Life span of positive solutions for a semilinear heat equation with non-decaying initial data

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Problem.

We consider the Cauchy problem of semilinear heat equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + u^p, & x \in \mathbb{R}^n,\ t > 0, \\ u(x,0) = \phi(x), & x \in \mathbb{R}^n. \end{cases} \tag{1}$$

· $p > 1$.

· $\phi \in BC(\mathbb{R}^n)$, $\phi \geq 0$, $\not\equiv 0$. $(\Rightarrow u$: positive)

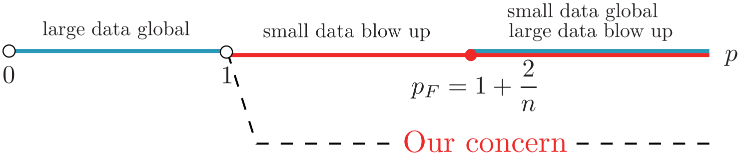
Our goal.

To give information on the life span T^* of the sol. of (1).

Fujita-type results

Since 1960's, many researchers have analysed positive sol.s on (1).

· H. Fujita, 1966 ($n \in \mathbb{N}$, $p > 1$, $p \neq p_F$)
· K. Hayakawa, 1973 ($n = 1, 2$, $p = p_F$)
· K. Kobayashi, T. Sirao, H. Tanaka, 1977 ($n \in \mathbb{N}$, $p > 1$)
· F.B. Weissler, 1981 ($n \in \mathbb{N}$, $p > 1$, L^q-framework)
· J. Aguirre, M. Escobedo, 1981 ($n \in \mathbb{N}$, $0 < p < 1$)



1.3 Blow-up for slowly decaying initial data

For slowly decaying (nondecaying) initial data,
it is well known that the classical solution blows up in finite time. $(p > 1)$

· P. Baras, R. Kersner, 1987
· T. Lee, W.-M. Ni, 1992
· P. Souplet, F.B. Weissler, 1997
· N. Mizoguchi, E. Yanagida, 1998
· F. Rouchon, 2001

Theorem (Lee-Ni 1992)

$$\liminf_{|x|\to\infty} |x|^{\frac{2}{p-1}}\phi(x) > \mu_1^{1/(p-1)}$$
$$\Longrightarrow T^* < \infty.$$

In particular, for non-decayng initial data ϕ, the solution blows up in finite time.

2.1 Life span

Life span T^* of the solution $u(x,t)$

$T^* := \sup\{T > 0 |\ (1)$ possesses
a unique classical solution in $\mathbb{R}^n \times [0,T)\}$.

Remark. (i) For any $T' < T^*$, u is bounded on $\mathbb{R}^n \times [0,T']$.
(ii) $||u(\cdot,t;\lambda)||_{L^\infty(\mathbb{R}^n)} \to \infty$ as $t \to T^*$ if $T^* < \infty$.

2.2 Asymptotics of life span for large (small) data

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + u^p, & x \in \mathbb{R}^n,\ t > 0, \\ u(x,0) = \lambda \phi(x), & x \in \mathbb{R}^n. \end{cases} \quad (2)$$

- $p > 1$.
- $\phi \in BC(\mathbb{R}^n)$, $\phi \geq 0, \not\equiv 0$.
- λ is a positive parameter.

Life span T_λ^* of the solution $u(x,t;\lambda)$

$T_\lambda^* := \sup\{T > 0 |\ (2)$ possesses a unique classical solution in $\mathbb{R}^n \times [0,T)\}$.

2.2 Asymptotics of life span for large (small) data

Theorem (Lee-Ni 1992)

If $\liminf_{|x|\to\infty} \phi(x) > 0$, *then*

$$T_\lambda^* \sim \lambda^{1-p} \quad (\lambda \to \infty),$$
$$T_\lambda^* \sim \lambda^{1-p} \quad (\lambda \to 0).$$

Theorem (Gui-Wang 1995)

(i) $\lim_{\lambda\to\infty} T_\lambda^* \cdot \lambda^{p-1} = \dfrac{1}{p-1} \|\phi\|_{L^\infty}{}^{1-p}$.

(ii) *If* $\lim_{|x|\to\infty} \phi(x) = \phi_\infty > 0$, *then* $\lim_{\lambda\to 0} T_\lambda^* \cdot \lambda^{p-1} = \dfrac{1}{p-1} \phi_\infty{}^{1-p}$.

2.3 Minimal time blow up

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + u^p, & x \in \mathbb{R}^n,\ t > 0, \\ u(x,0) = \phi(x), & x \in \mathbb{R}^n. \end{cases}$$

Minimal blow up time

$$T^* = \frac{1}{p-1}\|\phi\|_{L^\infty(\mathbb{R}^n)}^{1-p}$$

Remark. From comparison principle and $u(x,0) \leq \|\phi\|_{L^\infty(\mathbb{R}^n)}$, we always have

$$T^* \geq \frac{1}{p-1}\|\phi\|_{L^\infty(\mathbb{R}^n)}^{1-p}.$$

2.3 Minimal time blow up

- Y. Giga, N. Umeda, 2006 (semilinear case)
- Y. Seki, R. Suzuki, N. Umeda, 2008 (quasilinear case)
- Y. Seki, 2008 (quasilinear case)

Theorem (sufficient condition for minimal time blow up)

$\exists \{x_j\} \subset \mathbb{R}^n$ *s.t.*
- $|x_j| \to \infty \ (j \to \infty)$
- $\phi(x + x_j) \to \|\phi\|_{L^\infty(\mathbb{R}^n)}$ *a.e. in* $\mathbb{R}^n$

$\Longrightarrow$ *minimal time blow up occurs.*

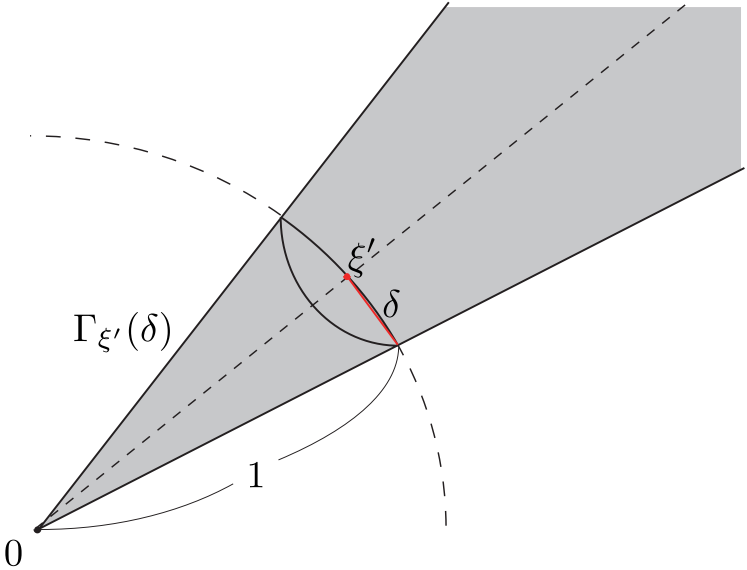
3.1 Definition of M_∞

Conic neighborhood $\Gamma_{\xi'}(\delta)$ and $S_{\xi'}(\delta)$

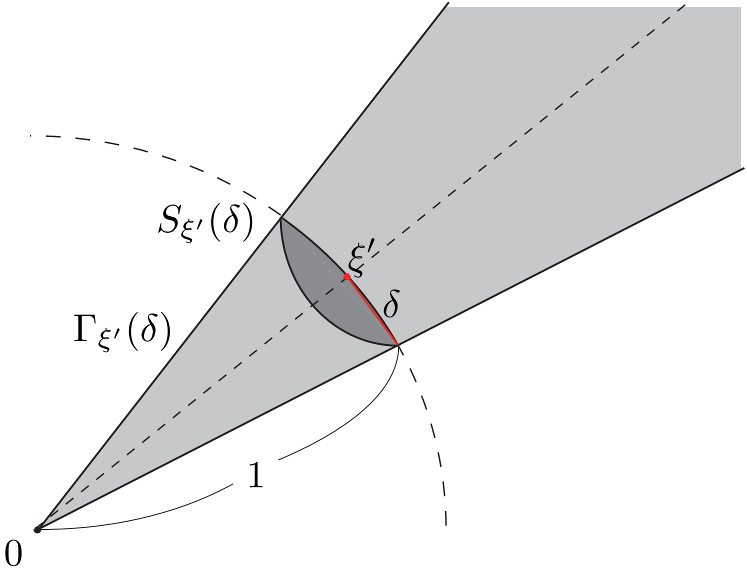
For $\xi' \in \mathbb{S}^{n-1}$ and $\delta \in (0, \sqrt{2})$,

$$\Gamma_{\xi'}(\delta) := \left\{ \eta \in \mathbb{R}^n \setminus \{0\};\ \left| \xi' - \frac{\eta}{|\eta|} \right| < \delta \right\},$$
$$S_{\xi'}(\delta) := \Gamma_{\xi'}(\delta) \cap \mathbb{S}^{n-1}.$$

$\Gamma_{\xi'}(\delta)$ and $S_{\xi'}(\delta)$



$\Gamma_{\xi'}(\delta)$ and $S_{\xi'}(\delta)$



Conic neighborhood $\Gamma_{\xi'}(\delta)$ and $S_{\xi'}(\delta)$

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$$S_{\xi'}(\delta) := \Gamma_{\xi'}(\delta) \cap \mathbb{S}^{n-1}.$$

M_∞

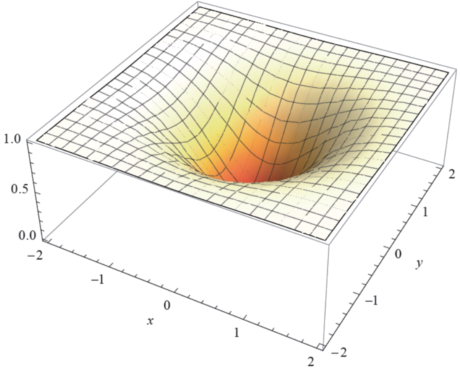
$$M_\infty := \sup_{\xi', \delta} \left\{ \operatorname*{ess.inf}_{x' \in S_{\xi'}(\delta)} \left(\liminf_{r \to \infty} \phi(rx') \right) \right\}.$$

In this talk, we assume that

$$M_\infty > 0. \quad \text{(non-decaying data)}$$

Examples of initial data ϕ and M_∞ in 2-dim.

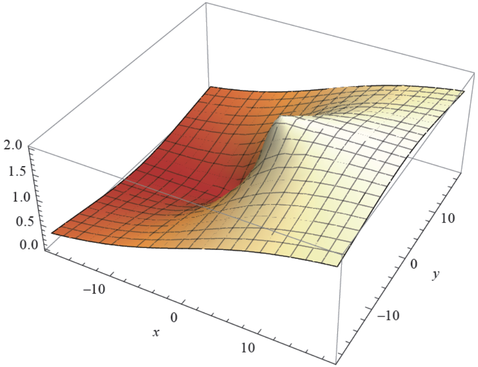
Example 1. $\phi(r,\theta) = 1 - \exp(-r^2)$



$$\liminf_{r\to+\infty} \phi(rx') = 1, \quad M_\infty = 1.$$

Examples of initial data ϕ and M_∞ in 2-dim.

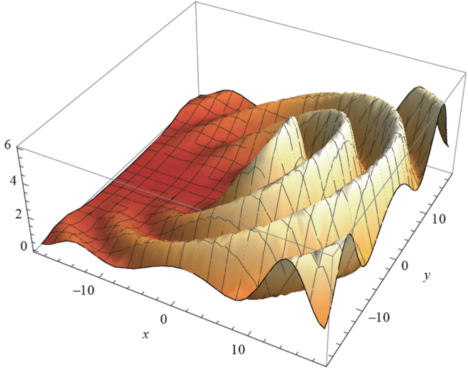
Example 2. $\phi(r,\theta) = \{1 - \exp(-r^2)\}\,(1 + \cos\theta)$



$$\liminf_{r\to+\infty} \phi(rx') = 1 + \cos\theta, \qquad M_\infty = 2.$$

Examples of initial data ϕ and M_∞ in 2-dim.

<u>Example 3.</u> $\phi(r,\theta) = \{1-\exp(-r^2)\}(1+\cos\theta)(2-\cos r)$



$$\liminf_{r\to+\infty} \phi(rx') = 1+\cos\theta, \qquad M_\infty = 2.$$

Theorem 1. $(n \geq 2)$

Assume that $M_\infty > 0$. Then the solution of (1) blows up in finite time T^*, and we have

$$T^* \leq \frac{1}{p-1} M_\infty^{1-p},$$

where

$$M_\infty := \sup_{\xi',\, \delta} \left\{ \operatorname*{ess.inf}_{x' \in S_{\xi'}(\delta)} \left(\liminf_{r \to \infty} \phi(rx') \right) \right\}$$

4.1 Preliminary

$\{a_j\}$ and $\{R_j\}$

For fixed ξ' and δ, determine $\{a_j\} \subset \mathbb{R}^n$ and $\{R_j\} \subset (0,\infty)$ as follows:

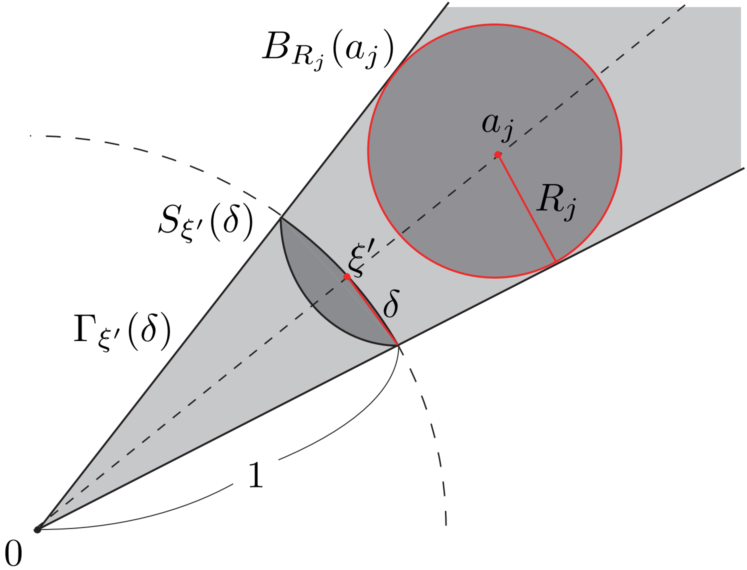
- $|a_j| \to \infty$ as $j \to \infty$
- $a_j/|a_j| = \xi'$ for any $j \in \mathbb{N}$
- $R_j := (\delta\sqrt{4-\delta^2}/2)|a_j|$.

$\{\rho_{R_j}(x)\}$ and $\{\mu_{R_j}\}$

$\rho_{R_j}(x)$: the first eigenfunction of $-\Delta$ on $B_{R_j}(0)$
with zero Dirichlet boundary condition
under the normalization $\int_{B_{R_j}(0)} \rho_{R_j}(x)dx = 1$.

μ_{R_j} : the corresponding first eigenvalue.

$B_{R_j}(0)$ and $\Gamma_{\xi'}(\delta)$



$\{w_j(t)\}$

Using a_j, R_j and $\rho_{R_j}(x)$, we put

$$w_j(t) := \int_{B_{R_j}(0)} u(x+a_j,t)\rho_{R_j}(x)dx.$$

Remark. If u is bounded on $\mathbb{R}^n \times (0,T)$, then w_j is also bounded on $(0,T)$. Hence, $T^*_{w_j} \geq T^*$.

All we have to do is to obtain the estimate of $T^*_{w_j}$.

4.2 Proof of Theorem 1.

Translating both sides of the equation (1) by a_j, multiplying by ρ_{R_j} and integrating over $B_{R_j}(0)$, we obtain the following O.D.I:

$$\begin{cases} w_j' \geq w_j^p - \mu_{R_j} w_j, \quad t \in (0, T_{w_j}), \\ w_j(0) = \displaystyle\int_{B_{R_j}(0)} \phi(x + a_j)\rho_{R_j}(x)dx. \end{cases}$$

$$w_j(t) \geq \left\{ w_j^{1-p}(0) - \frac{1 - \exp((1-p)\mu_{R_j} t)}{\mu_{R_j}} \right\}^{-\frac{1}{p-1}} \exp(-\mu_{R_j} t).$$

$$T_{w_j}^* \leq \frac{\log\left(1 - \mu_{R_j} w_j^{1-p}(0)\right)}{(1-p)\mu_{R_j}}.$$

Lemma. (Properties of $\{w_j(0)\}$)

(i)

$$\liminf_{j\to+\infty} w_j(0) \geq \operatorname*{ess.inf}_{x'\in S_{\xi'}(\delta)} \left(\liminf_{r\to\infty} \phi(rx') \right).$$

(ii)

$$\lim_{j\to+\infty} \frac{\log\left(1-\mu_{R_j} w_j^{1-p}(0)\right)}{-\mu_{R_j} w_j^{1-p}(0)} = 1.$$

4.2 Proof of Theorem 1.

From the Lemma, we see that

$$\begin{aligned}
\limsup_{j\to\infty} T^*_{w_j} &\le \limsup_{j\to\infty} \frac{\log\left(1-\mu_{R_j} w_j^{1-p}(0)\right)}{-(p-1)\mu_{R_j}} \\
&= \frac{1}{p-1} \lim_{j\to\infty} \frac{\log\left(1-\mu_{R_j} w_j^{1-p}(0)\right)}{-\mu_{R_j} w_j^{1-p}(0)} \left(\liminf_{j\to\infty} w_j(0)\right)^{1-p} \\
&\le \frac{1}{p-1} \left(\operatorname*{ess.inf}_{x'\in S_{\xi'}(\delta)} \left(\liminf_{r\to\infty} \phi(rx')\right)\right)^{1-p}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\limsup_{j\to\infty} T^*_{w_j} &\ge \limsup_{j\to\infty} T^* \\
&= T^*.
\end{aligned}$$

Hence we have

$$T^* \leq \frac{1}{p-1}\left(\operatorname*{ess.inf}_{x' \in S_{\xi'}(\delta)}\left(\liminf_{r \to \infty} \phi(rx)\right)\right)^{1-p}.$$

From arbitrariness of ξ' and δ, we obtain

$$\begin{aligned} T^* &\leq \frac{1}{p-1}\left\{\sup_{\xi',\, \delta}\left(\operatorname*{ess.inf}_{x' \in S_{\xi'}(\delta)}(\liminf_{r \to \infty} \phi(rx))\right)\right\}^{1-p} \\ &= \frac{1}{p-1} M_{\infty}{}^{1-p}. \end{aligned}$$

This completes the proof.□

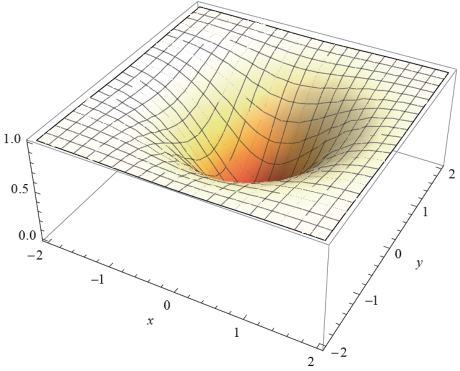
Gap between upper bound and lower bound

$$\frac{1}{p-1}\|\phi\|_{L^\infty(\mathbb{R}^n)}^{1-p} \le T^* \le \frac{1}{p-1}M_\infty^{1-p}$$

Remark. For initial data in Example 1 and 2, $\|\phi\|_{L^\infty(\mathbb{R}^n)} = M_\infty$ holds. Hence, minimal time blow up occurs.

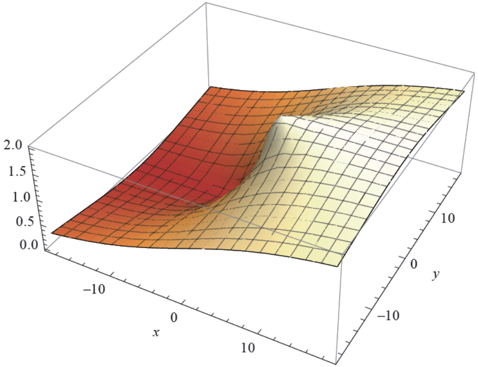
Examples of initial data ϕ and M_∞ in 2-dim.

Example 1. $\phi(r,\theta) = 1 - \exp(-r^2)$



$$\liminf_{r\to+\infty} \phi(rx') = 1, \quad M_\infty = 1 = \|\phi\|_{L^\infty(\mathbb{R}^n)}.$$

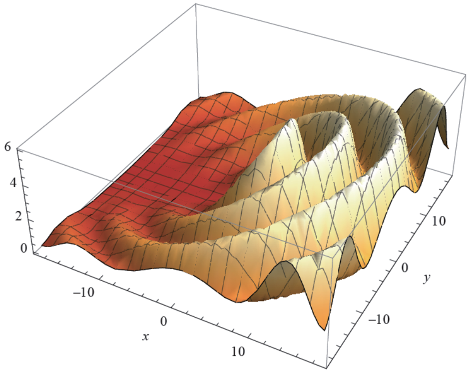
Example 2. $\phi(r,\theta) = \{1 - \exp(-r^2)\}\,(1+\cos\theta)$



$$\liminf_{r\to+\infty} \phi(rx') = 1+\cos\theta, \qquad M_\infty = 2 = \|\phi\|_{L^\infty(\mathbb{R}^n)}.$$

Examples of initial data ϕ and M_∞ in 2-dim.

Example 3. $\phi(r,\theta) = \left\{1 - \exp(-r^2)\right\}(1+\cos\theta)(2-\cos r)$



$$\liminf_{r\to+\infty} \phi(rx') = 1 + \cos\theta, \quad M_\infty = 2 \neq 6 = \|\phi\|_{L^\infty(\mathbb{R}^n)}.$$

Main result in 1-dim.

Theorem 1'. ($n = 1$)

$$\max\left\{\liminf_{x\to+\infty}\phi(x), \liminf_{x\to-\infty}\phi(x)\right\} > 0,$$

$$\Longrightarrow T^* \le \frac{1}{p-1}\left(\max\left\{\liminf_{x\to+\infty}\phi(x), \liminf_{x\to-\infty}\phi(x)\right\}\right)^{1-p}.$$

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