega t)\delta_{i,j} - \sin(\frac{\xi_3}{|\xi|}\Omega t)\mathsf{R}_{i,j}(\xi)\big]\Big).$$

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For illustration we pay our attention on the term

$$\begin{aligned} D^{(1)}(x,t) &:= \mathcal{F}^{-1}\left(\frac{1}{8\pi^3}e^{-4\pi^2 t|\xi|^2}\cos(\frac{\xi_3}{|\xi|}\Omega t)\right) \\ &= \mathcal{G}_t(x) + \sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!(n-1)!}(\Omega t)^{2n}\int_t^{\infty}(s-t)^{n-1}\partial_3^{2n}\mathcal{G}_s(x)ds, \end{aligned}$$

where $\mathcal{G}_t$ denotes the heat kernel.

Let the polynomials $p_n \in C^\infty(\mathbb{R})$, $n \in \mathbb{N}$, be defined by

$$p_1(y) := -y \quad \text{and} \quad p_{n+1} := p_n'(y) - y \cdot p_n(y).$$

We note that the polynomial $p_{n,t}(y) := (2t)^{-\frac{n}{2}} p_n\left(\frac{y}{\sqrt{2t}}\right)$ has degree n. Hence we obtain

$$\partial_3^n \mathcal{G}_t(x) = p_{n,t}(x_3)\mathcal{G}_t(x) = \left((\partial_3 - \frac{x_3}{2t})^n 1\right) \mathcal{G}_t(x).$$

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The polynomial p_{2n} can be respresented by the formula

$$p_{2n}(y) = \sum_{l=0}^{n} (-1)^{n+l} a_{n,l} y^{2l},$$

where the coefficients $a_{n,l} \geq 1$ are recursively defined by

$$\begin{aligned} a_{0,l} &:= \delta_{0,l}, \qquad a_{n,-1} := 0 \\ a_{n,l} &:= (2l+1)(2l+2)a_{n-1,l+1} + (4l+1)a_{n-1,l} + a_{n-1,l-1}. \end{aligned}$$

In partilcular, there holds $a_{n,n} = 1$ and $a_{n,k} = 0$ for all $k > n$.
Furthermore, for all $1 \leq l \leq n$ there holds: $a_{n,l} \leq (15l)^{n-l}$.

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Considering the following even ordered derivatives of $h(y) := e^{-\frac{y^2}{2}}$:

$$\begin{array}{lclllll} \partial^0 h(y) & = & [1 & +0\cdot y^2 & +0\cdot y^4] & \cdot h(y) \\ \partial^2 h(y) & = & [-1 & +1\cdot y^2 & +0\cdot y^4] & \cdot h(y) \\ \partial^4 h(y) & = & [3 & -6\cdot y^2 & +1\cdot y^4] & \cdot h(y) \\ \vdots & & & & & \end{array}$$

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leads to the scheme:

$$\begin{array}{l|cccccc} l \setminus n & 0 & 1 & 2 & 3 & & \cdots \\ \hline 0 & 1 & -1 & 3 & -15 & \searrow & \\ 1 & 0 & 1 & -6 & 45 & \rightarrow & a_{4,1} \\ 2 & 0 & 0 & 1 & -15 & \nearrow & \\ 3 & 0 & 0 & 0 & 1 & & \\ \vdots & & & & & & \ddots \end{array}$$

Lemma

Let $n \in \mathbb{N}$ and $|x| \geq \sqrt{t}$. Then there holds

$$\int_t^\infty (s-t)^{n-1} \partial_3^{2n} \mathcal{G}_s(x) ds$$

$$= 2\pi^{-\frac{3}{2}} |x|^{-3} \sum_{l=0}^{n} a_{n,l} \frac{(-1)^{n+l} l!}{2^{n-1-l}} \left(\frac{x_3}{|x|}\right)^{2l} + |x|^{-3} \Psi_n \left(\frac{x}{\sqrt{t}}\right)$$

with an exponentially decaying remainder term Ψ_n.

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We define

$$V^{(1)}(x_3, t) := \sum_{n=1}^{\infty} \sum_{l=0}^{n} a_{n,l} \frac{(-1)^l 2^{-n+l+1} l!}{(2n)!(n-1)!} (\Omega t)^{2n} \left(\frac{x_3}{|x|} \right)^{2l}$$

and $V^{(j)}(x_3, t)$, $j = 2, 3$, in a slightly different way.

Let $\varepsilon > 0$. We consider the summands of this series $V^{(j)}$ with respect to $l = 0$:

$$V_{0,t-1}^{(1)} := \sum_{n=1}^{\infty} a_{n,0} \frac{2}{2^n(2n)!(n-1)!} (\Omega t)^{2n}$$

and $V_{0,t-1}^{(j)}$ for the corresponding series $V^{(j)}(x_3, t)$, $j = 2, 3$, we get away from the rotating axis, i.e. for all $|x_3|^{1+\varepsilon} \leq |x|$:

$$|x|^{-3} \left| V^{(j)}(x_3, t) - V_{0,t-1}^{(j)} \right| = \mathcal{O}_t(|x|^{-3-\delta})$$

with an suitable $\delta(\varepsilon) > 0$.

Let us define the auxiliary function v_j, $j = 1, 2, 3$, as follows:

$$u_{0,j} = \mathcal{G}_1(x) \int_{\mathbb{R}^3} u_{0,j}(y)dy + v_j(x).$$

This leads us to

$$(D^{(1)}(t) * u_{0,j})(x) = (D^{(1)}(t) * \mathcal{G}_1)(x) \int_{\mathbb{R}^3} u_{0,j}(y)dy + (D^{(1)}(t) * v_j)(x)$$

and the Fourier transform yields $D^{(1)}(t) * \mathcal{G}_1 = D^{(1)}(t+1)$.

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and the Fourier transform yields $D^{(1)}(t) * \mathcal{G}_1 = D^{(1)}(t+1)$.
Therefore, we can establish the leading term of the considered convolution with the kernel $D^{(1)}(x,t)$ for all $|x_3|^{1+\varepsilon} \leq |x|$:

$$\left(D^{(1)}(t) * u_{0,j}\right)(x) = \frac{2}{\sqrt{\pi}^3 |x|^3} V_{0,t}^{(1)} \int_{\mathbb{R}^3} u_{0,j}(y)dy + \mathcal{O}_t(|x|^{-3-\delta}).$$

Summarising all appearing convolutions of this type we obtain:

Theorem (Spatial Asymptotic away from the Rotating Axis)

Let $i = 1, 2$, $\varepsilon > 0$ *and* $\delta := \frac{2\varepsilon}{1+\varepsilon}$. *For* $\mu > 4$ *and an initial velocity* $u_0 \in L^\infty_4(\mathbb{R}^3)^3$ *with* $\mathrm{div} u_0 = 0$, *let* u *be the mild solution. Then the following profile holds for almost all* $|x| \gg \sqrt{t}$ *and* $x_3^{1+\varepsilon} \leq |x|$:

$$u_i(x,t) = \frac{C^{(1)}_{i,t}}{|x|^3} + (-1)^{i+1} \frac{C^{(2)}_{i,t} x_{3-i} x_3}{|x|^5} + \mathcal{O}_t(|x|^{-\min\{3+\delta,4\}}),$$

$$u_3(x,t) = \frac{C^{(1)}_{3,t}}{|x|^3} - \frac{C^{(2)}_{3,t} x_2 x_3}{|x|^5} + \frac{C^{(3)}_{3,t} x_1 x_3}{|x|^5} + \mathcal{O}_t(|x|^{-\min\{3+\delta,4\}}).$$

$$C^{(1)}_{i,t} := \frac{2}{\sqrt{\pi}^3} V^{(1)}_{0,t} \int_{\mathbb{R}^3} u_{0,i}(y) dy + (-1)^{i+1} V^{(2)}_{0,t} \int_{\mathbb{R}^3} u_{0,3-i}(y) dy,$$

$$C^{(1)}_{3,t} := \frac{2}{\sqrt{\pi}^3} V^{(1)}_{0,t} \int_{\mathbb{R}^3} u_{0,3}(y) dy, \qquad \cdots$$

Literature

- [1] Bae, H.-O., Brandolese, L., *On the Effect of External Forces on Incompressible Fluid Motions at Large Distances*, Ann. Univ. Ferrara Sez. VII Sci. Mat., **55**, 225-238 (2009).
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Thanks for your attention!