Stochastic and Variational Approach to the Lax-Friedrichs Scheme

Kohei Soga
(Waseda University)

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1. Introduction

Let $c, h(c) \in \mathbb{R}$ be given constants.

$$\text{(HJ)} \quad \begin{cases} v_t + H(x,t,c+v_x) = h(c) \text{ in } \mathbb{T} \times (0,T], \\ v(x,0) = v^0(x) \in Lip(\mathbb{T}) \text{ on } \mathbb{T} := \mathbb{R}/\mathbb{Z}, \end{cases}$$

$$\text{(CL)} \quad \begin{cases} u_t + H(x,t,c+u)_x = 0 \text{ in } \mathbb{T} \times (0,T], \\ u(x,0) = u^0(x) \in L^\infty(\mathbb{T}) \text{ on } \mathbb{T}. \end{cases}$$

Suppose that $u^0 = v^0_x$. Then

$\exists/\, v \in Lip$: viscosity sol.
$\exists/\, u \in C^0((0,T];L^\infty)$: entropy sol. s.t. $v_x = u$.

Consider the variational problems

$$\inf_{\gamma \in AC, \gamma(t)=x} \left[\int_0^t \{L^c(\gamma(s), s, \gamma'(s))\} ds + v^0(\gamma(0)) \right] + h(c)t,$$

where $L(x,t,\cdot) := H^*(x,t,\cdot)$ and $L^c(x,t,\xi) := L(x,t,\xi) - c\xi$. Then

$$\forall\, (x,t) \in \mathbb{T} \times (0,T],\ \exists\, V(x,t)\text{: infimum and } \gamma^*\text{: minimizer.}$$

Variational characterization of (HJ) and (CL):

(1) $v(x,t) = V(x,t)$.

(2) (x,t): regular point of v (i.e. $\exists\, v_x(x,t)$) and γ^*: minimizer

$$\Rightarrow \quad u(x,t) = \int_0^t L_x^c(\gamma^*(s), s, \gamma^{*\prime}(s))ds + u^0(\gamma^*(0)).$$

$*$ Lax, Hopf, Conway, Krushkov, Crandall, Lions, etc.

Variational characterization of (HJ) and (CL) is a powerful tool for:

- Investigation of detailed properties of v, u.
- Application of (HJ) and (CL) to dynamical systems (weak KAM).
- Approximation theories of v, u.

Vanishing viscosity method: Fleming ('69)

Finite difference method: Soga ('11)

Consider discretization of (HJ) and (CL) by the Lax-Friedrichs scheme

$$(\mathrm{HJ})_\triangle \quad \begin{cases} D_t v_m^{k+1} + H(x_m, t_k, c + D_x v_{m+1}^k) = h(c) \\ v_{m+1\pm 2N}^k = v_{m+1}^k, \quad v_{m+1}^0 = v_\triangle^0(x_{m+1}). \end{cases}$$

$$(\mathrm{CL})_\triangle \quad \begin{cases} D_t u_{m+1}^{k+1} + D_x H(x_{m+2}, t_k, c + u_{m+2}^k) = 0 \\ u_{m\pm 2N}^k = v_m^k, \quad u_m^0 = u_\triangle^0(x_m). \end{cases}$$

$$D_t v_m^{k+1} := \frac{v_m^{k+1} - \frac{v_{m-1}^k + v_{m+1}^k}{2}}{\Delta t}, \; D_x v_{m+1}^k := \frac{v_{m+1}^k - v_{m-1}^l}{2\Delta x} \text{ (Lax-Friedrichs scheme)}$$

$(\mathrm{HJ})_\triangle$ and $(\mathrm{CL})_\triangle$ are equivalent: $u_m^k = D_x v_{m+1}^k$, if $u^0 = v_x^0$.

Problem.

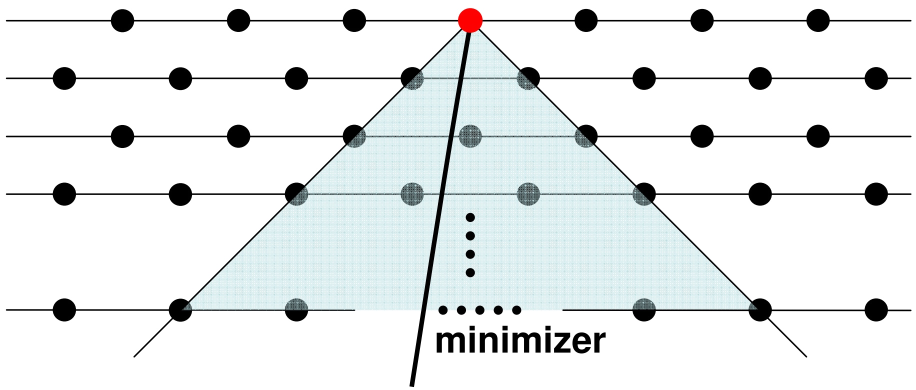
- Find variational problems for $(\mathrm{HJ})_\triangle$ and $(\mathrm{CL})_\triangle$.
  $\longrightarrow$ Stochastic calculus of variations with random walks.
- Prove convergence of approximation.
  $\longrightarrow$ Scaling limit of random walks (the law of large numbers).

2. Idea and Results

The values of solutions at the red point are determine by

(HC),(CL): Information on a minimizing curve.

$(HC)_\triangle,(CL)_\triangle$: Information on all the points of the blue triangle.



Idea.

- Introduce random walks in the triangle starting from the red point.
- Formulate stochastic and variational structure of the scheme.
- Prove concentration of the probability on a minimizing curve as $\Delta x, \Delta t \to 0$ under hyperbolic scaling $0 < \lambda_0 \le \lambda := \frac{\Delta t}{\Delta x} < \lambda_1$.

Results.

- Formulation of stochastic calculus of variations, equivalent to $(\mathrm{HJ})_\Delta$ and $(\mathrm{CL})_\Delta$.
- Uniform convergence of $v^k_{m+1} \to v$ with an error estimate.
- Uniform convergence of $u^k_m \to u$, except "small nbhd." of shocks.
- Approximation of u, v up to an arbitrarily large $(0, T]$.
- Approximation of characteristic curves as well.
- Simpler proofs for convergence of approximation.

Usual functional analytic approach with a priori estimates:

- Convergence of $u^k_m \to u$ is in the L^1-norm.
- Approximation of u up to an arbitrarily large $(0, T]$ is hard.
- There is no way to approximate characteristic curves.

3. Random walks for the Lax-Friedrichs scheme

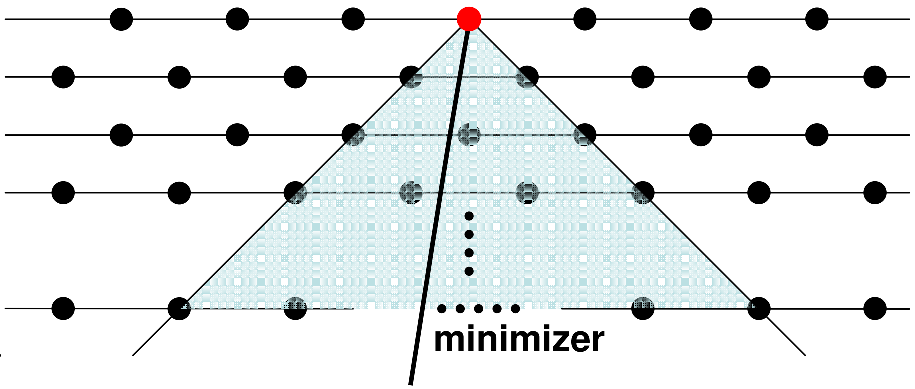
Fix arbitrary $T > 0$. Take $\Delta x, \Delta t > 0$.

Take $K = K(T) \in \mathbb{N}$ so that $t_K := K\Delta t \in (T - \Delta t, T]$.

Consider backward random walk for $0 \le k \le l+1 \le K$

$$\gamma^{l+1} = x_n, \;\; \gamma^k = \gamma^{k+1} \pm \Delta x \text{ with a transition probability } \bar{\bar{\rho}},\ \bar{\rho}.$$

Each step takes time Δt and we consider γ^k in $(\Delta x \mathbb{Z}) \times (\Delta t \mathbb{Z}_{\ge 0})$.



More precisely

$$G := \text{set of all } (x_m, t_k) \text{ in the blue triangle for } 0 < k \le l+1,$$

$$\xi : G \ni (x_m, t_k) \mapsto \xi_m^k \in [-\lambda^{-1}, \lambda^{-1}], \;\; \lambda = \Delta t / \Delta x,$$

$$\bar{\bar{\rho}} : G \ni (x_m, t_k) \mapsto \bar{\bar{\rho}}_m^k := \frac{1}{2} - \frac{1}{2}\lambda \xi_m^k \in [0,1], \;\; \bar{\rho} := 1 - \bar{\bar{\rho}},$$

$\gamma : \{0, 1, 2 \cdots, l+1\} \ni k \mapsto \gamma^k \in \Delta x \mathbb{Z},\ \gamma^{l+1} = x_n,\ \gamma^k - \gamma^{k+1} = \pm \Delta x,$

Ω : the family of all γ,

$\mu(\gamma) = \mu(\gamma; \xi)$: the product of transition probabilities $\bar{\bar{\rho}}, \bar{\rho}$ along γ,

$$\mathrm{Prob}(A) := \sum_{\gamma \in A} \mu(\gamma),\ A \subset \Omega \text{ is a probability measure of } \Omega,$$

$$\eta^k(\gamma) := x_n + \sum_{k < k' \le l+1} \xi^{k'}_{m(\gamma^{k'})} \Delta t \text{ for } \gamma \in \Omega.$$

Scaling limit for $\Delta x, \Delta t \to 0$ under $0 < \lambda_0 \le \lambda = \Delta t / \Delta x < \lambda_1$.

Thm. Set $\tilde{\sigma}^k := E_{\mu(\cdot;\xi)}\big[|\gamma^k - \eta^k(\gamma)|^2\big] := \sum_{\gamma \in \Omega} \mu(\gamma; \xi) |\gamma^k - \eta^k(\gamma)|^2.$

Then for $\forall \xi$

1. $\tilde{\sigma}^{k-1} = \tilde{\sigma}^k + 4 E_{\mu(\cdot;\xi)}\big[\bar{\bar{\rho}}^k_{m(\gamma^k)} \bar{\rho}^k_{m(\gamma^k)}\big] \Delta x^2.$
2. $\tilde{\sigma}^k \le \dfrac{t_{l+1} - t_k}{\lambda} \Delta x \le \dfrac{T}{\lambda} \Delta x.$
3. $\tilde{\sigma}^k \to 0$ for each $0 \le k \le l+1 \le K$ as $\Delta x, \Delta t \to 0.$

4. Stochastic and variational approach

Suppose that the C^2-flux function $H(x,t,p):\mathbb{T}^2\times\mathbb{R}\to\mathbb{R}$ satisfies

$$H_{pp}>0,\quad \lim_{|p|\to\infty}\frac{H(x,t,p)}{p}=\infty,\quad |L_x|\le\alpha(1+|L|)\ (L:=H^*).$$

Consider the stochastic calculus of variations for each (x_n,t_{l+1})

$$(\sharp)\qquad \inf_{\xi} E_{\mu(\cdot;\xi)}\Big[\sum_{0<k\le l+1} L^c(\gamma^k,t_{k-1},\xi^k_{m(\gamma^k)})\Delta t+v^0_\Delta(\gamma^0)\Big]+h(c)t_{l+1}.$$

Thm. For each $T>0$, $\exists\,\lambda_1>0$ s.t. if $\lambda=\Delta t/\Delta x<\lambda_1$ then

1. $(\sharp)$ has the infimum V^{l+1}_n w.r.t. $\xi:G\to[-\lambda^{-1},\lambda^{-1}]$.
2. V^{l+1}_n is attained by $\xi^*:G\to(-\lambda_1^{-1},\lambda_1^{-1})\subset[-\lambda^{-1},\lambda^{-1}]$.
3. ${\xi^*}^{k+1}_m=H_p(x_m,t_k,c+D_xV^k_{m+1})$.
4. $v^{k+1}_m:=V^{k+1}_m$, $v^0_{m+1}:=v^0_\Delta(x_{m+1})$ is the sol. of $(\mathrm{HJ})_\Delta$.
5. $u^{k+1}_{m+1}:=D_xv^{k+1}_{m+2}$, $u^0_m:=u^0_\Delta(x_m)$ is the sol. of $(\mathrm{CL})_\Delta$ which satisfies the CFL-condition up to $k\le K(T)$.

Convergence.
Let $\Delta x, \Delta t \to 0$ under hyperbolic scaling $0 < \lambda_0 \leq \lambda = \Delta t/\Delta x < \lambda_1$.

Thm. The linear interpolation v_Δ of v^k_{m+1} satisfies

$$v_\Delta(x,t) \to v(x,t) = \inf_\gamma \Big[\int_0^t \{L^c(\gamma(s), s, \gamma'(s))\} ds + v^0(\gamma(0)) \Big] + h(c)t,$$

$$|v_\Delta(x,t) - v(x,t)| \leq A\sqrt{\Delta x} \text{ on } \mathbb{T} \times [0,T].$$

Thm. Suppose that

$(x,t) \in \mathbb{T} \times [0,T]$: regular point of v (i.e. $\exists\, v_x(x,t)$),

$\gamma^* : [0,t] \to \mathbb{R}$: minimizer for $v(x,t)$,

$x_n \in [x - 2\Delta x, x + 2\Delta x), t_{l+1} \in [t - \Delta t, t + \Delta t)$,

ξ^*: minimizer for v_n^{l+1},

$w_\Delta(\gamma)$: $[0,t] \to \mathbb{R}$: linear interpolation of γ generated by ξ^*.

Then

$$w_\Delta(\gamma) \to \gamma^* \text{ uniformly in probability.}$$

$*$ Approximation of characteristic curves, as well as PDE-sol.

Thm.

1. Let ξ^*, $\tilde{\xi}^*$ be the minimizer for v_n^{l+1}, v_{n+2}^{l+1}. Then u_{n+1}^{l+1} satisfies

$$u_{n+1}^{l+1} \le E_{\mu(\cdot;\xi^*)}\Big[\sum_{0<k\le l+1} L_x^c(\gamma^k, t_{k-1}, {\xi^*}^k_{m(\gamma^k)})\Delta t + u^0_{m(\gamma^0)+1}\Big] + O(\Delta x),$$

$$u_{n+1}^{l+1} \ge E_{\mu(\cdot;\tilde{\xi}^*)}\Big[\sum_{0<k\le l+1} L_x^c(\gamma^k, t_{k-1}, {\tilde{\xi}^*}{}^k_{m(\gamma^k)})\Delta t + u^0_{m(\gamma^0)-1}\Big] + O(\Delta x).$$

2. Let u_Δ be the linear interpolation of u_m^k. Then for each regular point (x,t)

$$u_\Delta(x,t) \to u(x,t) = \int_0^t L_x^c(\gamma^*(s), s, {\gamma^*}'(s))ds + u^0(\gamma^*(0)).$$

3. Except any "small" nbhd. of shocks, $u_\Delta \to u$ uniformly.

Comparison with the vanishing viscosity method (Fleming ('69))

$$\begin{cases} v^\nu_t + H(x,t,c+v^\nu_x) = h(c) + \nu v^\nu_{xx} \text{ in } \mathbb{T}\times(0,T], \\ v^\nu(x,0) = v^0(x) \in Lip(\mathbb{T}) \text{ on } \mathbb{T}. \end{cases}$$

$v^\nu \to v$ as $\nu \to 0+$.

$$v^\nu(x,t) = \inf_{\xi\in C^1} E\Big[\int_0^t L^c(\gamma^\nu(s), s, \xi(\gamma^\nu(s),s))ds + v_0(\gamma^\nu(0))\Big],$$

γ^ν : sol. of $d\gamma^\nu(s) = \xi(\gamma^\nu(s),s)ds + \sqrt{2\nu}dB(t-s)$, $\gamma^\nu(t) = x$,

B : Brownian motion,

E : expectation w.r.t. Wiener measure.

Key. Stochastic process η: $\eta'(s) = \xi(\gamma^\nu(s),s)$, $\eta(t) = x$ satisfies

($\flat$) $\qquad E[|\gamma^\nu(s) - \eta(s)|] = \sqrt{2\nu}E[|B(t-s)|]$, $\forall\, s \in [0,t]$.

$*$ ($\flat$) corresponds to $\sqrt{\tilde\sigma^k} = \sqrt{E_{\mu(\cdot;\xi)}[|\gamma^k - \eta^k(\gamma)|^2]} \le \beta\sqrt{\Delta x}$.

This yields $v^\nu \to v$ with $|v^\nu(x,t) - v(x,t)| \le a\sqrt{\nu}$ on $\mathbb{T}\times[0,T]$.