# Local well-posedness for the Navier-Stokes equations in the rotational framework

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# Introduction

**The Navier-Stokes equations with the Coriolis force**

$$
\text{(NSC)}\quad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + \Omega e_3 \times u + (u \cdot \nabla) u + \nabla p = 0 & \text{in } \mathbb{R}^3 \times (0, \infty), \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^3 \times (0, \infty), \\ u(x, 0) = u_0(x) & \text{in } \mathbb{R}^3, \end{cases}
$$

where

$u = u(x,t) = (u^1(x,t), u^2(x,t), u^3(x,t))$ : velocity filed

$p = p(x,t)$ : pressure

$u_0 = u_0(x) = (u_0^1(x), u_0^2(x), u_0^3(x))$ : initial velocity field

$\Omega \in \mathbb{R}$ : the Coriolis parameter

$e_3 := (0, 0, 1)$

# Known Results (Global existence)

## The global solvability for large $|\Omega|$

- Chemin-Desjardins-Gallagher-Grenier (2002, 2006)
  $\forall u_0 \in L^2(\mathbb{R}^2)^2 + H^{\frac{1}{2}}(\mathbb{R}^3)^3$ *with* $\operatorname{div} u_0 = 0$
  $\exists \Omega_0 = \Omega_0(u_0) > 0$ *s.t.*
  $|\Omega| \geqslant \Omega_0 \implies \exists 1 u$ : *global sol. for* (NSC)

## Uniform global solvability for small $u_0 \in X$

$\exists \delta > 0$ : *independent of* $\Omega \in \mathbb{R}$ *s.t.*
$\forall u_0 \in X$ *with* $\|u_0\|_X \leqslant \delta$, $\exists 1 u \in C([0,\infty); X)$ : *sol. to* (NSC)

- Giga-Inui-Mahalov-Saal (2008) : $X = \mathrm{FM}_0^{-1}(\mathbb{R}^3)$
- Hieber-Shibata (2010) : $X = H^{\frac{1}{2}}(\mathbb{R}^3)$
- Konieczny-Yoneda (2011) : $X = F\dot{B}_{p,\infty}^{2-\frac{3}{p}}(\mathbb{R}^3)$, $1 < p \leqslant \infty$

# Known Results (Local existence)

## Uniform local solvability for large $u_0$

- Giga-Inui-Mahalov-Matsui (2005)

  $\forall u_0 \in \mathrm{FM}_0(\mathbb{R}^3)^3$ *with* $\operatorname{div} u_0 = 0$,
  $\exists T = T(\|u_0\|_{\mathrm{FM}_0}) > 0$ : *independent of* $\Omega \in \mathbb{R}$
  *s.t.* $\exists 1 u \in C([0,T]; \mathrm{FM}_0(\mathbb{R}^3))^3$ : *sol. to* (NSC)

## Local solvability of (NSC)

- Giga-Inui-Mahalov-Matsui (2006)

  $\forall u_0 \in L^\infty_{\sigma,a}(\mathbb{R}^3) \subset L^\infty_\sigma(\mathbb{R}^3)$, $\forall \Omega \in \mathbb{R}$,
  $\exists T = T(\|u_0\|_{L^\infty_{\sigma,a}}, \Omega) > 0$ *s.t.*
  $\exists 1 u \in C_w([0,T]; L^\infty_\sigma(\mathbb{R}^3))$ : *sol. to* (NSC)

  *Furthermore*

$$T(1+\Omega T)^{6+4\delta} \geqslant \frac{C}{\|u_0\|^2_{L^\infty_{\sigma,a}}} \quad \forall \delta > 0$$

## Mild solution for (NSC)

$$u(t) = T_\Omega(t)u_0 - \int_0^t T_\Omega(t-\tau)\mathbb{P}\left[(u\cdot\nabla)u\right](\tau)d\tau \qquad t \geqslant 0$$

where

$$\mathbb{P} := (\delta_{ij} + R_i R_j)_{1\leqslant i,j\leqslant 3}\text{: Helmholtz projection}$$

$$T_\Omega(t)f = \mathscr{F}^{-1}\left[\cos\left(\Omega\frac{\xi_3}{|\xi|}t\right)e^{-|\xi|^2 t}\widehat{f}(\xi) + \sin\left(\Omega\frac{\xi_3}{|\xi|}t\right)e^{-|\xi|^2 t}R(\xi)\widehat{f}(\xi)\right]$$

$$R(\xi) := \frac{1}{|\xi|}\begin{pmatrix} 0 & \xi_3 & -\xi_2 \\ -\xi_3 & 0 & \xi_1 \\ \xi_2 & -\xi_1 & 0 \end{pmatrix} \qquad \xi \in \mathbb{R}^3 \setminus \{0\}$$

# Main Result

**Assumption (A).** $\frac{1}{2} < s < \frac{5}{4}$

**Assumption (B).**

$$0 < \frac{1}{p} < \frac{3-2s}{6}, \quad \max\left\{\frac{3-2s}{6}, \frac{s}{3}\right\} < \frac{1}{q} < \min\left\{\frac{1}{2}, \frac{5-2s}{6}\right\}$$

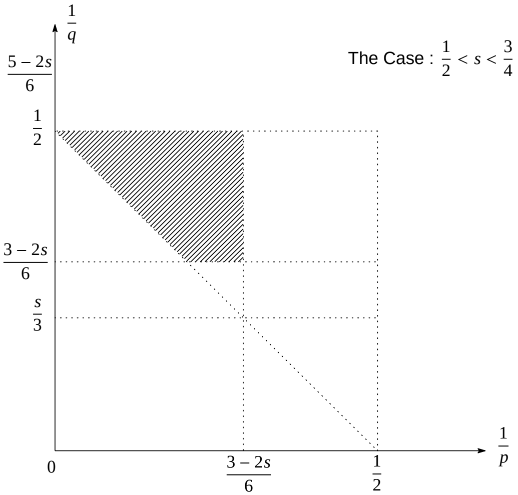
$$\text{with} \quad \frac{1}{p} + \frac{1}{q} > \frac{1}{2}$$

**Assumption (C).**

$$\frac{3}{4} - \frac{3}{2p} - \frac{s}{2} \leqslant \frac{1}{\theta_1} < \min\left\{1 - \frac{2}{p} - \frac{s}{2}, \frac{1}{2} - \frac{3}{2p}\right\},$$

$$\frac{5}{4} - \frac{3}{2q} - \frac{s}{2} \leqslant \frac{1}{\theta_2} < \min\left\{\frac{1}{2}, \frac{3}{2} - \frac{2}{q} - \frac{s}{2}, 1 - \frac{3}{2q}\right\}.$$

# Main Result




**The Figure of $(p, q)$ satisfying Assumption (B)**

# Main Result

## Theorem (Main Theorem)

$\forall(s, p, q, \theta_1, \theta_2)$ : *Assumptions (A), (B) and (C)*

$\exists C = C(s, p, q, \theta_1, \theta_2) > 0$

*s.t.*

$\forall \Omega \in \mathbb{R} \setminus \{0\}$ & $\forall u_0 \in \dot{H}^s(\mathbb{R}^3)^3$ *with* $\operatorname{div} u_0 = 0$

$\exists T = T(s, p, q, \theta_1, \theta_2, |\Omega|, \|u_0\|_{\dot{H}^s}) > 0$

*s.t.* $\exists 1 u \in X_T$ : *mild sol. to (NSC), where*

$$X_T := \left\{ u \in C([0,T]; \dot{H}^s(\mathbb{R}^3))^3 \;\middle|\; \|u\|_{X_T} \leqslant C\|u_0\|_{\dot{H}^s}, \operatorname{div} u = 0 \right\}$$

*with*

$$\|u\|_{X_T} := \sup_{0<t<T} \|u(t)\|_{\dot{H}^s} + |\Omega|^{\frac{1}{\theta_1} - \left(\frac{3}{4} - \frac{3}{2p} - \frac{s}{2}\right)} \|u\|_{L^{\theta_1}(0,T;L^p(\mathbb{R}^3))}$$
$$+ |\Omega|^{\frac{1}{\theta_2} - \left(\frac{5}{4} - \frac{3}{2q} - \frac{s}{2}\right)} \|\nabla u\|_{L^{\theta_2}(0,T;L^q(\mathbb{R}^3))}$$

# Main Result

**Theorem (Main Theorem)**

*Furthermore* $\exists C' = C'(s,p,q,\theta_1,\theta_2) > 0$ *s.t.*

$$T \geqslant C' \min\left\{ \frac{|\Omega|^{\frac{\frac{1}{\theta_1}-\left(\frac{3}{4}-\frac{3}{2p}-\frac{s}{2}\right)}{\frac{1}{2}-\frac{3}{2p}-\frac{1}{\theta_1}}}}{\|u_0\|_{\dot{H}^s}^{\frac{1}{\frac{1}{2}-\frac{3}{2p}-\frac{1}{\theta_1}}}}, \frac{|\Omega|^{\frac{\frac{1}{\theta_2}-\left(\frac{5}{4}-\frac{3}{2q}-\frac{s}{2}\right)}{1-\frac{3}{2q}-\frac{1}{\theta_2}}}}{\|u_0\|_{\dot{H}^s}^{\frac{1}{1-\frac{3}{2q}-\frac{1}{\theta_2}}}} \right\}$$

**Assumption (C).**

$$\frac{3}{4}-\frac{3}{2p}-\frac{s}{2} \leqslant \frac{1}{\theta_1} < \min\left\{1-\frac{2}{p}-\frac{s}{2}, \frac{1}{2}-\frac{3}{2p}\right\},$$

$$\frac{5}{4}-\frac{3}{2q}-\frac{s}{2} \leqslant \frac{1}{\theta_2} < \min\left\{\frac{1}{2}, \frac{3}{2}-\frac{2}{q}-\frac{s}{2}, 1-\frac{3}{2q}\right\}.$$

## Remarks

- In the case $\Omega = 0$, it is known that

$$T \geqslant \frac{C}{\|u_0\|_{H^s}^{\frac{2}{s-1/2}}} \quad \textit{for } u_0 \in H^s(\mathbb{R}^3)^3 \textit{ with } s > \frac{1}{2}.$$

- In the case

$$\frac{1}{\theta_1} = \frac{3}{4} - \frac{3}{2p} - \frac{s}{2} \quad \& \quad \frac{1}{\theta_2} = \frac{5}{4} - \frac{3}{2q} - \frac{s}{2},$$

our theorem holds for all $\Omega \in \mathbb{R}$, and

$$T \geqslant C' \min\left\{ \frac{|\Omega|^{\frac{\frac{1}{\theta_1} - \left(\frac{3}{4} - \frac{3}{2p} - \frac{s}{2}\right)}{\frac{1}{2} - \frac{3}{2p} - \frac{1}{\theta_1}}}}{\|u_0\|_{\dot{H}^s}^{\frac{1}{\frac{1}{2} - \frac{3}{2p} - \frac{1}{\theta_1}}}}, \frac{|\Omega|^{\frac{\frac{1}{\theta_2} - \left(\frac{5}{4} - \frac{3}{2q} - \frac{s}{2}\right)}{1 - \frac{3}{2q} - \frac{1}{\theta_2}}}}{\|u_0\|_{\dot{H}^s}^{\frac{1}{1 - \frac{3}{2q} - \frac{1}{\theta_2}}}} \right\} = \frac{C'}{\|u_0\|_{\dot{H}^s}^{\frac{2}{s-1/2}}}$$

# Linear Estimates

**The operator $\mathscr{G}_\pm(\tau)$ $(\tau \in \mathbb{R})$ of oscillatory integral type**

$$\mathscr{G}_\pm(\tau)[f](x) := \mathscr{F}^{-1}\left[e^{\pm i\tau\frac{\xi_3}{|\xi|}}\mathscr{F}[f]\right](x) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{\pm i\tau\frac{\xi_3}{|\xi|}} e^{ix\cdot\xi}\widehat{f}(\xi)d\xi$$

**Decomposition of the semigroup $T_\Omega(t)$**

$$\begin{aligned} T_\Omega(t)f &= \mathscr{F}^{-1}\left[\cos\left(\Omega\frac{\xi_3}{|\xi|}t\right)e^{-|\xi|^2t}\widehat{f}(\xi) + \sin\left(\Omega\frac{\xi_3}{|\xi|}t\right)e^{-|\xi|^2t}R(\xi)\widehat{f}(\xi)\right] \\ &= \frac{1}{2}\mathscr{G}_+(\Omega t)\left[e^{t\Delta}(I+\mathscr{R})f\right] + \frac{1}{2}\mathscr{G}_-(\Omega t)\left[e^{t\Delta}(I-\mathscr{R})f\right] \end{aligned}$$

where

$$\mathscr{R} := \begin{pmatrix} 0 & R_3 & -R_2 \\ -R_3 & 0 & R_1 \\ R_2 & -R_1 & 0 \end{pmatrix}, \qquad R_j : \text{the Riesz transform}$$

# Linear Estimates

## Dispersive estimate for $\mathscr{G}_{\pm}(\tau)$

**Lemma (Dispersive estimate)**

$2 \leqslant \forall p \leqslant \infty,\ \exists C = C(p) >$ *s.t.*

$$\left\|\mathscr{G}_{\pm}(\tau)[f]\right\|_{\dot{B}^{s}_{p,q}} \leqslant C \left\{ \frac{\log(e+|\tau|)}{1+|\tau|} \right\}^{\frac{1}{2}\left(1-\frac{2}{p}\right)} \left\| f \right\|_{\dot{B}^{s+3(1-\frac{2}{p})}_{p',q}}$$

*for* $\forall \tau \in \mathbb{R}, \forall s \in \mathbb{R}, 1 \leqslant \forall q \leqslant \infty, \forall f \in \dot{B}^{s+3(1-\frac{2}{p})}_{p',q}(\mathbb{R}^3)$.

[ Idea of the proof ]

1. $L^1 - L^\infty$ estimates $\Longleftarrow$ stationary phase method
2. $L^2 - L^2$ estimates $\Longleftarrow$ the Plancherel theorem
3. the Riesz-Thorin real interpolation theorem

# Linear Estimates

**Estimate of the Strichartz type for $T_\Omega(t)$**

**Assumption (L1).** $0 \leqslant s < 3/2$.
**Assumption (L2).**

$$\max\left\{\frac{1-2s}{6}, 0\right\} < \frac{1}{p} < \frac{3-2s}{6}, \quad \frac{3}{4} - \frac{3}{2p} - \frac{s}{2} \leqslant \frac{1}{\theta} < \min\left\{\frac{1}{2}, 1 - \frac{2}{p} - \frac{s}{2}\right\}$$

**Lemma (Estimate of the Strichartz type)**

$\forall (s, p, \theta)$ : *Assumptions (L1) and (L2)*
$\exists C = C(s, p, \theta) > 0$ *s.t.*

$$\|T_\Omega(\cdot) f\|_{L^\theta(0,\infty;L^p(\mathbb{R}^3))} \leqslant C|\Omega|^{-\left\{\frac{1}{\theta} - \left(\frac{3}{4} - \frac{3}{2p} - \frac{s}{2}\right)\right\}} \|f\|_{\dot{H}^s}$$

*for* $\forall \Omega \in \mathbb{R} \setminus \{0\}$ & $\forall f \in \dot{H}^s(\mathbb{R}^3)^3$.

# Linear Estimates

**Lemma (Estimate of the Strichartz type)**

$\forall(s,p,\theta)$ : *Assumptions (L1) and (L2)*
$\exists C = C(s,p,\theta) > 0$ *s.t.*

$$\|T_\Omega(\cdot)f\|_{L^\theta(0,\infty;L^p(\mathbb{R}^3))} \leqslant C|\Omega|^{-\left\{\frac{1}{\theta}-\left(\frac{3}{4}-\frac{3}{2p}-\frac{s}{2}\right)\right\}}\|f\|_{\dot{H}^s}$$

*for* $\forall\Omega \in \mathbb{R}\setminus\{0\}$ & $\forall f \in \dot{H}^s(\mathbb{R}^3)^3$.

**Remark**

$$\frac{1}{\theta} = \frac{3}{4} - \frac{3}{2p} - \frac{s}{2} \iff \underline{\frac{2}{\theta} + \frac{3}{p} = \frac{3}{2} - s}$$

$H^s$ admissible pair of the Strichartz estimates for $e^{it\Delta}$ :

$$\left\|e^{it\Delta}f\right\|_{L^\theta(\mathbb{R};L^p(\mathbb{R}^3))} \leqslant C\|f\|_{H^s}$$