Weighted Trudinger-Moser type inequality and its application

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joint work with Michinori Ishiwata and Makoto Nakamura

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Contents

- Best constant of Weighted Trudinger-Moser inequality
- Asymptotic best constant of Caffarelli-Kohn-Nirenberg inequality
- Application to Klein-Gordon equation with weighted exponential type nonlinearlity

Trudinger (1967), Moser (1971)

$n \geq 2$, $\exists C_n > 0$ **s.t.**

$$\int_{\Omega} \exp\left(\alpha |u(x)|^{n'}\right) dx \leq C_n |\Omega| \tag{1}$$

for $\forall u \in H_0^{1,n}(\Omega)$ **with** $\|\nabla u\|_{L^n(\Omega)} \leq 1$,
$\forall \Omega$: **bounded**, $0 < \forall \alpha \leq n\omega_{n-1}^{\frac{1}{n-1}}$, **where** $\omega_{n-1} = |S_{n-1}|$.

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- **If** $\alpha > n\omega_{n-1}^{\frac{1}{n-1}}$, **then** (1) **fails.**

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- If $\alpha > n\omega_{n-1}^{\frac{1}{n-1}}$, then (1) fails.
- (1) has been **generalized** by many authors :

  Adams (1998) $\to H_0^{m,\frac{n}{m}}(\Omega)$ with **best constant**
  Ogawa (1990), Ogawa-Ozawa (1991) $\to H^{\frac{n}{2},2}(\mathbb{R}^n)$
  Ozawa (1995) $\to H^{\frac{n}{p},p}(\mathbb{R}^n)$
  Adachi-Tanaka (1999) $\to H^{1,n}(\mathbb{R}^n)$ with **best constant**

  $\cdots\cdots$

Adachi-Tanaka (1999)

(i) $n \geq 2$, $0 < \forall \alpha < n\omega_{n-1}^{\frac{1}{n-1}}$, $\exists C_{n,\alpha} > 0$ **s.t.**

$$\int_{\mathbb{R}^n} \Phi_n \left(\alpha |u(x)|^{n'} \right) dx \leq C_{n,\alpha} \|u\|^n_{L^n(\mathbb{R}^n)} \tag{2}$$

for $\forall u \in H^{1,n}(\mathbb{R}^n)$ **with** $\|\nabla u\|_{L^n(\mathbb{R}^n)} \leq 1$,

where $\Phi_n(t) := \sum_{j=n-1}^{\infty} \frac{t^j}{j!}$ **for** $t \geq 0$.

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- **Sobolev embedding :** $H^{1,n}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ **for** $n \leq \forall q < \infty$.
- $\int_{\mathbb{R}^n} \Phi_n \left(\alpha |u(x)|^{n'}\right) dx = \sum_{j=n-1}^{\infty} \frac{\alpha^j}{j!} \|u\|^{n'j}_{L^{n'j}(\mathbb{R}^n)}$.

Caffarelli-Kohn-Nirenberg (1984)

$n \geq 2$, $\tilde{s} \leq s < n$, $n \leq q < \infty$, $\exists C_{n,s,\tilde{s},q} > 0$ **s.t.**

$$\|u\|_{L^q(|x|^{-s})} \leq C_{n,s,\tilde{s},q} \|u\|_{L^n(|x|^{-\tilde{s}})}^{\frac{n(n-s)}{q(n-\tilde{s})}} \|\nabla u\|_{L^n(\mathbb{R}^n)}^{1-\frac{n(n-s)}{q(n-\tilde{s})}} \qquad (3)$$

for $\forall u \in L^n(|x|^{-\tilde{s}}) \cap \dot{H}^{1,n}(\mathbb{R}^n)$,
where $\|u\|_{L^n(|x|^{-\tilde{s}})} := \left(\int_{\mathbb{R}^n} |u(x)|^n |x|^{-\tilde{s}} dx\right)^{\frac{1}{n}}$.

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- $s = \tilde{s} = 0$ **in** (3) **implies** $H^{1,n}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ **for** $n \leq \forall q < \infty$.

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- $s = \tilde{s} = 0$ **in** (3) **implies** $H^{1,n}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ **for** $n \leq \forall q < \infty$.
- **First goal is to establish weighted Trudinger-Moser inequality crresponding to** (3).

Theorem 1.

(i) $n \geq 2$, $\tilde{s} \leq s < n$, $0 < \forall \alpha < (n-s)\omega_{n-1}^{\frac{1}{n-1}}$, $\exists C_{n,s,\tilde{s},\alpha} > 0$ **s.t.**

$$\int_{\mathbb{R}^n} \Phi_n \left(\alpha |u(x)|^{n'} \right) \frac{dx}{|x|^s} \leq C_{n,s,\tilde{s},\alpha} \|u\|_{L^n(|x|^{-\tilde{s}})}^{\frac{n(n-s)}{n-\tilde{s}}} \tag{4}$$

for radial $\forall u \in L^n(|x|^{-\tilde{s}}) \cap \dot{H}^{1,n}(\mathbb{R}^n)$ **with** $\|\nabla u\|_{L^n(\mathbb{R}^n)} \leq 1$.

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- **If** $\tilde{s} = 0$, **radial condition can be removed by rearrangement.**

Rearrangement inequalities

Let $n \geq 2$, $0 \leq s < n$, $1 \leq q < \infty$, $\alpha > 0$,

and let u^* be rearrangement of u.

$$\begin{cases} \int_{\mathbb{R}^n} |u(x)|^q |x|^{-s} dx \leq \int_{\mathbb{R}^n} |u^*(x)|^q |x|^{-s} dx, \\ \int_{\mathbb{R}^n} |\nabla u^*(x)|^2 dx \leq \int_{\mathbb{R}^n} |\nabla u(x)|^2 dx, \\ \int_{\mathbb{R}^n} \Phi_n\left(\alpha |u(x)|^{n'}\right) |x|^{-s} dx \leq \int_{\mathbb{R}^n} \Phi_n\left(\alpha |u^*(x)|^{n'}\right) |x|^{-s} dx. \end{cases}$$

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Corollary of Theorem 1.

$n \geq 2$, $0 \leq s < n$, $0 < \forall \alpha < (n-s)\omega_{n-1}^{\frac{1}{n-1}}$, $\exists C_{n,s,\alpha} > 0$ **s.t.**

$$\int_{\mathbb{R}^n} \Phi_n \left(\alpha |u(x)|^{n'} \right) \frac{dx}{|x|^s} \leq C_{n,s,\alpha} \|u\|_{L^n(\mathbb{R}^n)}^{n-s}$$

for $\forall u \in H^{1,n}(\mathbb{R}^n)$ **with** $\|\nabla u\|_{L^n(\mathbb{R}^n)} \leq 1$.

Comments on proof of Theorem 1.

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Proof of (i) **For** $s < n$, $0 < \alpha < (n-s)\omega_{n-1}^{\frac{1}{n-1}}$,

$$\int_{\mathbb{R}^n} \Phi_n\left(\alpha |u(x)|^{n'}\right) \frac{dx}{|x|^s} \leq C_{n,s,\alpha} \|u\|_{L^n(|x|^{-s})}^{n} \tag{5}$$

for radial $u \in L^n(|x|^{-s}) \cap \dot{H}^{1,n}(\mathbb{R}^n)$ **with** $\|\nabla u\|_{L^n(\mathbb{R}^n)} \leq 1$.

(5) **is equivalent to non-singular case $s = 0$ by scaling**

$$v(x) := \left(\frac{n-s}{n}\right)^{\frac{n-1}{n}} \tilde{u}(|x|^{\frac{n}{n-s}}) \quad \textbf{for} \quad \tilde{u}(|x|) = u(x) \quad \textbf{for} \quad x \in \mathbb{R}^n.$$

$$\begin{cases} \|\nabla u\|_{L^n(\mathbb{R}^n)} = \|\nabla v\|_{L^n(\mathbb{R}^n)}, \quad \|u\|_{L^n(|x|^{-s})} = \frac{n}{n-s}\|u\|_{L^n(\mathbb{R}^n)}, \\ \int_{\mathbb{R}^n} \Phi_n\left(\alpha|u(x)|^{n'}\right)\frac{dx}{|x|^s} = \frac{n}{n-s}\int_{\mathbb{R}^n}\Phi_n\left(\frac{n}{n-s}\alpha|u(x)|^{n'}\right)dx, \end{cases}$$

and note $0 < \frac{n}{n-s}\alpha < n\omega_{n-1}^{\frac{1}{n-1}}$.

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and note $0 < \frac{n}{n-s}\alpha < n\omega_{n-1}^{\frac{1}{n-1}}$.

- **Equation** (5) + **Caffarelli-Kohn-Nirenberg inequality** $\rightarrow$ **Theorem 1 (i).**

Proof of (ii) **Define** $\{u_k\}_{k\in\mathbb{N}} \subset L^n(|x|^{-\tilde{s}}) \cap \dot{H}^{1,n}(\mathbb{R}^n)$ **by**

$$u_k(x) = \begin{cases} 0 \quad \textbf{if} \ |x| \geq 1, \\ \left(\frac{n-s}{\omega_{n-1}k}\right)^{\frac{1}{n}} \log\left(\frac{1}{|x|}\right) \quad \textbf{if} \ e^{-\frac{k}{n-s}} < |x| < 1, \\ \left(\frac{1}{\omega_{n-1}}\right)^{\frac{1}{n}} \left(\frac{k}{n-s}\right)^{\frac{1}{n'}} \quad \textbf{if} \ 0 \leq |x| \leq e^{-\frac{k}{n-s}}. \end{cases}$$

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Then we see $\|\nabla u_k\|_{L^n(\mathbb{R}^n)} = 1$ for all $k \in \mathbb{N}$, and

$$\frac{\int_{\mathbb{R}^n} \Phi_n\left((n-s)\omega_{n-1}^{\frac{1}{n-1}}|u_k(x)|^{n'}\right)\frac{dx}{|x|^s}}{\|u_k\|_{L^n(|x|^{-\tilde{s}})}^{\frac{n(n-s)}{n-\tilde{s}}}} \to \infty$$

as $k \to \infty$, which implies (4) fails if $\alpha = (n-s)\omega_{n-1}^{\frac{1}{n-1}}$.

Asymptotic best-constant of Caffarelli-Kohn-Nirenberg inequality

Theorem 2.

(i) $n \geq 2$, $\tilde{s} \leq s < n$, $\forall \beta > \left(\frac{n-1}{en(n-s)\omega_{n-1}^{\frac{1}{n-1}}} \right)^{\frac{n-1}{n}}$, $\exists r_{n,s,\tilde{s},\beta} \geq n$ **s.t.**

$$\|u\|_{L^q(|x|^{-s})} \leq \beta q^{\frac{1}{n'}} \|u\|_{L^n(|x|^{-\tilde{s}})}^{\frac{n(n-s)}{q(n-\tilde{s})}} \|\nabla u\|_{L^n(\mathbb{R}^n)}^{1-\frac{n(n-s)}{q(n-\tilde{s})}} \qquad (6)$$

for radial $\forall u \in L^n(|x|^{-\tilde{s}}) \cap \dot{H}^{1,n}(\mathbb{R}^n)$ **and** $r_{n,s,\tilde{s},\beta} \leq \forall q < \infty$.

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for radial $\forall u \in L^n(|x|^{-\tilde{s}}) \cap \dot{H}^{1,n}(\mathbb{R}^n)$ **and** $r_{n,s,\tilde{s},\beta} \leq \forall q < \infty$.

(ii) If $0 < \forall \beta < \left(\frac{n-1}{en(n-s)\omega_{n-1}^{\frac{1}{n-1}}} \right)^{\frac{n-1}{n}}$, **then** (6) **fails**

in the above asymptotic sense.

Comments on proof of Theorem 2.

$\alpha_0 := \sup\{\alpha > 0 \,|\, \exists C_{n,s,\tilde{s},\alpha} > 0$ **s.t.**

$$\int_{\mathbb{R}^n} \Phi_n\left(\alpha|u(x)|^{n'}\right)\frac{dx}{|x|^s} \le C_{n,s,\tilde{s},\alpha}\|u\|_{L^n(|x|^{-\tilde{s}})}^{\frac{n(n-s)}{n-\tilde{s}}}$$

for radial $u \in L^n(|x|^{-\tilde{s}}) \cap \dot{H}^{1,n}(\mathbb{R}^n)$ **with** $\|\nabla u\|_{L^n(\mathbb{R}^n)} \le 1.\}$

$\beta_0 := \inf\{\beta > 0 \,|\, \exists r_{n,s,\tilde{s},\beta} \ge n$ **s.t.**

$$\|u\|_{L^q(|x|^{-s})} \le \beta\, q^{\frac{1}{n'}}\|u\|_{L^n(|x|^{-\tilde{s}})}^{\frac{n(n-s)}{q(n-\tilde{s})}}\|\nabla u\|_{L^n(\mathbb{R}^n)}^{1-\frac{n(n-s)}{q(n-\tilde{s})}}$$

for radial $u \in L^n(|x|^{-\tilde{s}}) \cap \dot{H}^{1,n}(\mathbb{R}^n)$ **and** $r_{n,s,\tilde{s},\beta} \le \forall q < \infty.\}$

Exact relation between α_0 and β_0 :

$$\beta_0 = \left(\frac{1}{en'\alpha_0} \right)^{\frac{1}{n'}}. \tag{7}$$

On the other hand, since $\alpha_0 = (n-s)\omega_{n-1}^{\frac{1}{n-1}}$ by Theorem 1,

$$\beta_0 = \left(\frac{n-1}{en(n-s)\omega_{n-1}^{\frac{1}{n-1}}} \right)^{\frac{n-1}{n}}.$$

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- **For non-singular case $s = \tilde{s} = 0$, (7) was proved by Ozawa (1997).**

Theorem 3.

Let $0 \le s < 1$ and $(f, g) \in H^{1,2}(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$ with $\|\nabla f\|_{L^2(\mathbb{R}^2)} < 1$.

$$\begin{cases} \Box u + u = -\frac{u}{|x|^s}\left(e^{4(1-s)\pi u^2} - 1\right) & \text{in } (0, T) \times \mathbb{R}^2, \\ u(0, \cdot) = f \quad \text{and} \quad \partial_t u(0, \cdot) = g & \text{in } \mathbb{R}^2 \end{cases}$$

has unique local solution for some $T = T(\|f\|_{L^2(\mathbb{R}^2)}, \|g\|_{L^2(\mathbb{R}^2)}) > 0$ in the class

$$\|u\|_{X_T} := \sup_{0<t<T}\left(\|u(t, \cdot)\|_{H^{1,2}(\mathbb{R}^2)} + \|\partial_t u(t, \cdot)\|_{L^2(\mathbb{R}^2)}\right) + \|u\|_{L^4\left(0,T;C^{\frac{1}{4}}(\mathbb{R}^2)\right)}.$$

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has unique local solution for some $T = T(\|f\|_{L^2(\mathbb{R}^2)}, \|g\|_{L^2(\mathbb{R}^2)}) > 0$ in the class

$$\|u\|_{X_T} := \sup_{0<t<T}\left(\|u(t,\cdot)\|_{H^{1,2}(\mathbb{R}^2)} + \|\partial_t u(t,\cdot)\|_{L^2(\mathbb{R}^2)}\right) + \|u\|_{L^4\left(0,T;C^{\frac{1}{4}}(\mathbb{R}^2)\right)}.$$

- Non-singular case $s = 0$ in Theorem 3 was proved by Ibrahim-Majdoub-Masmoudi (2006).

Comments on proof of Theorem 3.

We decompose solution $u = v + v_0$**, where**

$$\begin{cases} \Box v + v = -\frac{v+v_0}{|x|^s}\left(e^{4(1-s)\pi(v+v_0)^2} - 1\right) & \textbf{in } (0,T)\times\mathbb{R}^2, \\ v(0,\cdot) = \partial_t v(0,\cdot) = 0 \quad \textbf{in } \mathbb{R}^2, \end{cases}$$

$$\begin{cases} \Box v_0 + v_0 = 0 \quad \textbf{in } (0,T)\times\mathbb{R}^2, \\ v_0(0,\cdot) = f \quad \textbf{and} \quad \partial_t v_0(0,\cdot) = g \quad \textbf{in } \mathbb{R}^2. \end{cases}$$

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v_0 **can be solved explicitly by**

$$v_0(t,x) = \cos(t\sqrt{1-\Delta})f(x) + \frac{\sin(t\sqrt{1-\Delta})}{\sqrt{1-\Delta}}g(x).$$

Our function space : $X_T(\delta) := \{v \in X_T \mid \|v\|_{X_T} \le \delta\}$

for $T > 0$ **and** $\delta > 0$**, where**

$$\|u\|_{X_T} := \sup_{0<t<T} \left(\|u(t,\cdot)\|_{H^{1,2}(\mathbb{R}^2)} + \|\partial_t u(t,\cdot)\|_{L^2(\mathbb{R}^2)} \right) + \|u\|_{L^4\left(0,T\,;\,C^{\frac{1}{4}}(\mathbb{R}^2)\right)}$$

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To apply fixed point theorem, define a map $\Phi(v)$ **for** $v \in X_T(\delta)$ **by**

$$\begin{cases} \Box\Phi(v) + \Phi(v) = -\frac{v+v_0}{|x|^s}\left(e^{4(1-s)\pi(v+v_0)^2} - 1\right) & \text{in } (0,T)\times\mathbb{R}^2, \\ \Phi(v)(0,\cdot) = \partial_t\Phi(v)(0,\cdot) = 0 & \text{on } \mathbb{R}^2. \end{cases} \tag{8}$$

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Our goal : Φ becomes contraction from $X_T(\delta)$ to $X_T(\delta)$

for sufficiently small T and δ.

- **Energy estimate for** (8) **:**

$$\sup_{0<t<T} \left(\|\Phi(v)(t,\cdot)\|_{H^{1,2}(\mathbb{R}^2)} + \|\partial_t \Phi(v)(t,\cdot)\|_{L^2(\mathbb{R}^2)} \right)$$
$$\leq C \left\| \frac{v+v_0}{|x|^s} \left(e^{4(1-s)\pi(v+v_0)^2} - 1 \right) \right\|_{L^1(0,T\,;\,L^2(\mathbb{R}^2))}$$

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- **Strichartz estimate for (8) by Ginibre-Velo (1995) :**

$$\|\Phi(v)\|_{L^4\left(0,T\,;\,C^{\frac{1}{4}}(\mathbb{R}^2)\right)} \leq C \left\| \frac{v+v_0}{|x|^s} \left(e^{4(1-s)\pi(v+v_0)^2 - 1} \right) \right\|_{L^1(0,T\,;\,L^2(\mathbb{R}^2))}.$$

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Combining Energy estimate with Strichartz estimate,

$$\|u\|_{X_T} \leq C \left\| \frac{v+v_0}{|x|^s} \left(e^{4(1-s)\pi(v+v_0)^2 - 1} \right) \right\|_{L^1(0,T\,;\,L^2(\mathbb{R}^2))}.$$

By Hölder and Sobolev inequalities, for any $\varepsilon > 0$ and $0 < t < T$,

$$\left\| \frac{v+v_0}{|x|^s} \left(e^{4(1-s)\pi(v+v_0)^2} - 1 \right) \right\|_{L^2(\mathbb{R}^2)}^2$$

$$\leq \|(v+v_0)^2\|_{L^{(1+\varepsilon)'}(\mathbb{R}^2)} \left\| \left(\frac{e^{4(1-s)\pi(v+v_0)^2}}{|x|^s} - 1 \right)^2 \right\|_{L^{1+\varepsilon}(\mathbb{R}^2)}$$

$$= \|v+v_0\|_{L^{2+\frac{2}{\varepsilon}}(\mathbb{R}^2)}^2 \left\| \left(e^{4(1-s)\pi(v+v_0)^2} - 1 \right) \left(\frac{e^{4(1-s)\pi(v+v_0)^2} - 1}{|x|^{2s}} \right) \right\|_{L^{1+\varepsilon}(\mathbb{R}^2)}$$

$$\leq C\|v+v_0\|_{H^{1,2}(\mathbb{R}^2)}^2 e^{4(1-s)\pi\|v+v_0\|_{L^\infty(\mathbb{R}^2)}^2} \left\| \frac{e^{4(1-s)\pi(v+v_0)^2} - 1}{|x|^{2s}} \right\|_{L^{1+\varepsilon}(\mathbb{R}^2)}.$$

$\tilde{s} = 0$ and $n = 2$ in Theorem 1.

$0 \leq s < 2$, $0 < \alpha < 2(2-s)\pi$, $\exists C_{s,\alpha} > 0$ **s.t.**

$$\int_{\mathbb{R}^2} \left(e^{\alpha |u(x)|^2} - 1 \right) \frac{dx}{|x|^s} \leq C_{s,\alpha} \|u\|_{L^2(\mathbb{R}^2)}^{2-s}$$

for $\forall u \in H^{1,2}(\mathbb{R}^2)$ **with** $\|\nabla u\|_{L^2(\mathbb{R}^2)} \leq 1$.

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Brézis-Gallouët-Wainger type inequality by Ibrahim-Majdoub-Masmoudi (2007)

$\lambda > \frac{2}{\pi}$, $\exists C_\lambda > 0$ **s.t.**

$$\|u\|_{L^\infty(\mathbb{R}^2)}^2 \leq \lambda \|u\|_{H^{1,2}(\mathbb{R}^2)}^2 \log \left(C_\lambda + \frac{\|u\|_{C^{\frac{1}{4}}(\mathbb{R}^2)}}{\|u\|_{H^{1,2}(\mathbb{R}^2)}} \right)$$

for $\forall u \in \left(H^{1,2}(\mathbb{R}^2) \cap C^{\frac{1}{4}}(\mathbb{R}^2) \right) \setminus \{0\}$.