On a mathematical justification of the penalty method for the Stokes and Navier-Stokes equations

Norikazu YAMAGUCHI

University of Toyama, JAPAN

November 28, 2011
4th Japan-Germany Workshop on Mathematical Fluid Dynamics,
Waseda University, Tokyo, Japan

Navier-Stokes equation

Navier-Stokes equation (NS)

$$\partial_t \boldsymbol{u} - \Delta \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p = 0, \qquad x \in \Omega, t > 0, \tag{1a}$$

$$\operatorname{div} \boldsymbol{u} = 0, \qquad x \in \Omega, t > 0., \tag{1b}$$

$$\boldsymbol{u}(x,0) = \boldsymbol{u}_0(x), \qquad x \in \Omega. \tag{1c}$$

$\Omega \subseteq \mathbb{R}^d$ $(d \geq 2)$ (if $\partial\Omega \neq \emptyset$ some boundary condition is imposed).
$\boldsymbol{u} = (u^1, \ldots, u^d)$: velocity, p: pressure (unknown).

Navier-Stokes equation

Navier-Stokes equation (NS)

$$\partial_t \boldsymbol{u} - \Delta \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p = 0, \qquad x \in \Omega, t > 0, \qquad (1a)$$

$$\operatorname{div} \boldsymbol{u} = 0, \qquad x \in \Omega, t > 0., \qquad (1b)$$

$$\boldsymbol{u}(x,0) = \boldsymbol{u}_0(x), \qquad x \in \Omega. \qquad (1c)$$

$\Omega \subseteq \mathbb{R}^d$ $(d \geq 2)$ (if $\partial\Omega \neq \emptyset$ some boundary condition is imposed).
$\boldsymbol{u} = (u^1, \dots, u^d)$: velocity, p: pressure (unknown).

Main difficulties of (NS)

- The pressure has **no** time evolution (1a)
- Divergence free constraint (1b)

Difficulties in numerical computation

- $T > 0, N \in \mathbb{N}$. $h = T/N$ (time step size)
- $\boldsymbol{U}^n(x) \approx \boldsymbol{u}(x, t_n)$, $P^n(x) \approx p(x, t_n)$ ($t_n = nh$): difference approximation of (NS) at $t = t_n$.

Difficulties in numerical computation

- $T > 0, N \in \mathbb{N}$. $h = T/N$ (time step size)
- $\boldsymbol{U}^n(x) \approx \boldsymbol{u}(x, t_n)$, $P^n(x) \approx p(x, t_n)$ ($t_n = nh$): difference approximation of (NS) at $t = t_n$.

Time discretization of (NS): Direct method

$$\frac{\boldsymbol{U}^{n+1} - \boldsymbol{U}^n}{h} - \Delta \boldsymbol{U}^n + \boldsymbol{U}^n \cdot \nabla \boldsymbol{U}^n + \nabla P^n = 0, \quad n = 0, 1, \ldots, N-1,$$

$$\operatorname{div} \boldsymbol{U}^n = 0.$$

Difficulties in numerical computation

- $T > 0, N \in \mathbb{N}$. $h = T/N$ (time step size)
- $\boldsymbol{U}^n(x) \approx \boldsymbol{u}(x, t_n)$, $P^n(x) \approx p(x, t_n)$ ($t_n = nh$): difference approximation of (NS) at $t = t_n$.

Time discretization of (NS): Direct method

$$\frac{\boldsymbol{U}^{n+1} - \boldsymbol{U}^n}{h} - \Delta \boldsymbol{U}^n + \boldsymbol{U}^n \cdot \nabla \boldsymbol{U}^n + \nabla P^n = 0, \quad n = 0, 1, \ldots, N-1,$$

$$\operatorname{div} \boldsymbol{U}^n = 0.$$

Solving the above difference eq. w.r.t $\boldsymbol{U}^{n+1}$ we have

$$\boldsymbol{U}^{n+1} = \boldsymbol{U}^n + h\Delta \boldsymbol{U}^n - h\boldsymbol{U}^n \cdot \nabla \boldsymbol{U}^n - h\nabla P^n, \quad n = 0, 1, \ldots, N-1,$$

$$\operatorname{div} \boldsymbol{U}^n = 0, \quad n = 0, 1, \ldots, N.$$

Difficulties in numerical computation

Time discretization of (NS): Direct method

$$\boldsymbol{U}^{n+1} = \boldsymbol{U}^n + h(\Delta \boldsymbol{U}^n - \boldsymbol{U}^n \cdot \nabla \boldsymbol{U}^n - \nabla P^n), \quad n = 0, 1, \ldots, N-1,$$
$$\operatorname{div} \boldsymbol{U}^n = 0, \quad n = 0, 1, \ldots, N.$$

- To obtain $\boldsymbol{U}^{n+1}$, we need not only $\boldsymbol{U}^n$, but also P^n.

Difficulties in numerical computation

Time discretization of (NS): Direct method

$$\boldsymbol{U}^{n+1} = \boldsymbol{U}^n + h(\Delta \boldsymbol{U}^n - \boldsymbol{U}^n \cdot \nabla \boldsymbol{U}^n - \nabla P^n), \quad n = 0, 1, \ldots, N-1,$$
$$\operatorname{div} \boldsymbol{U}^n = 0, \quad n = 0, 1, \ldots, N.$$

- To obtain $\boldsymbol{U}^{n+1}$, we need not only $\boldsymbol{U}^n$, but also P^n.

Since $\operatorname{div} \boldsymbol{U}^n = 0$, we have

$$-\Delta P^n = \operatorname{div}(\boldsymbol{U}^n \cdot \nabla \boldsymbol{U}^n), \quad x \in \Omega, \tag{NLP}$$

Difficulties in numerical computation

Time discretization of (NS): Direct method

$$\boldsymbol{U}^{n+1} = \boldsymbol{U}^n + h(\Delta \boldsymbol{U}^n - \boldsymbol{U}^n \cdot \nabla \boldsymbol{U}^n - \nabla P^n), \quad n = 0, 1, \dots, N-1,$$
$$\operatorname{div} \boldsymbol{U}^n = 0, \quad n = 0, 1, \dots, N.$$

- To obtain $\boldsymbol{U}^{n+1}$, we need not only $\boldsymbol{U}^n$, but also P^n.

Since $\operatorname{div} \boldsymbol{U}^n = 0$, we have

$$-\Delta P^n = \operatorname{div}(\boldsymbol{U}^n \cdot \nabla \boldsymbol{U}^n), \quad x \in \Omega, \tag{NLP}$$

- P^n is represented by $\boldsymbol{U}^n$. Formally, $P^n = (-\Delta_\Omega)^{-1} \operatorname{div}(\boldsymbol{U}^n \cdot \nabla \boldsymbol{U}^n)$. This representation is **non-local**.
- Boundary condition for P^n ?
- Does $\operatorname{div} \boldsymbol{U}^n = 0$ hold for any $n \geq 1$, if we apply some space discretization ?

Difficulties in numerical computation

Time discretization of (NS): Direct method

$$\boldsymbol{U}^{n+1} = \boldsymbol{U}^n + h(\Delta \boldsymbol{U}^n - \boldsymbol{U}^n \cdot \nabla \boldsymbol{U}^n - \nabla P^n), \quad n = 0, 1, \ldots, N-1,$$
$$\operatorname{div} \boldsymbol{U}^n = 0, \quad n = 0, 1, \ldots, N.$$

- To obtain $\boldsymbol{U}^{n+1}$, we need not only $\boldsymbol{U}^n$, but also P^n.

Since $\operatorname{div} \boldsymbol{U}^n = 0$, we have

$$-\Delta P^n = \operatorname{div}(\boldsymbol{U}^n \cdot \nabla \boldsymbol{U}^n), \quad x \in \Omega, \tag{NLP}$$

- P^n is represented by $\boldsymbol{U}^n$. Formally, $P^n = (-\Delta_\Omega)^{-1} \operatorname{div}(\boldsymbol{U}^n \cdot \nabla \boldsymbol{U}^n)$. This representation is **non-local**.
- Boundary condition for P^n ?
- Does $\operatorname{div} \boldsymbol{U}^n = 0$ hold for any $n \geq 1$, if we apply some space discretization ?

Pressure makes *direct* numerical computation of (NS) complicate.

Penalty method

One of the standard ways to eliminate the pressure from (NS) is **penalty method** (Temam (1968)):

Penalty method

One of the standard ways to eliminate the pressure from (NS) is **penalty method** (Temam (1968)):
Replacing $\operatorname{div} \boldsymbol{u} = 0$ by

$$\operatorname{div} \boldsymbol{u} = -p/\eta \quad (\eta > 0), \tag{PEN}$$

Penalty method

One of the standard ways to eliminate the pressure from (NS) is **penalty method** (Temam (1968)):
Replacing $\operatorname{div} \boldsymbol{u} = 0$ by

$$\operatorname{div} \boldsymbol{u} = -p/\eta \quad (\eta > 0), \tag{PEN}$$

and substituting $p = -\eta \operatorname{div} \boldsymbol{u}$ into (1a), we have a penalized (NS).

Penalized (NS)

$$\partial_t \boldsymbol{u}^\eta - \Delta \boldsymbol{u}^\eta + \boldsymbol{u}^\eta \cdot \nabla \boldsymbol{u}^\eta - \eta \nabla \operatorname{div} \boldsymbol{u}^\eta = 0. \tag{NS$_\eta$}$$

- $(\text{NS})_\eta$ does not include the pressure p
- Formally $\eta \to +\infty$, (PEN) becomes $\operatorname{div} \boldsymbol{u}^\eta = 0$
- Since we do not need to solve Poisson equation (NLP), Penalty method is *indirect* method.

Topics and Known results

Penalized (NS)

$$\partial_t \boldsymbol{u}^\eta - \Delta \boldsymbol{u}^\eta + \boldsymbol{u}^\eta \cdot \nabla \boldsymbol{u}^\eta - \eta \nabla \operatorname{div} \boldsymbol{u}^\eta = 0. \qquad \text{(NS)}_\eta$$

(NS)_η: approximate problem of (NS).

Topics and Known results

Penalized (NS)

$$\partial_t \boldsymbol{u}^\eta - \Delta \boldsymbol{u}^\eta + \boldsymbol{u}^\eta \cdot \nabla \boldsymbol{u}^\eta - \eta \nabla \operatorname{div} \boldsymbol{u}^\eta = 0. \qquad \text{(NS)}_\eta$$

(NS)_η: approximate problem of (NS).

Question

$\eta \to \infty \Longrightarrow (\boldsymbol{u}^\eta, p^\eta) \to (\boldsymbol{u}, p)$?

Topics and Known results

Penalized (NS)

$$\partial_t \boldsymbol{u}^\eta - \Delta \boldsymbol{u}^\eta + \boldsymbol{u}^\eta \cdot \nabla \boldsymbol{u}^\eta - \eta \nabla \operatorname{div} \boldsymbol{u}^\eta = 0. \qquad \text{(NS)}_\eta$$

(NS)_η: approximate problem of (NS).

Question

$\eta \to \infty \Longrightarrow (\boldsymbol{u}^\eta, p^\eta) \to (\boldsymbol{u}, p)$?

To answer this question, it is worthwhile well to get *error* estimates between $(\boldsymbol{u}^\eta, p^\eta)$ and $(\boldsymbol{u}, p)$.

Topics and Known results

Penalized (NS)

$$\partial_t \boldsymbol{u}^\eta - \Delta \boldsymbol{u}^\eta + \boldsymbol{u}^\eta \cdot \nabla \boldsymbol{u}^\eta - \eta \nabla \operatorname{div} \boldsymbol{u}^\eta = 0. \qquad \text{(NS)}_\eta$$

(NS)_η: approximate problem of (NS).

Question

$\eta \to \infty \Longrightarrow (\boldsymbol{u}^\eta, p^\eta) \to (\boldsymbol{u}, p)$?

To answer this question, it is worthwhile well to get *error* estimates between $(\boldsymbol{u}^\eta, p^\eta)$ and $(\boldsymbol{u}, p)$.

Known results

- Temam (1968): error estimate for stationary Stokes and Navier-Stokes in bounded domain (L^2 theory)
- Shen (1995): error estimate for nonstationary Stokes and Navier-Stokes in bounded domain (L^2 theory)

Topics and Known results

Penalized (NS)

$$\partial_t \boldsymbol{u}^\eta - \Delta \boldsymbol{u}^\eta + \boldsymbol{u}^\eta \cdot \nabla \boldsymbol{u}^\eta - \eta \nabla \operatorname{div} \boldsymbol{u}^\eta = 0. \qquad \text{(NS)}_\eta$$

(NS)_η: approximate problem of (NS).

Question

$\eta \to \infty \Longrightarrow (\boldsymbol{u}^\eta, p^\eta) \to (\boldsymbol{u}, p)$?

To answer this question, it is worthwhile well to get *error* estimates between $(\boldsymbol{u}^\eta, p^\eta)$ and $(\boldsymbol{u}, p)$.

Known results

- Temam (1968): error estimate for stationary Stokes and Navier-Stokes in bounded domain (L^2 theory)
- Shen (1995): error estimate for nonstationary Stokes and Navier-Stokes in bounded domain (L^2 theory)
- Y. Saito (2010): error estimate for Stokes resolvent problem in $\mathbb{R}^d$ (L^r theory)

Main topic

Topic:

- η-dependence of solution to penalized system
- Error estimate between solutions to original problem and penalized problem.

Main topic

Topic:

- η-dependence of solution to penalized system
- Error estimate between solutions to original problem and penalized problem.

Contents

1. Estimate solution to penalized **Stokes** equation which is linearized problem of (NS)$_\eta$
2. Error estimate for the Stokes equation case
3. Error estimate for the mild solution of (NS)$_\eta$. In particular, we are going to show that

$$\lim_{\eta \to \infty} \|\boldsymbol{u}^\eta(t) - \boldsymbol{u}(t)\|_d \leq C(\boldsymbol{u}_0, \boldsymbol{u}_0^\eta).$$

Stokes equation

Let $d \geq 2$. We consider the Cauchy problems.

Stokes equation (ST)

$$\partial_t \boldsymbol{u} - \Delta \boldsymbol{u} + \nabla p = 0, \qquad x \in \mathbb{R}^d, t > 0, \tag{2a}$$

$$\operatorname{div} \boldsymbol{u} = 0, \qquad x \in \mathbb{R}^d, t > 0, \tag{2b}$$

$$\boldsymbol{u}(x, 0) = \boldsymbol{u}_0, \qquad x \in \mathbb{R}^d. \tag{2c}$$

Stokes equation

Let $d \geq 2$. We consider the Cauchy problems.

Stokes equation (ST)

$$\partial_t \boldsymbol{u} - \Delta \boldsymbol{u} + \nabla p = 0, \qquad x \in \mathbb{R}^d, t > 0, \tag{2a}$$

$$\operatorname{div} \boldsymbol{u} = 0, \qquad x \in \mathbb{R}^d, t > 0, \tag{2b}$$

$$\boldsymbol{u}(x, 0) = \boldsymbol{u}_0, \qquad x \in \mathbb{R}^d. \tag{2c}$$

Applying penalty method to (ST) we have

Penalized Stokes equation $(\mathrm{PST})_\eta$

$$\partial_t \boldsymbol{u}^\eta - \Delta \boldsymbol{u}^\eta - \eta \nabla \operatorname{div} \boldsymbol{u}^\eta = 0, \qquad x \in \mathbb{R}^d, t > 0, \tag{3a}$$

$$p^\eta = -\eta \operatorname{div} \boldsymbol{u}^\eta, \qquad x \in \mathbb{R}^d, t > 0, \tag{3b}$$

$$\boldsymbol{u}^\eta(x, 0) = \boldsymbol{u}_0^\eta, \qquad x \in \mathbb{R}^d. \tag{3c}$$

Helmholtz decomposition for $L^r(\mathbb{R}^d)$

To reformulate $(\mathrm{PST})_\eta$, we use the Helmholtz decomposition.

Helmholtz decomposition for $L^r(\mathbb{R}^d)$

To reformulate $(\mathrm{PST})_\eta$, we use the Helmholtz decomposition.

Helmholtz decomposition in $\mathbb{R}^d$

Let $1 < r < \infty \Longrightarrow L^r(\mathbb{R}^d) = L^r_\sigma(\mathbb{R}^d) \oplus G^r(\mathbb{R}^d)$, where

$$L^r_\sigma(\mathbb{R}^d) = \{\boldsymbol{u} \in L^r(\mathbb{R}^d) \mid \operatorname{div} \boldsymbol{u} = 0\},$$
$$G^r(\mathbb{R}^d) = \{\nabla\phi \mid \phi \in \hat{W}^{1,r}(\mathbb{R}^d)\},$$
$$\hat{W}^{1,r}(\mathbb{R}^d) = \{\phi \in L^1_{\mathrm{loc}}(\mathbb{R}^d) \mid \nabla\phi \in L^r(\mathbb{R}^d)\}.$$

- $P = P_r : L^r(\mathbb{R}^d) \to L^r_\sigma(\mathbb{R}^d)$: solenoidal projection
- $Q = Q_r := I - P_r$.

Reformulation of (PST)$_\eta$

Applying P and Q to (PST)$_\eta$ we have the following equations for $\boldsymbol{v}^\eta = P\boldsymbol{u}^\eta$ and $\boldsymbol{w}^\eta = Q\boldsymbol{u}^\eta$.

Reformulation of $(\mathrm{PST})_\eta$

Applying P and Q to $(\mathrm{PST})_\eta$ we have the following equations for $\boldsymbol{v}^\eta = P\boldsymbol{u}^\eta$ and $\boldsymbol{w}^\eta = Q\boldsymbol{u}^\eta$.

Eq. for solenoidal part

$$\begin{aligned}&\partial_t \boldsymbol{v}^\eta - \Delta \boldsymbol{v}^\eta = 0, \quad \operatorname{div} \boldsymbol{v}^\eta = 0, && x \in \mathbb{R}^d, t > 0,\\ &\boldsymbol{v}^\eta|_{t=0} = \boldsymbol{v}_0^\eta =: P\boldsymbol{u}^\eta\end{aligned}$$

Reformulation of $(\mathrm{PST})_\eta$

Applying P and Q to $(\mathrm{PST})_\eta$ we have the following equations for $\boldsymbol{v}^\eta = P\boldsymbol{u}^\eta$ and $\boldsymbol{w}^\eta = Q\boldsymbol{u}^\eta$.

Eq. for solenoidal part

$$\begin{aligned} &\partial_t \boldsymbol{v}^\eta - \Delta \boldsymbol{v}^\eta = 0, \quad \operatorname{div} \boldsymbol{v}^\eta = 0, \qquad x \in \mathbb{R}^d, t > 0, \\ &\boldsymbol{v}^\eta|_{t=0} = \boldsymbol{v}_0^\eta =: P\boldsymbol{u}^\eta \end{aligned}$$

and

Eq. for scalar potential part

$$\begin{aligned} &\partial_t \boldsymbol{w}^\eta - (1+\eta)\Delta \boldsymbol{w}^\eta = 0, \quad \boldsymbol{w}^\eta = \nabla \varphi^\eta \qquad x \in \mathbb{R}^d, t > 0, \\ &\boldsymbol{w}^\eta|_{t=0} = \boldsymbol{w}_0^\eta =: Q\boldsymbol{u}_0^\eta \end{aligned}$$

Note:

$$-\Delta \boldsymbol{w} - \eta \nabla \operatorname{div} \boldsymbol{w} = -\Delta \nabla \varphi - \eta \nabla \operatorname{div} \nabla \varphi = -(1+\eta)\Delta \nabla \varphi = -(1+\eta)\boldsymbol{w}.$$

Linear heat equation

Linear heat eq.

$$\partial_t z - \nu \Delta z = 0, \quad x \in \mathbb{R}^d, t > 0,$$
$$z(x,0) = z_0(x), \quad x \in \mathbb{R}^d.$$

$\nu > 0$: heat diffusivity.

Linear heat equation

Linear heat eq.

$$\partial_t z - \nu \Delta z = 0, \quad x \in \mathbb{R}^d, t > 0,$$
$$z(x,0) = z_0(x), \quad x \in \mathbb{R}^d.$$

$\nu > 0$: heat diffusivity.

$$z(x,t;\nu) = e^{\nu t \Delta} z_0 := \frac{1}{4\pi \nu t} \int_{\mathbb{R}^d} \exp\left(\frac{|x-\xi|^2}{4\nu t} \right) z_0(\xi)\, d\xi$$

Linear heat equation

Linear heat eq.

$$\partial_t z - \nu \Delta z = 0, \quad x \in \mathbb{R}^d, t > 0,$$
$$z(x,0) = z_0(x), \quad x \in \mathbb{R}^d.$$

$\nu > 0$: heat diffusivity.

$$z(x,t;\nu) = e^{\nu t \Delta} z_0 := \frac{1}{4\pi \nu t} \int_{\mathbb{R}^d} \exp\left(\frac{|x-\xi|^2}{4\nu t} \right) z_0(\xi)\, d\xi$$

Lemma (L^r-L^q estimate)

Let $\nu > 0$, $1 \le r \le q \le \infty$, $j \in \mathbb{N}_0$, $\alpha \in \mathbb{N}_0^d$. Then the following estimate holds for any $t > 0$.

$$\|\partial_t^j \partial_x^\alpha z(\cdot, t; \nu)\|_q \le C_{q,r,\alpha,j}\, \nu^{-\frac{d}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{|\alpha|}{2}}\, t^{-\frac{d}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{|\alpha|}{2}-j} \|z_0\|_r$$

Estimates for $v^\eta(t)$ and $w^\eta(t)$

Let $\boldsymbol{u}_0^\eta \in L^r(\mathbb{R}^d)$ $(1 < r < \infty)$ and set $\boldsymbol{v}_0^\eta := P\boldsymbol{u}_0^\eta \in L^r_\sigma$ and $\boldsymbol{w}_0^\eta := Q\boldsymbol{u}_0^\eta \in G^r \Longrightarrow$

- $\boldsymbol{v}^\eta(t) = e^{t\Delta}\boldsymbol{v}_0^\eta$
- $\boldsymbol{w}^\eta(t) = e^{(1+\eta)t\Delta}\boldsymbol{w}_0^\eta$

Estimates for $v^\eta(t)$ and $w^\eta(t)$

Let $\boldsymbol{u}_0^\eta \in L^r(\mathbb{R}^d)$ ($1 < r < \infty$) and set $\boldsymbol{v}_0^\eta := P\boldsymbol{u}_0^\eta \in L^r_\sigma$ and $\boldsymbol{w}_0^\eta := Q\boldsymbol{u}_0^\eta \in G^r \Longrightarrow$

- $\boldsymbol{v}^\eta(t) = e^{t\Delta}\boldsymbol{v}_0^\eta$
- $\boldsymbol{w}^\eta(t) = e^{(1+\eta)t\Delta}\boldsymbol{w}_0^\eta$

As a consequence of Lemma (L^r-L^q estimate), we have

$$\|\partial_t^j \partial_x^\alpha \boldsymbol{v}^\eta(t)\|_q \le C_{q,r} t^{-\frac{d}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{|\alpha|}{2}-j}\|\boldsymbol{v}_0^\eta\|_r,$$

$$\|\partial_t^j \partial_x^\alpha \boldsymbol{w}^\eta(t)\|_q \le C_{q,r}(1+\eta)^{-\frac{d}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{|\alpha|}{2}} t^{-\frac{d}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{|\alpha|}{2}-j}\|\boldsymbol{v}_0^\eta\|_r$$

for $1 < r \le q \le \infty$ $(r \neq \infty)$, $t > 0$, $j \in \mathbb{N}_0, \alpha \in \mathbb{N}_0^d$.

Estimates for $v^\eta(t)$ and $w^\eta(t)$

Let $\boldsymbol{u}_0^\eta \in L^r(\mathbb{R}^d)$ $(1 < r < \infty)$ and set $\boldsymbol{v}_0^\eta := P\boldsymbol{u}_0^\eta \in L^r_\sigma$ and $\boldsymbol{w}_0^\eta := Q\boldsymbol{u}_0^\eta \in G^r \Longrightarrow$

- $\boldsymbol{v}^\eta(t) = e^{t\Delta}\boldsymbol{v}_0^\eta$
- $\boldsymbol{w}^\eta(t) = e^{(1+\eta)t\Delta}\boldsymbol{w}_0^\eta$

As a consequence of Lemma (L^r-L^q estimate), we have

$$\|\partial_t^j \partial_x^\alpha \boldsymbol{v}^\eta(t)\|_q \le C_{q,r} t^{-\frac{d}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{|\alpha|}{2}-j} \|\boldsymbol{v}_0^\eta\|_r,$$

$$\|\partial_t^j \partial_x^\alpha \boldsymbol{w}^\eta(t)\|_q \le C_{q,r} (1+\eta)^{-\frac{d}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{|\alpha|}{2}} t^{-\frac{d}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{|\alpha|}{2}-j} \|\boldsymbol{v}_0^\eta\|_r$$

for $1 < r \le q \le \infty$ $(r \ne \infty)$, $t > 0$, $j \in \mathbb{N}_0, \alpha \in \mathbb{N}_0^d$.
In particular $q = r$, $j = 0, \alpha = (0, \ldots, 0) \Longrightarrow$

$$\|\boldsymbol{w}^\eta(t)\|_r \le C_{q,r} \|\boldsymbol{w}_0^\eta\|_r.$$

Estimates for $v^\eta(t)$ and $w^\eta(t)$

Let $\boldsymbol{u}_0^\eta \in L^r(\mathbb{R}^d)$ $(1 < r < \infty)$ and set $\boldsymbol{v}_0^\eta := P\boldsymbol{u}_0^\eta \in L^r_\sigma$ and $\boldsymbol{w}_0^\eta := Q\boldsymbol{u}_0^\eta \in G^r \Longrightarrow$

- $\boldsymbol{v}^\eta(t) = e^{t\Delta}\boldsymbol{v}_0^\eta$
- $\boldsymbol{w}^\eta(t) = e^{(1+\eta)t\Delta}\boldsymbol{w}_0^\eta$

As a consequence of Lemma (L^r-L^q estimate), we have

$$\|\partial_t^j\partial_x^\alpha \boldsymbol{v}^\eta(t)\|_q \le C_{q,r}t^{-\frac{d}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{|\alpha|}{2}-j}\|\boldsymbol{v}_0^\eta\|_r,$$

$$\|\partial_t^j\partial_x^\alpha \boldsymbol{w}^\eta(t)\|_q \le C_{q,r}(1+\eta)^{-\frac{d}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{|\alpha|}{2}}t^{-\frac{d}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{|\alpha|}{2}-j}\|\boldsymbol{v}_0^\eta\|_r$$

for $1 < r \le q \le \infty$ $(r \ne \infty)$, $t > 0$, $j \in \mathbb{N}_0, \alpha \in \mathbb{N}_0^d$.
In particular $q = r$, $j = 0, \alpha = (0, \dots, 0) \Longrightarrow$

$$\|\boldsymbol{w}^\eta(t)\|_r \le C_{q,r}\|\boldsymbol{w}_0^\eta\|_r.$$

Remark

For any $\eta > 0$, $\|\boldsymbol{w}^\eta(t)\|_r$ is bounded, provided that $\boldsymbol{u}_0^\eta \in L^r(\mathbb{R}^d)$.

Estimate for $\boldsymbol{w}^\eta(t)$

For $\boldsymbol{w}_0^\eta = Q_r \boldsymbol{u}_0^\eta \in G^r(\mathbb{R}^d)$, put $\boldsymbol{w}_0^\eta = \nabla \varphi_0^\eta, \varphi_0^\eta \in \hat{W}^{1,r}(\mathbb{R}^d)$.

Estimate for $w^\eta(t)$

For $\boldsymbol{w}_0^\eta = Q_r \boldsymbol{u}_0^\eta \in G^r(\mathbb{R}^d)$, put $\boldsymbol{w}_0^\eta = \nabla \varphi_0^\eta, \varphi_0^\eta \in \hat{W}^{1,r}(\mathbb{R}^d)$.

Lemma

$C_0^\infty(\mathbb{R}^d) \subset \hat{W}^{1,r}(\mathbb{R}^d)$: dense w.r.t $\|\nabla \cdot\|_r$.

Estimate for $w^\eta(t)$

For $\boldsymbol{w}_0^\eta = Q_r \boldsymbol{u}_0^\eta \in G^r(\mathbb{R}^d)$, put $\boldsymbol{w}_0^\eta = \nabla \varphi_0^\eta, \varphi_0^\eta \in \hat{W}^{1,r}(\mathbb{R}^d)$.

Lemma

$C_0^\infty(\mathbb{R}^d) \subset \hat{W}^{1,r}(\mathbb{R}^d)$: dense w.r.t $\|\nabla \cdot \|_r$.

For any $\varepsilon > 0$, there exists $\varphi_{0,\varepsilon} \in C_0^\infty(\mathbb{R}^d)$ such that

$$\|\boldsymbol{w}_0^\eta - \nabla \varphi_{0,\varepsilon}\|_r = \|\nabla \varphi_0 - \nabla \varphi_{0,\varepsilon}\|_r < \varepsilon.$$

Estimate for $w^\eta(t)$

For $\boldsymbol{w}_0^\eta = Q_r \boldsymbol{u}_0^\eta \in G^r(\mathbb{R}^d)$, put $\boldsymbol{w}_0^\eta = \nabla \varphi_0^\eta, \varphi_0^\eta \in \hat{W}^{1,r}(\mathbb{R}^d)$.

Lemma

$C_0^\infty(\mathbb{R}^d) \subset \hat{W}^{1,r}(\mathbb{R}^d)$: dense w.r.t $\|\nabla \cdot \|_r$.

For any $\varepsilon > 0$, there exists $\varphi_{0,\varepsilon} \in C_0^\infty(\mathbb{R}^d)$ such that

$$\|\boldsymbol{w}_0^\eta - \nabla \varphi_{0,\varepsilon}\|_r = \|\nabla \varphi_0 - \nabla \varphi_{0,\varepsilon}\|_r < \varepsilon.$$

By (13) and L^r-L^q estimate, we have

$$\begin{aligned}
\|\boldsymbol{w}^\eta(t)\|_r &= \|e^{t(1+\eta)\Delta}(\boldsymbol{w}_0^\eta - \nabla \varphi_{0,\varepsilon}^\eta)\|_r + \|e^{(1+\eta)\Delta} \nabla \varphi_{0,\varepsilon}^\eta\|_r \\
&\leq C_r \|\boldsymbol{w}_0^\eta - \nabla \varphi_{0,\varepsilon}^\eta\|_r + \|\nabla e^{(1+\eta)\Delta} \varphi_{0,\varepsilon}^\eta\|_r \\
&\leq C_r \varepsilon + C_r (1+\eta)^{-\frac{d}{2}(\frac{1}{s}-\frac{1}{r})-\frac{1}{2}} t^{-\frac{d}{2}(\frac{1}{s}-\frac{1}{r})-\frac{1}{2}} \|\varphi_{0,\varepsilon}^\eta\|_s, \ ({}^\exists s \in [1, r]) \\
&\leq C_r \varepsilon + C_{r,t_0,d} (1+\eta)^{-\frac{d}{2}(\frac{1}{s}-\frac{1}{r})-\frac{1}{2}}, \quad t \geq t_0 > 0.
\end{aligned}$$

Estimate for $w^\eta(t)$

By density argument, we obtained

$$\|w^\eta(t)\|_r \leq C_r \varepsilon + C_{r,t_0,d} (1+\eta)^{-\frac{1}{2}}, \quad t \geq t_0 > 0 \quad (s = r, \text{for simplicity}).$$

For each $t \geq t_0 > 0$, we have

$$\lim_{\eta \to \infty} \|w^\eta(t)\|_r = 0.$$

Estimate for $w^\eta(t)$

By density argument, we obtained

$$\|\boldsymbol{w}^\eta(t)\|_r \le C_r\varepsilon + C_{r,t_0,d}(1+\eta)^{-\frac{1}{2}}, \quad t \ge t_0 > 0 \quad (s = r, \text{for simplicity}).$$

For each $t \ge t_0 > 0$, we have

$$\lim_{\eta\to\infty} \|\boldsymbol{w}^\eta(t)\|_r = 0.$$

- Since $p^\eta(t) = -\eta\,\mathrm{div}\,\boldsymbol{u}^\eta(t) = -\eta\,\mathrm{div}\,\boldsymbol{w}^\eta(t)$ (because $\mathrm{div}\,\boldsymbol{v}^\eta(t) = 0$), it suffices to estimate $\eta\nabla^2\boldsymbol{w}^\eta(t)$.
- To get estimate for the pressure p^η, the above estimate plays an essential role.

Error estimates

Let $1 < r < \infty$ and

- $(\boldsymbol{u}(t), p(t))$: solution to Stokes equation with initial data $\boldsymbol{u_0} \in L^r_\sigma(\mathbb{R}^d)$ (compatibility condition)
- $\boldsymbol{u}^\eta(t)$: solution to penalized Stokes equation with initial data $\boldsymbol{u}_{\boldsymbol{0}}^\eta \in L^r(\mathbb{R}^d)$

Set

$$\boldsymbol{U}^\eta(t) := \boldsymbol{u}^\eta(t) - \boldsymbol{u}(t), \quad \Pi^\eta(t) := p^\eta(t) - p(t).$$

$(\boldsymbol{U}, P)$ satisfies

$$\begin{aligned}
&\partial_t \boldsymbol{U}^\eta - \Delta \boldsymbol{U}^\eta + \nabla \Pi = 0, && x \in \mathbb{R}^d, t > 0,\\
&\operatorname{div} \boldsymbol{U}^\eta = -p^\eta/\eta, && x \in \mathbb{R}^d, t > 0,\\
&\boldsymbol{U}^\eta|_{t=0} = \boldsymbol{U} =: \boldsymbol{u}_{\boldsymbol{0}}^\eta - \boldsymbol{u}_{\boldsymbol{0}}, && x \in \mathbb{R}^d.
\end{aligned}$$

Error estimates

By Helmholtz projection P_r and $Q_r := I - P_r$, equation for $\boldsymbol{U}^\eta, \Pi^\eta$ is decomposed into

$$\partial_t(\boldsymbol{v}^\eta - \boldsymbol{u}) - \Delta(\boldsymbol{v}^\eta - \boldsymbol{u}) = 0, \quad \operatorname{div}(\boldsymbol{v}^\eta - \boldsymbol{u}) = 0, \qquad x \in \mathbb{R}^d, t > 0,$$
$$(\boldsymbol{v}^\eta - \boldsymbol{u})|_{t=0} = (v_{\boldsymbol{0}}^\eta - \boldsymbol{u}_0) \in L_\sigma^r(\mathbb{R}^d).$$

and

$$\partial_t \boldsymbol{w}^\eta - (1+\eta)\Delta \boldsymbol{w}^\eta = 0, \qquad\qquad x \in \mathbb{R}^d, t > 0,$$
$$\boldsymbol{w}^\eta|_t = 0 = \boldsymbol{w}_{\boldsymbol{0}}^\eta \in G^r(\mathbb{R}^d).$$

Here we have used the fact that $\nabla p = 0$ in G^r.

Since

- $\|U^{\eta}(t)\| \leq \|v^{\eta}(t) - u(t)\| + \|w^{\eta}(t)\|$
- $\|\nabla \Pi^{\eta}(t)\| = \|\nabla p^{\eta}(t)\|$

we have by previous estimate,

Theorem 1 (Error estimate).

(i) Let $1 < r \leq q \leq \infty$ $(r \neq \infty)$. Then for any $\varepsilon > 0$, ${}^{\exists}\varphi_{\mathbf{0},\varepsilon} \in C_{\mathbf{0}}^{\infty}(\mathbb{R}^d)$ such that the following estimate holds for any $\eta > 0, t > 0$.

$$\begin{aligned}\|\partial_t^j \partial_x^\alpha U^{\eta}(t)\|_q \leq\ & C_{q,r} t^{-\frac{d}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{|\alpha|}{2}-j} \|v_{\mathbf{0}}^{\eta} - u_{\mathbf{0}}\|_r \\ & + C_{q,r}\varepsilon t^{-\frac{d}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{|\alpha|}{2}-j} (1+\eta)^{-\frac{d}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{|\alpha|}{2}} \\ & + C_{q,r} t^{-\frac{d}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{|\alpha|+1}{2}-j} (1+\eta)^{-\frac{d}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{|\alpha|+1}{2}} \|\varphi_{\mathbf{0},\varepsilon}^{\eta}\|_r\end{aligned}$$

Since

- $\|U^\eta(t)\| \le \|v^\eta(t) - u(t)\| + \|w^\eta(t)\|$
- $\|\nabla \Pi^\eta(t)\| = \|\nabla p^\eta(t)\|$

we have by previous estimate,

Theorem 1 (Error estimate).

(i) Let $1 < r \le q \le \infty$ $(r \ne \infty)$. Then for any $\varepsilon > 0$, $^\exists \varphi_{0,\varepsilon} \in C_0^\infty(\mathbb{R}^d)$ such that the following estimate holds for any $\eta > 0, t > 0$.

$$\begin{aligned} \|\partial_t^j \partial_x^\alpha U^\eta(t)\|_q \le\ & C_{q,r} t^{-\frac{d}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{|\alpha|}{2}-j} \|v_0^\eta - u_0\|_r \\ & + C_{q,r} \varepsilon t^{-\frac{d}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{|\alpha|}{2}-j} (1+\eta)^{-\frac{d}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{|\alpha|}{2}} \\ & + C_{q,r} t^{-\frac{d}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{|\alpha|+1}{2}-j} (1+\eta)^{-\frac{d}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{|\alpha|+1}{2}} \|\varphi_{0,\varepsilon}^\eta\|_r \end{aligned}$$

(ii) In particular $q = r, j = 0, \alpha = (0, \ldots, 0)$,

$$\limsup_{\eta \to \infty} \|U^\eta(t)\|_r \le C_r \|v_0^\eta - u_0\|_r, \quad t \ge t_0 > 0,$$

$$\limsup_{\eta \to \infty} \|\Pi^\eta(t)\|_r = 0, \quad t \ge t_0 > 0$$

Remarks on Theorem 1

- If $\|\boldsymbol{v}_0^\eta - \boldsymbol{u}_0\|_r \ll 1, \eta \gg 1 \Longrightarrow \|\boldsymbol{U}^\eta(t)\|_r \ll 1$. In particular if $\boldsymbol{u}_0 = \boldsymbol{v}_0^\eta$, error is managed by only $\boldsymbol{w}^\eta(t)$.
- If $\boldsymbol{u}_0^\eta = \boldsymbol{u}_0 \in L^r_\sigma \Longrightarrow \boldsymbol{w}_0^\eta = 0$. Hence, there is **no** error.
- We have used the fact that P_r and ∂_{x_j} commute each other.
  Our argument does not work in $\Omega \not\equiv \mathbb{R}^d$.
- Our argument deeply depends on explicit formula of $e^{\nu \Delta t}$.

Navier-Stokes equation

Penalized Navier-Stokes equation (NS)$_\eta$

$$\partial_t \boldsymbol{u}^\eta - \Delta \boldsymbol{u}^\eta - \eta \nabla \operatorname{div} \boldsymbol{u}^\eta + \boldsymbol{u}^\eta \cdot \nabla \boldsymbol{u}^\eta = 0, \qquad x \in \mathbb{R}^d, t > 0, \tag{5a}$$

$$p^\eta = -\eta \operatorname{div} \boldsymbol{u}^\eta, \qquad x \in \mathbb{R}^d, t > 0, \tag{5b}$$

$$\boldsymbol{u}^\eta(x,0) = \boldsymbol{u}_0^\eta, \qquad x \in \mathbb{R}^d. \tag{5c}$$

Navier-Stokes equation

Penalized Navier-Stokes equation (NS)$_\eta$

$$\partial_t \boldsymbol{u}^\eta - \Delta \boldsymbol{u}^\eta - \eta \nabla \operatorname{div} \boldsymbol{u}^\eta + \boldsymbol{u}^\eta \cdot \nabla \boldsymbol{u}^\eta = 0, \qquad x \in \mathbb{R}^d, t > 0, \tag{5a}$$

$$p^\eta = -\eta \operatorname{div} \boldsymbol{u}^\eta, \qquad x \in \mathbb{R}^d, t > 0, \tag{5b}$$

$$\boldsymbol{u}^\eta(x, 0) = \boldsymbol{u}_0^\eta, \qquad x \in \mathbb{R}^d. \tag{5c}$$

Let $L_{r,\eta}\boldsymbol{u} := -\Delta \boldsymbol{u} - \eta \nabla \operatorname{div} \boldsymbol{u}$ $(1 < r < \infty)$. Then $-L_{r,\eta}$ generates an analytic semigroup $(e^{-tL_{r,\eta}})_{t \geq 0}$ on $L^r(\mathbb{R}^d)$ and the semigroup satisfies standard L^r-L^q type estimates. Therefore

Proposition

$\boldsymbol{u}_0^\eta \in L^d(\mathbb{R}^d) \Longrightarrow \exists T > 0$ such that $\boldsymbol{u}^\eta(t) \in C([0,T); L^d(\mathbb{R}^d))$: mild sol. to (NS)$_\eta$ uniquely exists.
In particular $\|\boldsymbol{u}_0^\eta\|_d \ll 1 \Longrightarrow$ mild solution exists globally in time.

$\star$ $L^d(\mathbb{R}^d)$ is scale invariant space of (NS) and (NS)$_\eta$.

Reformulation

Put $\boldsymbol{u}^\eta = \boldsymbol{v}^\eta + \boldsymbol{w}^\eta$, $\operatorname{div} \boldsymbol{v}^\eta = 0$, $\boldsymbol{w}^\eta = \nabla \varphi^\eta$.

Abstract form of $(\mathrm{NS})_\eta$: $(\mathrm{ABS})_\eta$

$$
\begin{aligned}
&\partial_t \boldsymbol{v}^\eta - \Delta \boldsymbol{v}^\eta + P(\boldsymbol{u} \cdot \nabla \boldsymbol{u}) = 0, && x \in \mathbb{R}^d, t > 0,\\
&\partial_t \boldsymbol{w}^\eta - (1+\eta)\Delta \boldsymbol{w} + Q(\boldsymbol{u} \cdot \nabla \boldsymbol{u}) = 0, && x \in \mathbb{R}^d, t > 0,\\
&\boldsymbol{v}^\eta(x,0) = \boldsymbol{v}_0^\eta =: P\boldsymbol{u}_0^\eta, \quad \boldsymbol{w}^\eta(x,0) = \boldsymbol{w}_0^\eta =: Q\boldsymbol{u}_0^\eta.
\end{aligned}
$$

Reformulation

Put $\boldsymbol{u}^\eta = \boldsymbol{v}^\eta + \boldsymbol{w}^\eta$, $\operatorname{div} \boldsymbol{v}^\eta = 0$, $\boldsymbol{w}^\eta = \nabla\varphi^\eta$.

Abstract form of (NS)$_\eta$: (ABS)$_\eta$

$$\begin{aligned} &\partial_t \boldsymbol{v}^\eta - \Delta \boldsymbol{v}^\eta + P(\boldsymbol{u}\cdot\nabla\boldsymbol{u}) = 0, && x\in\mathbb{R}^d, t>0,\\ &\partial_t \boldsymbol{w}^\eta - (1+\eta)\Delta \boldsymbol{w} + Q(\boldsymbol{u}\cdot\nabla\boldsymbol{u}) = 0, && x\in\mathbb{R}^d, t>0,\\ &\boldsymbol{v}^\eta(x,0) = \boldsymbol{v}_0^\eta =: P\boldsymbol{u}_0^\eta, \quad \boldsymbol{w}^\eta(x,0) = \boldsymbol{w}_0^\eta =: Q\boldsymbol{u}_0^\eta. \end{aligned}$$

Remark

In the Navier-Stokes equation, there are nonlinear interactions between $\boldsymbol{v}^\eta$ and $\boldsymbol{w}^\eta$.

Mild formulation

By Duhamel's principle (ABS)$_\eta$ is converted into integral equations.

Integral equations (INT)$_\eta$

$$\boldsymbol{v}(t) = e^{t\Delta}\boldsymbol{v}_0 - \int_0^t e^{(t-s)\Delta} P(\boldsymbol{u}(s)\cdot\nabla\boldsymbol{u}(s))\,ds =: \boldsymbol{v}^0(t) + N_1(\boldsymbol{u})(t),$$

$$\begin{aligned}\boldsymbol{w}(t) &= e^{(1+\eta)t\Delta}\boldsymbol{w}_0 - \int_0^t e^{(1+\eta)t\Delta} Q(\boldsymbol{u}(s)\cdot\nabla\boldsymbol{u}(s))\,ds \\ &= \boldsymbol{w}^0(t) + N_2(\boldsymbol{u})(t).\end{aligned}$$

Define mapping Φ by

$$\Phi(\boldsymbol{v},\boldsymbol{w}) = \begin{pmatrix}\boldsymbol{v}^0(t)\\ \boldsymbol{w}^0(t)\end{pmatrix} + \begin{pmatrix}N_1(\boldsymbol{u})(t)\\ N_2(\boldsymbol{u})(t)\end{pmatrix}.$$

Task

Show Φ has a fixed point, provided that $\|(\boldsymbol{v}_0,\boldsymbol{w}_0)\|_d \le {}^\exists\delta$.

Small data global existence

Theorem 4 (Small data global existence)

Let $(\boldsymbol{v}_0, \boldsymbol{w}_0) \in L^d_\sigma(\mathbb{R}^d) \times G^r(\mathbb{R}^d)$. Then $\exists \delta > 0$ s.t. if $\|(\boldsymbol{v}_0, \boldsymbol{w}_0)\| < \delta \Longrightarrow \exists 1\ (\boldsymbol{v}(t), \boldsymbol{w}(t)) \in C([0,\infty); L^d_\sigma(\mathbb{R}^d) \times G^d(\mathbb{R}^d))$ which enjoys

$$\lim_{t \to +0} \|(\boldsymbol{v}(t), \boldsymbol{w}(t)) - (\boldsymbol{v}_0, \boldsymbol{w}_0)\|_d = 0,$$

$$\|(\boldsymbol{v}(t), \boldsymbol{w}(t))\|_r = O\big(t^{-\frac{1}{2} + \frac{d}{2r}}\big), \quad d \le r < \infty,$$

$$\|\nabla(\boldsymbol{v}(t), \boldsymbol{w}(t))\|_d = O\big(t^{-\frac{1}{2}}\big)$$

as $t \to \infty$ for any fixed $\eta > 0$.

Small data global existence

Theorem 4 (Small data global existence)

Let $(\boldsymbol{v}_0, \boldsymbol{w}_0) \in L^d_\sigma(\mathbb{R}^d) \times G^r(\mathbb{R}^d)$. Then $\exists \delta > 0$ s.t. if $\|(\boldsymbol{v}_0, \boldsymbol{w}_0)\| < \delta \Longrightarrow$ $\exists 1\ (\boldsymbol{v}(t), \boldsymbol{w}(t)) \in C([0,\infty); L^d_\sigma(\mathbb{R}^d) \times G^d(\mathbb{R}^d))$ which enjoys

$$\lim_{t \to +0} \|(\boldsymbol{v}(t), \boldsymbol{w}(t)) - (\boldsymbol{v}_0, \boldsymbol{w}_0)\|_d = 0,$$

$$\|(\boldsymbol{v}(t), \boldsymbol{w}(t))\|_r = O\big(t^{-\frac{1}{2}+\frac{d}{2r}}\big), \quad d \le r < \infty,$$

$$\|\nabla(\boldsymbol{v}(t), \boldsymbol{w}(t))\|_d = O\big(t^{-\frac{1}{2}}\big)$$

as $t \to \infty$ for any fixed $\eta > 0$.

Furthermore, the above mild solution satisfies

$$\|\boldsymbol{w}(t)\|_r = O(\eta^{-\frac{1}{2}+\frac{d}{2r}}), \quad d \le r < \infty$$

as $\eta \to +\infty$ for fixed $t \ge t_0 > 0$.

Kato's argument to (INS)$_\eta$

As an underlying space, set

$$
\begin{aligned}
X_R := \{(\boldsymbol{v}(t), \boldsymbol{w}(t)) \in C([0,\infty); L^d_\sigma(\mathbb{R}^d) \times G^r(\mathbb{R}^d)) \mid \\
\lim_{t\to+0} \|\boldsymbol{v}(t) - \boldsymbol{v}_0\|_d = 0, \quad \lim_{t\to+0} \|\boldsymbol{w}(t) - \boldsymbol{w}_0\|_d = 0, \\
\lim_{t\to+0} |\boldsymbol{u}|_{\frac{1}{2}-\frac{d}{2r},r,t} = 0, \quad \lim_{t\to+0} |\nabla \boldsymbol{u}|_{\frac{1}{2},d,t} = 0, \\
\sup_{t>0,\eta>0} |||\Phi(\boldsymbol{v},\boldsymbol{w})(t)||| \le 2R\|(\boldsymbol{v}_0,\boldsymbol{w}_0)\|_d\}
\end{aligned}
$$

where $r \in (d,\infty)$ and constant $R > 0$ will be determined later.

- $|\boldsymbol{u}|_{\ell,q,t} := \sup_{0<s\le t} s^\ell(\|\boldsymbol{v}(s)\|_q + \sup_\eta (1+\eta)^\ell \|\boldsymbol{w}(s)\|_q)$
- $[\boldsymbol{u}]_t := |\boldsymbol{u}|_{1/2-d/2r,r,t} + |\nabla \boldsymbol{u}|_{1/2,d,t}$
- $|||\boldsymbol{u}|||_t := |\boldsymbol{u}|_{0,d,t} + [\boldsymbol{u}]_t$

Estimates for Duhamel terms

Let $r > d$ and $1/q = 1/r + 1/d$. Then

$$\begin{aligned}
\|N_1(\boldsymbol{u})(t)\|_r &\le \int_0^t \|e^{(t-s)\Delta} P(\boldsymbol{u}(s)\cdot\nabla\boldsymbol{u}(s))\|_r\,ds \\
&\le C_{r,d}\int_0^t (t-s)^{-1/2}\|\boldsymbol{u}(s)\|_r\|\nabla\boldsymbol{u}(s)\|_d\,ds \\
&\le C_{r,d}\int_0^t (t-s)^{-1/2}s^{-1+d/2r}\,ds\,[\boldsymbol{u}]_t^2 \\
&\le C_{r,d}t^{-1/2+d/2r}B(1/2,d/2r)[\boldsymbol{u}]_t^2
\end{aligned}$$

By similar manners,

$$\begin{aligned}
\|N_2(\boldsymbol{u})(t)\|_d &\le C_{r,d}[\boldsymbol{u}]_t^2, \\
\|\nabla N_2(\boldsymbol{u})(t)\|_d &\le C_{r,d}t^{-1/2}[\boldsymbol{u}]_t^2
\end{aligned}$$

Let $r > d$ and $1/q = 1/r + 1/d$. Then

$$\begin{aligned}
\|N_2(\boldsymbol{u})(t)\|_r &\le \int_0^t \|e^{(1+\eta)(t-s)\Delta} P(\boldsymbol{u}(s)\cdot\nabla\boldsymbol{u}(s))\|_r \, ds \\
&\le C_{r,d}(1+\eta)^{-1/2} \int_0^t (t-s)^{-1/2} \|\boldsymbol{u}(s)\|_r \|\nabla \boldsymbol{u}(s)\|_d \, ds \\
&\le C_{r,d}(1+\eta)^{-1/2} \int_0^t (t-s)^{-1/2} s^{-1+d/2r} \, ds \, [\boldsymbol{u}]_t^2 \\
&\le C_{r,d}(1+\eta)^{-1/2} t^{-1/2+d/2r} B(1/2, d/2r)[\boldsymbol{u}]_t^2 \\
&\le C_{r,d}(1+\eta)^{-1/2+r/2d} [\boldsymbol{u}]_t^2
\end{aligned}$$

By similar manners,

$$\begin{aligned}
\|N_1(\boldsymbol{u})(t)\|_d &\le C_{r,d}(1+\eta)^{-d/2r}[\boldsymbol{u}]_t^2 \le C_{r,d}[\boldsymbol{u}]_t^2, \\
\|\nabla N_1(\boldsymbol{u})(t)\|_d &\le C_{r,d}(1+\eta)^{-1/2-d/2r} t^{-1/2}[\boldsymbol{u}]_t^2 \le C_{r,d}(1+\eta)^{-1/2}[\boldsymbol{u}]_t^2
\end{aligned}$$

Proof of Theorem 4

If $(\boldsymbol{v}, \boldsymbol{w}) \in X_R \Longrightarrow$

$$\begin{aligned} |||\Phi(\boldsymbol{v}, \boldsymbol{w})|||_t &\leq R\|(\boldsymbol{v}_0, \boldsymbol{w}_0)\|_d + C[\boldsymbol{u}]_t^2 \\ &\leq R\|(\boldsymbol{v}_0, \boldsymbol{w}_0)\|_d + 4CR^2\|(\boldsymbol{v}_0, \boldsymbol{w}_0)\|_d^2 \end{aligned}$$

for any $t > 0, \eta > 0$.

Proof of Theorem 4

If $(\boldsymbol{v}, \boldsymbol{w}) \in X_R \Longrightarrow$

$$\begin{aligned} |||\Phi(\boldsymbol{v}, \boldsymbol{w})|||_t &\leq R\|(\boldsymbol{v}_0, \boldsymbol{w}_0)\|_d + C[\boldsymbol{u}]_t^2 \\ &\leq R\|(\boldsymbol{v}_0, \boldsymbol{w}_0)\|_d + 4CR^2\|(\boldsymbol{v}_0, \boldsymbol{w}_0)\|_d^2 \end{aligned}$$

for any $t > 0, \eta > 0$. Choose $\delta > 0$ in such a way that $4CR\delta < 1$,

$$|||\Phi(\boldsymbol{v}, \boldsymbol{w})|||_t \leq 2R\|(\boldsymbol{v}_0, \boldsymbol{w}_0)\|_d$$

for any $t > 0, \eta > 0$.

Proof of Theorem 4

If $(\boldsymbol{v}, \boldsymbol{w}) \in X_R \Longrightarrow$

$$
\begin{aligned}
|||\Phi(\boldsymbol{v}, \boldsymbol{w})|||_t &\leq R\|(\boldsymbol{v}_0, \boldsymbol{w}_0)\|_d + C[\boldsymbol{u}]_t^2 \\
&\leq R\|(\boldsymbol{v}_0, \boldsymbol{w}_0)\|_d + 4CR^2\|(\boldsymbol{v}_0, \boldsymbol{w}_0)\|_d^2
\end{aligned}
$$

for any $t > 0, \eta > 0$. Choose $\delta > 0$ in such a way that $4CR\delta < 1$,

$$
|||\Phi(\boldsymbol{v}, \boldsymbol{w})|||_t \leq 2R\|(\boldsymbol{v}_0, \boldsymbol{w}_0)\|_d
$$

for any $t > 0, \eta > 0$.
Summing up the above, we have

Lemma

$\Phi(\boldsymbol{v}, \boldsymbol{w}) \in X_R$, provided $(\boldsymbol{v}, \boldsymbol{w}) \in X_R$.

Proof of Theorem 4

If $(\boldsymbol{v}, \boldsymbol{w}) \in X_R \Longrightarrow$

$$\begin{aligned} |||\Phi(\boldsymbol{v}, \boldsymbol{w})|||_t &\leq R\|(\boldsymbol{v}_0, \boldsymbol{w}_0)\|_d + C[\boldsymbol{u}]_t^2 \\ &\leq R\|(\boldsymbol{v}_0, \boldsymbol{w}_0)\|_d + 4CR^2\|(\boldsymbol{v}_0, \boldsymbol{w}_0)\|_d^2 \end{aligned}$$

for any $t > 0, \eta > 0$. Choose $\delta > 0$ in such a way that $4CR\delta < 1$,

$$|||\Phi(\boldsymbol{v}, \boldsymbol{w})|||_t \leq 2R\|(\boldsymbol{v}_0, \boldsymbol{w}_0)\|_d$$

for any $t > 0, \eta > 0$.
Summing up the above, we have

Lemma

$\Phi(\boldsymbol{v}, \boldsymbol{w}) \in X_R$, provided $(\boldsymbol{v}, \boldsymbol{w}) \in X_R$.

Since a similar arguments works well for the difference
$\Phi(\boldsymbol{v}_1, \boldsymbol{w}_1) - \Phi(\boldsymbol{v}_2, \boldsymbol{w}_2)$, we have Φ: contraction mapping on X_R into itself.

Estimate of $\|\boldsymbol{w}(t)\|_d$

Claim

$$\lim_{\eta\to\infty} \|\boldsymbol{w}(t)\|_d = 0, \quad t \geq t_0 > 0.$$

Estimate of $\|w(t)\|_d$

Claim

$$\lim_{\eta\to\infty} \|w(t)\|_d = 0, \quad t \geq t_0 > 0.$$

We first show the above claim for $(v_0, w_0) \in C^\infty_{0,\sigma}(\mathbb{R}^d) \times C^\infty_0(\mathbb{R}^d)$.
Take $q \in (d/2, d)$ and set $\sigma = d/2q - 1/2$ (i.e., $0 < \sigma < 1/2$). By small data global existence result, L^q-L^d est. and $L^{d/2}$-L^d est. we have,

$$\begin{aligned}\|w(t)\|_d &\leq C t^{-\sigma}(1+\eta)^{-\sigma}\|w_0\|_q \\ &\qquad + C(1+\eta)^{-1/2}\int_0^t (t-s)^{-\frac{1}{2}}\|u(s)\|_d\|\nabla u(s)\|_d\, ds \\ &\leq C t^{-\sigma}(1+\eta)^{-\sigma}(\|w_0\|_q + \tilde{C}[u]_{\sigma,d,t}\|(v_0, w_0)\|_d)\end{aligned}$$

Take initial data in such a way that $\tilde{C}\,\|(\boldsymbol{v}_0, \boldsymbol{w}_0)\|_d < 1/2$

$$\sup_{0<s\le t} s^{\sigma}\left(\|\boldsymbol{v}(s)\|_d + \sup_{\eta>0}(1+\eta)^{\sigma}\,\|\boldsymbol{w}(s)\|_d\right) \le 2C\,\|(\boldsymbol{v}_0, \boldsymbol{w}_0)\|_d .$$

This implies that the previous Claim for $t > 0$.

For general initial data Claim follows from density argument.

Remark

The above proof also refines the decay rate as $t \to \infty$.

Error estimate

- $\boldsymbol{u}(t)$: global mild sol. of (NS) with $\boldsymbol{u}_0 \in L^d_\sigma(\mathbb{R}^d)$, $\|\boldsymbol{u}_0\|_d \ll 1$
- $\boldsymbol{v}^\eta(t)$ and $\boldsymbol{w}^\eta(t)$: global mild solution of (NS)$_\eta$ with initial data $\|\boldsymbol{v}_0^\eta\|_d + \|\boldsymbol{w}_0^\eta\|_d \ll 1$.

Set $\mathcal{E}^\eta(t) := \boldsymbol{v}^\eta(t) - \boldsymbol{u}(t)$.

Claim

$\limsup_{\eta\to\infty} \|\mathcal{E}^\eta(t)\|_d \to 0$ (for any $t \geq t_0 > 0$).

$\mathcal{E}(t) := \boldsymbol{v}^\eta(t) - \boldsymbol{u}(t)$ satisfies

$$\begin{aligned}\mathcal{E}(t) = e^{t\Delta}\mathcal{E}_0 &- \int_0^t e^{(t-s)\Delta} P(\mathcal{E}\cdot\nabla\boldsymbol{u} + \boldsymbol{v}^\eta\cdot\nabla\mathcal{E})(s)\,ds \\ &- \int_0^t e^{(t-s)\Delta} P(\boldsymbol{w}^\eta\cdot\nabla\boldsymbol{v}^\eta + \boldsymbol{v}^\eta\cdot\nabla\boldsymbol{w}^\eta + \boldsymbol{w}^\eta + \nabla\boldsymbol{w}^\eta)(s)\,ds\end{aligned}$$

Since $\mathcal{E}, \boldsymbol{u}, \boldsymbol{v}^\eta$ are solenoidal,

$$\mathcal{E}(t) = e^{t\Delta}\mathcal{E}_0 - \int_0^t e^{(t-s)\Delta} P(\mathrm{div}(\mathcal{E} \otimes \boldsymbol{u}) + \mathrm{div}(\boldsymbol{v}^\eta \otimes \mathcal{E}))(s)\, ds$$
$$- \int_0^t e^{(t-s)\Delta} P(\boldsymbol{w}^\eta \cdot \nabla \boldsymbol{v}^\eta + \boldsymbol{v}^\eta \cdot \nabla \boldsymbol{w}^\eta + \boldsymbol{w}^\eta + \nabla \boldsymbol{w}^\eta)(s)\, ds.$$

If we choose $\|\boldsymbol{u}_0\|_d$ and $\|\boldsymbol{v}_0^\eta\|_d$ small enough (if necessary), we have by estimate for $\boldsymbol{w}^\eta(t)$,

$$\|\mathcal{E}^\eta(t)\|_d \leq C\|\mathcal{E}_0\|_d + C(1+\eta)^{-\frac{1}{2}+\frac{d}{2r}}.$$

This implies the Claim.