Construction of a weak solution of a certain stochastic Navier Stokes equation

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This talk is organized as

1. Introduction
    - Motivation, the equation derived from the variational problem
2. Stochastic Navier-Stokes equation
    - Main result
3. Outline of the proof

We study $u = u(t,x) = (u^1(t,x), u^2(t,x))$, $t \geq 0$, $x = (x_1, x_2)$ and the pressure term $p = p(t,x)$ on a two-dimensional torus $\mathbf{T}^2$:

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)\, u + \sqrt{2\mu} \nabla u \cdot \dot{B}(t) - \mu \Delta u + \nabla p = 0, \quad t > 0, x \in \mathbf{T}^2, \tag{1}$$

with the incompressibility condition:

$$\operatorname{div} u = 0, \quad t > 0, x \in \mathbf{T}^2, \tag{2}$$

under the initial condition:

$$u(0,x) = u_0(x), \quad x \in \mathbf{T}^2, \tag{3}$$

where $\mu > 0$ is a constant and $\dot{B}(t) = \frac{d}{dt} B(t)$ is a formal derivative of the two dimensional Brownian motion $B(t) = (B^1(t), B^2(t))$.

1. Introduction

Background :

Inoue, A., Funaki, T., On a new derivation of the Navier-Stokes equation. Comm. Math. Phys. **65** (1979).

Variational problem;

Diff($\mathbf{R}^n$) : the set of volume preserving diffeomorphisms of $\mathbf{R}^n$.

$\{\Phi(t)\}_{t\in[0,1]}$: 1-parameter group with values in Diff($\mathbf{R}^n$).

Let Ψ^0 and $\Psi^1 \in$ Diff($\mathbf{R}^n$) be given and

$$\Phi(t,x) = (\Phi_1(t,x), \cdots, \Phi_n(t,x)), \quad t \in [0,1],$$

be an integral curve which takes values in Diff($\mathbf{R}^n$). A stationary point $\tilde{\Phi}$ of the following action functional J:

$$J(\Phi) := \sum_{j=1}^{n} \int_0^1 \int_{\mathbf{R^n}} |\frac{\partial \Phi_j}{\partial t}(t,x)|^2 dx\, dt,$$

satisfying $\Phi(0) = \Psi^0$ and $\Phi(1) = \Psi^1$.

Stationary point of J:

$$\tilde{\Phi}(t,x) := (\tilde{\Phi}_1(t,x), \cdots, \tilde{\Phi}_n(t,x)),$$

Velocity field u defined by

$$(u^1(t,\tilde{\Phi}(t,x)), \cdots, u^n(t,\tilde{\Phi}(t,x))) = (\frac{\partial \tilde{\Phi}_1}{\partial t}(t,x), \cdots, \frac{\partial \tilde{\Phi}_n}{\partial t}(t,x)),$$

that is,

$$\underline{(u^1(t,x), \cdots, u^n(t,x)) = (\frac{\partial \tilde{\Phi}_1}{\partial t}(t,\tilde{\Phi}^{-1}(x)), \cdots, \frac{\partial \tilde{\Phi}_n}{\partial t}(t,\tilde{\Phi}^{-1}(x))),}$$

Then, u satisfies the Euler equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = 0 & t > 0, \quad x \in \mathbf{R}^n, \\ \mathrm{div}\, u = 0, & t > 0, \quad x \in \mathbf{R}^n. \end{cases} \tag{4}$$

The case where a fluctuation is added to the "Euler flow" $\Phi(t)$:

$$\Phi(t,x) \Longrightarrow \underline{\Phi(t,x) + \sqrt{2\mu}B(t,\omega)} \quad \text{"} \leftarrow \text{random force"},$$

Let us consider the following Random action functional:

$$J_B(\Phi)(\omega) = \int_{\mathbf{R^n}} \int_0^1 \sum_{j=1}^{n} |\frac{\partial \Phi_j}{\partial t}(t,x) + \sqrt{2\mu}\frac{dB^j}{dt}(t,\omega)|^2 dxdt,$$

where $\mu > 0$, $\Phi(0) = \Psi^0(\omega)$ and $\Phi(1) = \Psi^1(\omega)$ and $B = (B^1, \cdots, B^n)$ is an n-dimensional Brownian motion defined on some probability space.

Remark 1. $J_B(\Phi)$ *is formally defined for each* ω. B_t *is not differentiable at any* $t > 0$.

Stationary point of J_B (for each ω) $\bar{\Phi}_B(t,x,\omega)$ is related by

$$\tilde{\Phi}(t,x,\omega) = \bar{\Phi}_B(t,x,\omega) + \sqrt{2\mu}B(t,\omega),$$

where $\tilde{\Phi}$ is the stationary point of the "Euler flow".

$$\bar{\Phi}_B(t,x,\omega) = \tilde{\Phi}(t,x,\omega) - \sqrt{2\mu}B(t,\omega),$$

Set $x = \tilde{\Phi}^{-1}(t,y)$. Then,

$$\bar{\Phi}_B(t,\tilde{\Phi}^{-1}(t,x),\omega) = \tilde{\Phi}(t,\tilde{\Phi}^{-1}(t,x),\omega) - \sqrt{2\mu}B(t,\omega),$$

Thus, random velocity field is defined by

$$\begin{aligned}\bar{u}(t,x,\omega) &:= \frac{d}{dt}\bar{\Phi}_B(t,\tilde{\Phi}^{-1}(t,x),\omega) \\ &= u(t,x) - \sqrt{2\mu}\dot{B}(t,\omega),\end{aligned}$$

where u is the solution of the Euler equation.

Thus, we arrive at

$$\begin{cases} \dfrac{\partial(\bar{u}(t,x) + \sqrt{2\mu}\dot{B}_t)}{\partial t} + \nabla\bar{u}(t,x) \circ (\bar{u}(t,x) + \sqrt{2\mu}\dot{B}_t) + \nabla p(t,x) = 0, \\ \text{div } \bar{u}(t,x) = 0, \end{cases}$$

where $\circ$ means the Storatonovich sense.

Stratonovich integral

For $f \in C_b(\mathbb{R})$ and $\{B^\epsilon\}_{\epsilon>0}$ the family of smooth approximation of B, (e.g. $B^\epsilon(t) = (B * \rho_\epsilon)(t)$, $\int_0^\infty \rho_\epsilon dt = 1$, $\rho_\epsilon \geq 0$, supp$\rho_\epsilon \subset (t-\epsilon, t+\epsilon)$).

$$\int_0^t f(B^\epsilon(s))\dot{B}^\epsilon(s)ds \to \int_0^t f(B(s)) \circ dB(s), \quad \text{uniformly in } t \in [0,T]$$

in $L^2(\Omega)$ as $\epsilon \to 0$.

The term of $\ddot{B}(t)$

For $\phi \in \mathbf{C}_{\sigma,0}(\mathbb{R}^n;\mathbb{R}^n)$,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \sqrt{2\mu}\ddot{B}(t) \cdot \phi(x)dx \\
&= \int_{\mathbb{R}^n} \sqrt{2\mu}\nabla(x \cdot \ddot{B}(t)) \cdot \phi(x)dx = -\int_{\mathbb{R}^n} \sqrt{2\mu}(x \cdot \ddot{B}(t))\text{div } \phi(x)dx = 0,
\end{aligned}$$

Thus the term $\sqrt{2\mu}\ddot{B}(t)$ is disregarded. In consequence,

$$\begin{cases} \dfrac{\partial \bar{u}(t,x)}{\partial t} + (\bar{u}(t,x) \cdot \nabla)\bar{u}(t,x) + \sqrt{2\mu}\nabla \bar{u}(t,x) \circ \dot{B}_t + \nabla p(t,x) = 0, \\ \text{div } \bar{u}(t,x) = 0, \end{cases} \tag{5}$$

From a Stratonovich integral to an Itô integral

$$\underbrace{\int_0^t \frac{\partial u}{\partial x_j}(s) \circ dB^j(s)}_{Stratonivich} = \underbrace{\int_0^t \frac{\partial u}{\partial x_j}(s) dB^j(s)}_{It\hat{o},(martingale)} + \frac{1}{2}\langle\langle M_{\frac{\partial u}{\partial x_j}}, B^j\rangle\rangle(t), \qquad j = 1, \cdots, n,$$

where $M_{\frac{\partial u}{\partial x_j}}$ denotes the martingale part determined uniquely by the decomposition of the process $\frac{\partial u}{\partial x_j}$ and $\langle\langle M_{\frac{\partial u}{\partial x_j}}, B^j\rangle\rangle$ the quadratic variation of $M_{\frac{\partial u}{\partial x_j}}$ and B^j. Integral form of (5):

$$\begin{aligned}
\bar{u}(t,x) &= \bar{u}(0,x) - \int_0^t (\bar{u}(s,x)\cdot\nabla)\bar{u}(s,x)ds \\
&- \sqrt{2\mu}\int_0^t \nabla\bar{u}(s,x)\circ dB_s - \int_0^t \nabla p(t,x)ds, \\
&= \bar{u}(0,x) - \int_0^t (\bar{u}(s,x)\cdot\nabla)\bar{u}(s,x)ds \\
&\underbrace{-\sqrt{2\mu}\int_0^t \nabla\bar{u}(s,x)\cdot dB_s}_{M_{\bar{u}}(t)} - \frac{\sqrt{2\mu}}{2}\langle\langle M_{\nabla\bar{u}}, B\rangle\rangle(t) - \int_0^t \nabla p(t,x)ds,
\end{aligned}$$

$$M_{\nabla \bar{u}}(t) = -\sqrt{2\mu}\int_0^t \triangle \bar{u}(s,x)\cdot dB_s,$$

Note that $dB_s^i dB_s^j = \delta_{ij}dt$. Thus,

$$\langle\langle M_{\nabla \bar{u}}, B\rangle\rangle(t) = -\sqrt{2\mu}\int_0^t \triangle \bar{u}(s,x)ds,$$

We arrive at the following stochastic Navier-Stokes equation:

$$\begin{cases} \dfrac{\partial \bar{u}}{\partial t} + \sum\limits_{j=1}^{n}\left(\bar{u}^j \dfrac{\partial \bar{u}}{\partial x_j} + \sqrt{2\mu}\dfrac{\partial \bar{u}}{\partial x_j}\dot{B}_t^j\right) - \mu\Delta\bar{u} + \nabla p(t,x) = 0, \\ \mathrm{div}\,\bar{u} = 0. \end{cases} \tag{6}$$

Assume that the existence of the solution $\bar{u}$ is shown.

Reynolds equation

Let us set $\langle \bar{u} \rangle(t,x) = \int_\Omega \bar{u}(t,x,\omega)P(d\omega)$. Then $\langle \bar{u} \rangle$ satisfies the Reynolds equation:

$$\begin{cases} \dfrac{\partial \langle \bar{u} \rangle}{\partial t} + (\langle \bar{u} \rangle \cdot \nabla)\langle \bar{u} \rangle - \mu \Delta \langle \bar{u} \rangle + \nabla \langle p \rangle = -\langle (\bar{u} - \langle \bar{u} \rangle) \cdot \nabla (\bar{u} - \langle \bar{u} \rangle) \rangle, \\ \text{div}\, \langle \bar{u} \rangle = 0. \end{cases}$$

This is easily shown by using the fact that the term of the Ito stochastic integral is martingale, that is, its expectation is equal to zero. However, <u>the existence of the weak solution of (6)</u> is not shown in [I-F].

$\Longrightarrow$ We want to prove the existence.

The weak form of (6)

$$\int_{\mathbb{R}^n} u(t,x)\cdot\phi(x)dx - \int_{\mathbb{R}^n} u_0(x)\cdot\phi(x)dx \tag{7}$$
$$= \int_0^t \int_{\mathbb{R}^n} (u(s,x)\cdot\nabla)\phi(x)\cdot u(s,x)dsdx$$
$$+\sqrt{2\mu}\int_0^t \left(\int_{\mathbb{R}^n} u(s,x)\cdot\nabla\phi(x)dx\right)\cdot dB_s + \mu\int_0^t\int_{\mathbb{R}^n} u(s,x)\cdot\Delta\phi(x)dsdx,$$

for all $\phi \in \mathbf{C}^\infty_{0,\sigma}$ and $t \geq 0$, where

$$\mathbf{C}^\infty_{0,\sigma} = \{\phi \in \mathbf{C}^\infty_0(\mathbb{R}^n;\mathbb{R}^n) \mid \mathrm{div}\,\phi = 0\}.$$

Note that the term containing $\frac{\partial p}{\partial x_i}$, $i = 1,\ldots,n$ vanishes because

$$\sum_{i=1}^n \int_{\mathbb{R}^n} \frac{\partial p}{\partial x_i}(t,x)\phi^i(x)dx = -\int_{\mathbb{R}^n} p(t,x)\ \mathrm{div}\ \phi(x)dx = 0,$$

holds. However, it is difficult to show the existence of (6) if $n \geq 3$.
In our case, $\mathbf{T}^2$: 2-dimensional torus.

2. Stochastic Navier-Stokes equation in $\mathbb{T}^2$.

Notations

$$\mathbf{C}_\sigma^\infty = \left\{ u \in \mathbf{C}^\infty(\mathbb{T}^2; \mathbb{R}^2) \mid \mathrm{div}\, u = 0, \int_{\mathbb{T}^2} u dx = 0 \right\},$$

$$\mathbf{H} = \left\{ u \in \mathbf{L}^2(\mathbb{T}^2; \mathbb{R}^2) \mid \mathrm{div}\, u = 0, \int_{\mathbb{T}^2} u dx = 0 \right\},$$

The inner product of $\mathbf{H}$:

$$\langle u, v \rangle = \sum_{j=1}^{2} \int_{\mathbb{T}^2} u^j(x) v^j(x) dx, \quad u, v \in \mathbf{H},$$

and the norm of $\mathbf{H}$: $|\cdot|_{\mathbf{H}}$.

$$\mathbf{V} = \mathbf{W}^{1,2}(\mathbb{T}^2; \mathbb{R}^2) \cap \mathbf{H}.$$

The inner product of $\mathbf{V}$:

$$\langle\langle u, v \rangle\rangle = \sum_{j=1}^{2} \langle \frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_j} \rangle, \quad u, v \in \mathbf{V},$$

and the norm of $\mathbf{V}$: $||\cdot||_{\mathbf{V}}$.

An abstract SDE

$$\begin{cases} du(t) + \{Au(t) + B(u(t), u(t))\}\, dt + Gu(t)dB_t = 0, & t > 0, \\ u(0) = u_0, \end{cases} \tag{8}$$

$D(A) = \mathbf{W}^{2,2}(\mathbb{T}^2; \mathbb{R}^2) \cap \mathbf{V}$ such that

$$A : D(A) \to \mathbf{H}, \quad Au = -\mu \mathbb{P} \Delta u,$$

where $\mathbb{P}$ is the Lelay projection, $(\lambda_j)_{j=1,2,\dots}$ its eigenvalues and $(e_j)_{j=1,2,\dots}$ the corresponding eigenfunctions. Note that $e_j \in \mathbf{C}_\sigma^\infty$, $\forall j$ and $0 < \lambda_1 \le \lambda_2 \le \cdots$.

$$B : \mathbf{V} \times \mathbf{V} \to \mathbf{V}', \quad B(v, w) = \mathbb{P}(v \cdot \nabla)w,$$

where $\mathbf{V}'$ is the dual space of $\mathbf{V}$.

$$G : \mathbf{V} \to \mathbf{L}_{\mathrm{H.S}}(\mathbb{R}^2; \mathbf{H}), \quad Gv = \sqrt{2\mu}\mathbb{P}\nabla v,$$

where $\mathbf{L}_{\mathrm{H.S}}(\mathbb{R}^2; \mathbf{H})$ denotes the family of Hilbert-Schmidt operators from $\mathbb{R}^2$ to $\mathbf{H}$.

Definition 1. *We say that* $\{u(t), B(t)\}_{t\geq 0}$ *is a weak solution of the stochastic Navier-Stokes equation (8) with the initial value* u_0 *if*

1. $\{u(t)\}_{t\geq 0}$ *is an adapted process defined on a probability space* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$.

2. $u \in L^2(0,T;\mathbf{V}) \cap L^\infty(0,T;\mathbf{H})$, P-$a.s.$ *for* $T > 0$.

3. $\{B(t), \{\mathcal{F}_t\}\}_{t\geq 0}$ *is a two-dimensional Brownian motion on* $(\Omega, \mathcal{F}, P)$.

4. *For every* $T > 0$ *and* $\phi \in \mathbf{C}^\infty_\sigma$, P-*a.s.*,

$$\langle u(t), \phi\rangle - \langle u_0, \phi\rangle =$$
$$-\int_0^t \langle A^*\phi, u(s)\rangle ds + \int_0^t \langle B(u(s),\phi), u(s)\rangle ds - \int_0^t (Gu(s))^*\phi\, dB(s),$$

holds for a.e.-$t \in [0,T]$.

Weak solution means

1. weak form: $\int u(t,x)\phi(x)dx$,

2. martingale solution: $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$ is a part of the solution, that is, $((\Omega, \mathcal{F}, P, (\mathcal{F}_t)), u, B)$ is the solution:

   (see G. Da Prato., J. Zabczyk, Stochastic equations in infinite dimensions, 1992, Chapter 8.)

Main result (Y. 2011)

Theorem 1. *There exists a weak solution* $\{u(t), B(t)\}_{t\geq 0}$ *of the stochastic Navier-Stokes equation (8) with the initial value* $u_0 \in \mathbf{V}$.

The properties of (6)

$$du(t) + \{Au(t) + B(u(t), u(t))\}\, dt + Gu(t) dB_t = 0,$$

For a suitable bounded set $D \subset \mathbb{R}^n$,

$$2\langle Au(t), u(t)\rangle - |Gu(t)|^2_{L_{\mathrm{H.S}}(\mathbb{R}^n;L^2(D))} = 0, \quad n \geq 2, \tag{9}$$

holds. By Itô's formula,

$$\begin{aligned}|u(t)|^2_{L^2} - |u_0|^2_{L^2} &= 2\int_0^t \langle u(s), du(s)\rangle ds + \int_0^t \langle du(s), du(s)\rangle \\ = &- 2\int_0^t \langle u(s), Au(s)\rangle ds - 2\int_0^t \langle u(s), B(u(s), u(s))\rangle ds \\ &- 2\langle \int_0^t Gu(s) \cdot dB_s, u(s)\rangle + \int_0^t |Gu(s)|^2_{L_{\mathrm{H.S}}(\mathbb{R}^n;L^2(D))} ds.\end{aligned}$$

Thus, if $u_0 \in L^2(D)$, $\exists C > 0$,

$$\mathbf{E}\left\{|u(t)|^2_{L^2}\right\} \leq C.$$

The condition (9) means that the term containing $\underline{|u(t)|^2_{H^1}}$ vanishes.

Difficulty

Let $\{u_n\}_{n\in\mathbb{N}}$ be the sequence of solutions of equations by Galerkin's argument. $\{u_n\}_{n\in\mathbb{N}}$ are bounded in $L^2(0,T;L^2(D))$. It follows that $\exists {u_{n(k)}}_{k\in\mathbb{N}}$ and $\overline{u}$ such that $u_n \to \overline{u}$ weakly in $L^2(0,T;L^2(D))$.

However,

$$\mathbf{E}\{\int_0^t \langle (u_{n(k)}(s)\cdot\nabla)\phi, u_{n(k)}(s)\rangle ds\} \to \mathbf{E}\{\int_0^t \langle (\overline{u}(s)\cdot\nabla)\phi, \overline{u}(s)\rangle ds\}$$

is not true.

$\Longrightarrow$ The uniform estimate with respect to stronger topology such as $L^2(0,T;H^1(D))$ is needed.

Coercivity condition

$$2\langle Au, u\rangle - |Gu|^2_{L_{\mathrm{H.S}}(\mathbb{R}^n, L^2(D))} \geq \delta |\nabla u|^2_{L^2(D)},$$

for some $\delta \in (0, 2]$, $\lambda_0 \geq 0$ and $\rho \geq 0$.

There are several results about the existence of the weak solution of equations satisfying the coercivity condition:

F. Flandoli and D. Gatarek, 1995.
bounded domain in $\mathbb{R}^n$.

M. Capinski and S. Peszat, 2001
$\mathbb{R}^n (n = 2, 3)$ or bounded domain with smooth $G(\cdot)$ and $||G(u(t))||_{L_{\mathrm{H.S}}} \leq C(1 + |u(t)|_{L^2})$.

R. Mikulevicius and B.L. Rozovskii, 2005
$\mathbb{R}^n (n \geq 2)$ or bounded domain,

Z. Brezezniak and S. Peszat, 2001

$n = 2$. They assume for a certain $c \in L^1(0,T)$ such that

$$|G(u)|_{L_{HS}(\mathbf{R}^n, H^1)} \leq C(t)(1 + |\nabla u|^2)$$

$\rightarrow$ the case of $G(u) = \nabla u$ is not included.

3. Outline of the proof

Our storategy

1. Consider $\frac{2+\delta}{2}\mu\triangle$ instead of $\mu\triangle$ and construct the strong solution u_n^δ, for each $\delta > 0$ by Galerkin's argument:

$$\sup_n \left\{ \mathbf{E}\left\{ |u_n^\delta(t)|_{\mathbf{H}}^2 \right\} + \delta\mu \int_0^t \mathbf{E}\left\{ ||u_n^\delta(s)||_{\mathbf{V}}^2 \right\} ds \right\} < \infty.$$

2. Obtain the uniform estimate with respect to $\delta > 0$, $n \geq 1$

3. Construct u^δ for each $\delta > 0$.

4. Take $\delta \to 0$ and find the solution u.

Step 1 : Finite dimensional equation,

Step 2 : A priori estimate,

Step 3 : tightness,

Step 4 : taking a limit.

Step 1: Finite dimensional equation.

We can expand $u_n^\delta(t) \in \mathbf{H}_n(:= \Pi_n \mathbf{H})$ as $u_n^\delta(t) = \sum_{j=1}^n u_j^{\delta,n}(t) e_j$, where $u_j^{\delta,n}(t) = \langle u_n^\delta(t), e_j \rangle$. Then,

$$u_j^{\delta,n}(t) = \langle \Pi_n u_0, e_j \rangle + \int_0^t F_j^{\delta,n}(u_n^\delta(s)) ds + \int_0^t \sigma_j^n(u_n^\delta(s))\, dB_s, \quad 1 \leq j \leq n,$$

where $F_j^{\delta,n}(u_n) = -\langle A_\delta u_n + \Pi_n B(u_n, u_n), e_j \rangle$ and $\sigma_j^n(u_n) = -(\Pi_n G u_n)^* e_j$ for $u_n \in \mathbf{H}_n$. We have

$$\begin{aligned} |\sigma_j^n(u_n) - \sigma_j^n(v_n)|_{\mathbf{R}^2} &\leq C_1 ||u_n - v_n||_{\mathbf{V}}, \\ |F_j^{\delta,n}(u_n) - F_j^{\delta,n}(v_n)| &\leq C_1 ||u_n - v_n||_{\mathbf{V}}, \end{aligned}$$

for every $u_n, v_n \in \mathbf{H}_n$ with some $C_1 = C_1(\delta, n) > 0$. Therefore, for any $\delta > 0$, $n \geq 1$ and $T > 0$, we see that there exists a unique strong solution $u_n^\delta \in C([0,T]; \mathbf{H}_n)$, a.s.

Step 2: A priori estimate.

By applying Itô's formula to $||u_n^\delta(t)||_{\mathbf{V}}^2$, we have

$$
\begin{aligned}
&||u_n^\delta(t)||_{\mathbf{V}}^2 - ||u_n^\delta(0)||_{\mathbf{V}}^2 \qquad (10)\\
&\leq -\delta\mu \sum_{k=1}^{2} \int_0^t \left\|\partial_k u_n^\delta(s)\right\|_{\mathbf{V}}^2 ds - 2\int_0^t \langle\langle u_n^\delta(s), \Pi_n(u_n^\delta(s)\cdot\nabla u_n^\delta(s))\rangle\rangle ds \\
&+ \text{(martingale)}
\end{aligned}
$$

for any $t \in [0, T]$, a.s. for $T > 0$.
Note that

$$\langle\langle u_n^\delta(s), \triangle u_n^\delta(s)\rangle\rangle = -\sum_{k=1}^{2} \left\|\partial_k u_n^\delta(s)\right\|_{\mathbf{V}}^2.$$

Furthermore,

$$\langle\langle u_n^\delta(s), \Pi_n(u_n^\delta(s)\cdot\nabla u_n^\delta(s))\rangle\rangle = 0,$$

holds, This is shown by using the fact u is written to $u = \nabla^\perp\phi$ with ϕ a scalar function in the case of $\mathbf{T}^2$, where $\nabla^\perp = (\partial x_2, -\partial x_1)$. Thus, if $u_0 \in \mathbf{V}$,

$$\sup_{n\geq 1, \delta>0} \mathbf{E}^P \left\{ ||u_n^\delta(t)||_{\mathbf{V}}^2 \right\} < \infty,$$

in particular,

$$\sup_{n\geq 1, \delta>0} \mathbf{E}^P \left\{ \int_0^T ||u_n^\delta(t)||_{\mathbf{V}}^2 dt \right\} < \infty,$$

hold for each $T > 0$.

Step 3: tightness.

Chebyshev's inequality:
Let $(E, |\cdot|)$ be a separable metric space and $(X_n)_n$ E-valued random variables. Then,

$$\mathbf{P}(X_n \in (B_R)^c) \leq \frac{1}{R}\mathbf{E}\{|X_n|\}.$$

holds, where $B_R = \{x : |x| < R\}$.

tightness:
A sequence of the probability law of $(X_n)_n$ is tight if, for any $\epsilon > 0$, there exists a compact set $K_\epsilon \subset E$ such that

$$\mathbf{P}(X_n \in K) \geq 1 - \epsilon.$$

Prohorov's theorem:
A family of the probability law of $(X_n)_n$, which is devoted by $\{\mathcal{L}(X_n)\}_n$ is tight if and only if $\{\mathcal{L}(X_n)\}_n$ is relatively compact, ($\exists\{\mathcal{L}(X_{n(k)})\}_k$, $\bar{\mu}$ s.t. $\langle f, \mathcal{L}(X_{n(k)})\rangle \to \langle f, \bar{\mu}\rangle$ for any continuous and bounded f.)

compactness:

Let $E_1 \subset\subset E \subset E_2$ and E_1, E_2 be reflective Banach space. $\alpha \in (0,1)$ and $p > 1$ given. Then, any bounded set in $L^2(0,T;E_1) \cap W^{\alpha,p}(0,T;E_2)$ is relative compact in $L^2(0,T;E)$.

We need to estimate

$$\mathbf{E}\{\int_0^T ||u_n^\delta(s)||^2_{\mathbf{V}} ds\},$$
$$\mathbf{E}\{||u_n^\delta||_{\mathbf{W}^{\alpha,2}(0,T;\mathbf{V}')}\},$$

where u_n^δ is the unique strong solution of the finite dimensional SDE and $\mathbf{W}^{\alpha,2}(0,T;E)$ the fractional Sobolev space.

$\Longrightarrow$ We obtain the tightness in $L^2(0,T;\mathbf{H})$.

Step 4: taking a limit.

Skorohod embedding theorem:
Set $\Omega_T = L^2(0,T;\mathbf{V}) \cap W^{\alpha,2}([0,T];\mathbf{V}')$, the cordinate process $\xi(t,w) = w(t)$, $w \in \Omega_T$. There exist another probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}})$ $L^2(0,T;\mathbf{H})$-valued random variables $\{X_k\}_{k\in\mathbb{N}}$, X such that $\mathcal{L}(u^{\delta_k}) = \mathcal{L}(X_k)$, $\bar{\mu} = \mathcal{L}(X)$ and $\tilde{\mathbf{P}}$-a.s., $X_k \to X$ in $L^2(0,T;\mathbf{H})$ holds.

The unbounded case.

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)\, u + \sqrt{2\mu}\nabla u \cdot \dot{B}(t) - \mu\Delta u + \nabla p = 0, \qquad t > 0, x \in \mathbb{R}^2,$$
$$\operatorname{div} u = 0, \qquad t > 0, x \in \mathbb{R}^2,$$
$$u(0,x) = u_0(x), \qquad x \in \mathbb{R}^2,$$

Let us set

$$\mathbf{H}(\mathbb{R}^2) = \{u \in \mathbf{L}^2(\mathbb{R}^2; \mathbb{R}^2) \,|\, \text{div } u = 0\},$$
$$\mathbf{V}(\mathbb{R}^2) = \mathbf{W}^{1,2}(\mathbb{R}^2; \mathbb{R}^2) \cap \mathbf{H}(\mathbb{R}^2).$$

Theorem 2. *(W.Stannat, Y, 2011) There exists a weak solution with the initial value* $u_0 \in \mathbf{V}(\mathbb{R}^2)$ *with compact support.*

- an a priori estimate for a sequence of periodic solutions defined on $[-l, l]^2$, $l \in \mathbf{N}$,

- the cutoff argument

Thank you for listening.