Analytical Aspects of Complex Fluids

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Outline

The problem (NS)

Consider

$$\begin{aligned} \rho(\partial_t u + (u\cdot\nabla)u) &= f + \operatorname{div} T(u) - \nabla\pi && \text{in } J\times\Omega,\\ \operatorname{div}\ u &= 0 && \text{in } J\times\Omega,\\ u &= 0, && \text{on } J\times\partial\Omega,\\ u(0) &= u_0 && \text{in } \Omega, \end{aligned}$$

Here:

- u velocity of the fluid, π pressure of the fluid,
- u_0 initial velocity of the fluid, f extra body force,
- ρ density, $J=(0,T)$, $\Omega\subset\mathbb{R}^n$ domain.
- $T(u)$ = extra stress tensor.

Newtonian Fluids/Navier-Stokes equations

We set

$$T(u) := T_N(u) = \mu Du$$

Then,

$$\text{div } T(u) = \Delta u.$$

Here:

- $\mu > 0$ viscosity,
- $Du = \frac{1}{2}\left(\nabla u + (\nabla u)^T\right)$.

Generalized Newtonian Fluids

We set

$$T(u) := T_{GN}(u) = \mu(|Du|_2^2)Du$$

Here,

- μ viscosity function.

Examples:

- *Power-Law*: $\mu(|D\tilde{u}|_2^2) = \mu_0(1 + |D\tilde{u}|_2^2)^{\frac{d}{2}-1}$, $\mu_0 > 0$, $d \geq 1$.

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Generalized Viscoelastic Fluids

We set

$$T(u) := T_{GN}(u) + \tau,$$

where

$$\begin{aligned} \partial_t\tau + (u \cdot \nabla)\tau + b\tau &= g(\nabla u, \tau) && \text{in } J \times \Omega, \\ \tau(0) &= \tau_0 && \text{in } \Omega. \end{aligned}$$

Here:

- τ: elastic part of the stress,

Examples

- *Oldroyd-B fluids*: $\mu > 0$,

$$g(\nabla u, \tau) = \beta Du - \tau Wu + Wu\tau + a(Du\tau + \tau Du)$$

for $\beta > 0$, $-1 \leq a \leq 1$ and $Wu = \frac{1}{2}(\nabla u - \nabla u^T)$,

- *Generalized Oldroyd-B*: Replace constant β by $\beta(|Du|^2)$,
- *White-Metzner*: $\mu > 0$, $b = 0$, and

$$g(\nabla u, \tau) = \beta(|Du|^2)Du + \gamma(|Du|^2)\tau - \tau Wu + Wu\tau + a(Du\tau + \tau Du)$$

for some functions β and γ.

The Helmholtz projection

- Let $1 < q < \infty$, $\Omega \subset \mathbb{R}^n$ be a domain.
- We say that the *Helmholtz decomposition* exists if

$$L^q(\Omega)^n = L^q_\sigma(\Omega) \oplus G_q(\Omega),$$

where

$$G_q(\Omega) := \{g \in L^q(\Omega)^n : \exists h \in \widehat{W}^{1,q}(\Omega) \text{ such that } g = \nabla h\},$$
$$L^q_\sigma(\Omega) := \overline{\{\varphi \in C_c^\infty(\Omega)^n : div\,\varphi = 0\}}^{\|\cdot\|_{L^q(\Omega)}}$$
$$= \{f \in L^q(\Omega)^n : \int_\Omega f \nabla\varphi = 0,\ \varphi \in \widehat{W}^{1,q'}(\Omega)\}$$

In this case there exists the *Helmholtz projection*

$$P_q : L^q(\Omega)^n \to L^q_\sigma(\Omega).$$

Existence of the Helmholtz projection

- Let $1 < q < \infty$. Then the Helmholtz projection exists on $L^q(\Omega)^n$, where
  - $\Omega = \mathbb{R}^n$, $\Omega = \mathbb{R}^n_+$,
  - Ω bounded with smooth boundary,
  - Ω exterior domain with smooth boundary,
  - Ω layer,
  - ...
- The Helmholtz projection exists $L^q(\Omega)^n \cap L^2(\Omega)^n$, $2 < q < \infty$, or $L^q(\Omega)^n + L^2(\Omega)^n$, $1 < q < 2$, Ω uniform C^1.

Contributors: Farwig, Fujiwara, Kozono, Miyakawa, Morimoto, Simader, Sohr, Thäter, von Wahl, Weyl, ...

Existence of the Helmholtz projection II

The Helmholtz projection exists on $L^q(\Omega)^n$, where

- $\Omega \subset \mathbb{R}^n$, bounded Lipschitz domain and $q \in (\frac{3}{2} - \varepsilon, 3 + \varepsilon)$, Fabes, Mendez and Mitrea.
- $\Omega \subset \mathbb{R}^2$ 'unbounded wedge' (smooth and non smooth), q depends on angle, Bogovskii.

Remark

The results above are sharp.

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The Stokes operator

Let $1 < q < \infty$ and $\Omega \subset \mathbb{R}^n$ be a domain such that the Helmholtz projection exists. Set

$$D(A_q) = W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) \cap L^q_\sigma(\Omega)$$

and define the *Stokes operator*

$$A_q : \begin{cases} D(A_q) & \rightarrow & L^q_\sigma(\Omega), \\ u & \mapsto & P_q \Delta u. \end{cases}$$

Maximal Regularity

We say that A_q has *maximal L^p-regularity in $L^q_\sigma(\Omega)$* if for

$$f \in L^p(J; L^q_\sigma(\Omega))$$

there exists a unique

$$u \in W^{1,p}(J; L^q_\sigma(\Omega)) \cap L^p(J; D(A_q))$$

satisfying

$$\begin{aligned} u'(t) - A_q u(t) &= f(t), \quad t \in J, \\ u(0) &= 0. \end{aligned}$$

Known results on the Stokes operator

- Let $1 < p, q < \infty$. Then A_q has maximal L^p-regularity in $L^q_\sigma(\Omega)$, where
  - $\Omega = \mathbb{R}^n$, $\Omega = \mathbb{R}^n_+$,
  - Ω bounded with smooth boundary,
  - Ω exterior domain with smooth boundary,
  - Ω layer,
  - ...
- A_q has maximal L^p-regularity on $L^q_\sigma(\Omega) \cap L^2_\sigma(\Omega)$, $2 < q < \infty$, or $L^q\sigma(\Omega) + L^2_\sigma(\Omega)$, $1 < q < 2$, uniform C^2.
- Helmholtz exists +(suitable decomposition of pressure) $\Rightarrow$ A_q has maximal L^p-regularity on $L^q_\sigma(\Omega)$

Contributors: Amann, Abels, Borchers, Desch, Farwig, Fujita, Fujiwara, Galdi, Giga, Grubb, Hieber, Hishida, Kato, Masuda, Miyakawa, Morimoto, Prüss, Shibata, Shimizu, Simader, Sohr, Solonnikov, Ukai, Varnhorn, Wiegner ...

Maximal L^p-Regularity

In this case, for $f \in L^p(J; L^q(\Omega)^n)$ and $u_0 \in (L^q_\sigma(\Omega), D(A_q))_{1-\frac{1}{p},p}$ there exists a unique

$$(u, \pi) \in X_u(T) \times X_\pi(T),$$

satisfying

$$\begin{array}{rcll} \partial_t u - \Delta u + \nabla \pi & = & f & \text{in } J \times \Omega, \\ \nabla \cdot u & = & 0 & \text{in } J \times \Omega, \\ u & = & 0 & \text{on } J \times \partial\Omega, \\ u(0, \cdot) & = & u_0 & \text{in } \Omega. \end{array}$$

Here,

$$X_u(T) := W^{1,p}(J; L^q_\sigma(\Omega)) \cap L^p(J; W^{2,q}(\Omega)), \; X_\pi(T) := L^p(J; \widehat{W}^{1,q}(\Omega)).$$

Maximal L^p-Regularity

In this case, for $f \in L^p(J; L^q(\Omega)^n)$ and $u_0 \in (L^q_\sigma(\Omega), D(A_q))_{1-\frac{1}{p},p}$ there exists a unique

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Moreover,

$$\|u\|_{X_u(T)} + \|\pi\|_{Y_u(T)} \leq C \left(\|f\|_{T,p,q} + \|u_0\|_{(L^q_\sigma(\Omega), D(A_q))_{1-\frac{1}{p},p}} \right), \quad T \in (0, T_0).$$

Proposition (WMR)

Let

- $p, q, q' \in (1, <\infty)$ *with* $\frac{1}{q} + \frac{1}{q'} = 1$, $T \in (0, T_0)$,
- Ω *uniformly* C^2*-domain,*
- $P_r : L_r(\Omega) \to L_{r,\sigma}(\Omega)$ *exists for* $r \in \{q, q'\}$,
- $\lambda + A_r \in BIP(\theta)$, $\theta < \pi/2$ *for* $r \in \{q, q'\}$ *and for some* $\lambda \geq 0$

Then for $u_0 = 0$ *and* $f = \operatorname{div} F$, $F \in L_p(0, T; H_q^1(\Omega))$ *the unique solution* $(u, \pi) \in X_u \times X_\pi$ *satisfies*

$$\|u\|_{Y_u(T)} \leq C\|F\|_{T,p,q},$$

where $C > 0$ *is independent of* F *and* T, $T \in (0, T_0)$.

Here:

$$Y_u(T) := H^{1/2,p}(0, T; L^q(\Omega)) \cap L^p(0, T; H^{1,q}(\Omega)).$$

Generalized Newtonian Fluids

Proposition

Let

- $p > n + 2$,
- $\Omega \subset \mathbb{R}^n$ *bounded, class* $C^{2,1}$,
- $\mu \in C^{1,1}(\mathbb{R}_+)$ *satisfying*

$$\mu(s) > 0, \qquad \mu(s) + 2s\mu'(s) > 0, \quad s \geq 0,$$

Then, for $f \in L^p(0, T, L^p(\Omega))$, $u_0 \in W^{2-2/p,p}(\Omega)$ *satisfying* $\operatorname{div} u_0 = 0$ *and* $u = 0$ *on* $\partial\Omega$, *there exists a unique solution* $(u, \pi) \in X_u(T) \times Y_u(T)$ *of (NS) with* $T(u) = T_{GN(u)}$.

Proposition

Let

- $p, q \in (1, \infty)$ *satisfy* $\frac{1}{p} + \frac{n}{2q} < \frac{1}{2}$, $T_0 > 0$,
- $\Omega \subset \mathbb{R}^n$ *uniform* C^2*-domain.*

Then for $T \in (0, T_0)$

$$X_u(T) \hookrightarrow L_\infty(0, T; W^1_\infty(\Omega)) \cap L_\infty(0, T; L_q(\Omega))$$
$$Y_u(T) \hookrightarrow L_\infty(0, T; L_\infty(\Omega)).$$

Moreover, if $u(0) = 0$

$$\|u\|_{L_\infty(0,T;W^1_\infty(\Omega))} + \|u\|_{T,\infty,q} \leq C\|u\|_{X_u(T)}, \quad T \in (0, T_0],\ u \in X_u(T),$$
$$\|u\|_{T,\infty,\infty} \leq C\|u\|_{Y_u(T)}, \quad T \in (0, T_0],\ u \in Y_u(T).$$

Consider

$$\begin{aligned} \partial_t \tau + (u \cdot \nabla)\tau + b\tau &= g \qquad \text{in } J \times \Omega, \\ \tau(0) &= \tau_0 \qquad \text{in } \Omega. \end{aligned} \tag{1}$$

for

- $1 < p < \infty$, $n < q < \infty$, $T \in (0, T_0)$,
- $u \in X_u(T)$ such that $u \cdot \nu = 0$ on $\partial\Omega$, $b \geq 0$,
- $\Omega \subset \mathbb{R}^n$ a uniform C^2-domain.

We set

$$\begin{aligned} X_\tau(T) &:= L^\infty(0, T; H^{1,q}(\Omega)), \\ Y_\tau(T) &:= L^\infty(0, T; L^q(\Omega)). \end{aligned}$$

Consider

$$\begin{aligned} \partial_t \tau + (u \cdot \nabla)\tau + b\tau &= g \qquad \text{in } J \times \Omega, \\ \tau(0) &= \tau_0 \qquad \text{in } \Omega. \end{aligned} \tag{1}$$

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Proposition

For

- $g \in L^1(0,T;H^{1,q}(\Omega)) \cap L^\infty(0,T;L^q(\Omega))$, $\tau_0 \in H^{1,q}(\Omega)$

there exists a unique solution $\tau \in X_\tau(T) \cap W^{1,\infty}(0,T;L^q(\Omega))$ *of* (1) *such that*

$$\|\tau\|_{X_\tau(T)} \leq C_1 \left(\|\tau_0\|_{H^{1,q}(\Omega)} + \|g\|_{L^1(0,T;H^{1,q}(\Omega))} \right) e^{C_1 T^{1-1/p} \|u\|_{L^p(0,T;H^{2,q}(\Omega))}}.$$

Moreover,

$$\|\tau\|_{Y_\tau(T)} \leq C_2 \left(\|\tau_0\|_q + \|g\|_{T,1,q} \right) e^{C_2 \|\operatorname{div} \tilde{u}\|_{L^1(0,T;H^{1,q}(\Omega))}}.$$

Here:

- C_1, C_2 independent of g, τ_0, u and $T \in (0, T_0)$,

Large initial data: Part I

Proposition

Let

- $p > n + 2$, $\Omega \subset \mathbb{R}^n$ *bounded, class* $C^{2,1}$, $g \in C^1$,
- $\mu \in C^{1,1}(\mathbb{R}_+)$ *satisfying*

$$\mu(s) > 0, \qquad \mu(s) + 2s\mu'(s) > 0, \quad s \geq 0.$$

Then, for $f \in L^p(0, T, L^p(\Omega))$, $u_0 \in W^{2-2/p,p}(\Omega)$ *satisfying* $\operatorname{div} u_0 = 0$ *and* $u = 0$ *on* $\partial\Omega$, $\tau_0 \in W^{1,p}(\Omega)$ *there exists a unique solution*

$$(u, \pi, \tau) \in X_u(T) \times Y_u(T) \times (X_\tau(T) \cap W^{1,\infty}(0, T; L^q(\Omega))$$

of (NS) with $T(u) = T_V(u)$.

Idea of Proof

Rewrite as fixed point problem:

$$
\begin{aligned}
\partial_t u - \operatorname{div} T_{GN}(u) + \nabla \pi &= f - (\tilde{u} \cdot \nabla)\tilde{u}) + \operatorname{div} \tilde{\tau} && \text{in } J \times \Omega, \\
\operatorname{div} \ u &= 0 && \text{in } J \times \Omega, \\
\partial_t \tau + (\tilde{u} \cdot \nabla)\tau + b\tau &= g(\nabla \tilde{u}, \tilde{\tau}) && \text{in } J \times \Omega, \\
u &= 0, && \text{on } J \times \partial\Omega, \\
u(0) &= u_0 && \text{in } \Omega, \\
\tau(0) &= \tau_0 && \text{in } \Omega.
\end{aligned}
$$

Define

$$
\Phi : \begin{cases} X_u^A(T_0) \times X_\tau(T_0) & \rightarrow X_u^A(T_0) \times X_\tau(T_0) \\ (\tilde{u}, \tilde{\tau}) & \mapsto (u, \tau) \end{cases} .
$$

Here:

$$
X_u^A(T) := X_u(T) \cap L^p(0, T; D(A_q))
$$

Fixed point argument

We set

$$K_u(T,R_1) := \{u \in X_u^A(T) : u(0) = u_0, \|u - u_*\|_{X_u(T)} \leq R_1\}$$
$$K_\tau(T,R_2) := \{\tau \in X_\tau(T) : \tau(0) = \tau_0, \|\tau\|_{X_\tau(T)} \leq R_2\}$$
$$K(T,R_1,R_2) := K_u(T_0,R_1) \times K_\tau(T,R_2)$$

Fixed point argument

Lemma

Let

- $\frac{1}{p} + \frac{n}{2q} < \frac{1}{2}$

Then, there exists $T_0, R_0 > 0$ *such that for* $T \in (0, T_0)$*,* $R \in (0, R_0)$*:*

$$\Phi(K(T,R,R)) \subset K(T,R,R).$$

Moreover,

$$\|g(\nabla u, \tau)\|_{T,\infty,q} \leq C, \quad (u,\tau) \in K(T,R,R).$$

Fixed point argument

Apply Schauder's fixed point theorem in

$$Z(T) := C([0,T]; C^1(\overline{\Omega})) \times C([0,T], C(\overline{\Omega}))$$

We show

- $K(T, R_0, R_0)$ closed w.r.t. topology of $Z(T)$.
- Continuity of Φ in topology of $Z(T)$.

Fixed point argument

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Apply Schauder's fixed point theorem in

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We show

- $K(T, R_0, R_0)$ closed w.r.t. topology of $Z(T)$.
- Continuity of Φ in topology of $Z(T)$.

Large initial data: Part II

Proposition

Let

- *assumptions of Proposition WMR fulfilled,*
- $g \in C^1$, $g(0,0) = 0$
- $\mu > 0$

Then, for $f \in L^p(0,T,L^p(\Omega))$, $u_0 \in W^{2-2/p,p}(\Omega)$ *satisfying* $\operatorname{div} u_0 = 0$ *and* $u = 0$ *on* $\partial\Omega$, $\tau_0 \in W^{1,p}(\Omega)$ *there exists a unique solution*

$$(u,\pi,\tau) \in X_u(T) \times Y_u(T) \times (X_\tau(T) \cap W^{1,\infty}(0,T;L^q(\Omega))$$

of (NS) with $T(u) = T_V(u)$.

Fixed point argument

Lemma

Let

- $\frac{1}{p} + \frac{n}{2q} < \frac{1}{2}$

Then, there exists $T_0, R_0 > 0$ *such that for* $T \in (0, T_0)$*,* $R \in (0, R_0)$*:*

$$\Phi(K(T,R,R)) \subset K(T,R,R).$$

Moreover,

$$\|g(\nabla u, \tau)\|_{T,\infty,q} \leq C, \quad (u,\tau) \in K(T,R,R).$$

Proposition

Let

- X *be a reflexive Banach space,*
- K *be a convex, closed and bounded subset of* X*,*
- $X \hookrightarrow Y$*, where* Y *is a Banach space,*
- $\Phi : X \to X$ *map* K *into* K*,*
- *for some* $\eta \in (0, 1)$*,*

$$\|\Phi(x) - \Phi(y)\|_Y \leq \eta \|x - y\|_Y, \quad x, y \in K.$$

Then there exists a unique fixed point of Φ *in* K*.*

Lemma

Let

- $\varepsilon > 0$, $p, q \in (1, \infty)$ *such that* $\frac{1}{p} + \frac{n}{2q} < \frac{1}{2}$.

Then, there exists $T_0, R_0 > 0$ *such that for* $T \in (0, T_0)$*,* $R \in (0, R_0)$

$$\|\tilde{u}_2 \otimes \tilde{u}_2 - \tilde{u}_1 \otimes \tilde{u}_1\|_{T,p,q} \leq \varepsilon \|\tilde{u}_2 - \tilde{u}_1\|_{Y_u(T)},$$

$$\|\tilde{\tau}_2 - \tilde{\tau}_1\|_{T,p,q} \leq \varepsilon \|\tilde{\tau}_2 - \tilde{\tau}_1\|_{Y_\tau(T)},$$

$$\|g(\nabla\tilde{u}_2, \tilde{\tau}_2) - g(\nabla\tilde{u}_1, \tilde{\tau}_1)\|_{T,1,q} \leq \varepsilon \left(\|\tilde{u}_2 - \tilde{u}_1\|_{Y_u(T)} + \|\tilde{\tau}_2 - \tilde{\tau}_1\|_{Y_\tau(T)}\right),$$

$$\|(\tilde{u}_2 - \tilde{u}_1).\nabla\tau_1\|_{T,1,q} \leq \varepsilon \|\tilde{u}_2 - \tilde{u}_1\|_{Y_u(T)}$$

for $(\tilde{u}_i, \tilde{\tau}_i), (\tilde{u}, \tilde{\tau}), (u_i, \tau_i) \in K(T, R, R)$*,* $i = 1, 2$*,*