

# Analytical Aspects of Complex Fluids

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# Outline

- 1 Introduction
- 2 Preliminaries
  - The Helmholtz decomposition
  - The Stokes operator
  - Generalized Newtonian Fluids
  - Embeddings
  - Transport Equation
- 3 Viscoelastic Fluids
  - Large data
  - Small data
  - Further/Related Results
- 4  $\mathcal{H}^\infty$ -calculus for the Stokes Operator

# The problem (NS)

Consider

$$\begin{aligned} \rho(\partial_t u + (u \cdot \nabla)u) &= f + \operatorname{div} T(u) - \nabla \pi && \text{in } J \times \Omega, \\ \operatorname{div} u &= 0 && \text{in } J \times \Omega, \\ u &= 0, && \text{on } J \times \partial\Omega, \\ u(0) &= u_0 && \text{in } \Omega, \end{aligned}$$

Here:

- $u$  velocity of the fluid,  $\pi$  pressure of the fluid,
- $u_0$  initial velocity of the fluid,  $f$  extra body force,
- $\rho$  density,  $J = (0, T)$ ,  $\Omega \subset \mathbb{R}^n$  domain.
- $T(u) =$  extra stress tensor.

# Newtonian Fluids/Navier-Stokes equations

We set

$$T(u) := T_N(u) = \mu Du$$

Then,

$$\operatorname{div} T(u) = \Delta u.$$

Here:

- $\mu > 0$  viscosity,
- $Du = \frac{1}{2} (\nabla u + (\nabla u)^T)$ .

# Generalized Newtonian Fluids

We set

$$T(u) := T_{GN}(u) = \mu(|Du|_2^2) Du$$

Here,

- $\mu$  viscosity function.

Examples:

- *Power-Law*:  $\mu(|D\tilde{u}|_2^2) = \mu_0(1 + |D\tilde{u}|_2^2)^{\frac{d}{2}-1}$ ,  $\mu_0 > 0$ ,  $d \geq 1$ .

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# Generalized Viscoelastic Fluids

We set

$$T(u) := T_{GN}(u) + \tau,$$

where

$$\begin{aligned} \partial_t \tau + (u \cdot \nabla) \tau + b\tau &= g(\nabla u, \tau) && \text{in } J \times \Omega, \\ \tau(0) &= \tau_0 && \text{in } \Omega. \end{aligned}$$

Here:

- $\tau$ : elastic part of the stress,

# Examples

- *Oldroyd-B fluids*:  $\mu > 0$ ,

$$g(\nabla u, \tau) = \beta Du - \tau Wu + Wu\tau + a(Du\tau + \tau Du)$$

for  $\beta > 0$ ,  $-1 \leq a \leq 1$  and  $Wu = \frac{1}{2}(\nabla u - \nabla u^T)$ ,

- *Generalized Oldroyd-B*: Replace constant  $\beta$  by  $\beta(|Du|^2)$ ,
- *White-Metzner*:  $\mu > 0$ ,  $b = 0$ , and

$$g(\nabla u, \tau) = \beta(|Du|^2)Du + \gamma(|Du|^2)\tau - \tau Wu + Wu\tau + a(Du\tau + \tau Du)$$

for some functions  $\beta$  and  $\gamma$ .



# The Helmholtz projection

- Let  $1 < q < \infty$ ,  $\Omega \subset \mathbb{R}^n$  be a domain.
- We say that the *Helmholtz decomposition* exists if

$$L^q(\Omega)^n = L_\sigma^q(\Omega) \oplus G_q(\Omega),$$

where

$$G_q(\Omega) := \{g \in L^q(\Omega)^n : \exists h \in \widehat{W}^{1,q}(\Omega) \text{ such that } g = \nabla h\},$$

$$L_\sigma^q(\Omega) := \{\varphi \in C_c^\infty(\Omega)^n : \operatorname{div} \varphi = 0\} \|\cdot\|_{L^q(\Omega)}$$

$$= \{f \in L^q(\Omega)^n : \int_\Omega f \nabla \varphi = 0, \varphi \in \widehat{W}^{1,q'}(\Omega)\}$$

In this case there exists the *Helmholtz projection*

$$P_q : L^q(\Omega)^n \rightarrow L_\sigma^q(\Omega).$$

# Existence of the Helmholtz projection

- Let  $1 < q < \infty$ . Then the Helmholtz projection exists on  $L^q(\Omega)^n$ , where
  - $\Omega = \mathbb{R}^n, \Omega = \mathbb{R}_+^n$ ,
  - $\Omega$  bounded with smooth boundary,
  - $\Omega$  exterior domain with smooth boundary,
  - $\Omega$  layer,
  - ...
- The Helmholtz projection exists  $L^q(\Omega)^n \cap L^2(\Omega)^n$ ,  
 $2 < q < \infty$ , or  $L^q(\Omega)^n + L^2(\Omega)^n$ ,  $1 < q < 2$ ,  $\Omega$  uniform  $C^1$ .

**Contributors:** Farwig, Fujiwara, Kozono, Miyakawa, Morimoto, Simader, Sohr, Thäter, von Wahl, Weyl, ...

# Existence of the Helmholtz projection II

The Helmholtz projection exists on  $L^q(\Omega)^n$ , where

- $\Omega \subset \mathbb{R}^n$ , bounded Lipschitz domain and  $q \in (\frac{3}{2} - \varepsilon, 3 + \varepsilon)$ , Fabes, Mendez and Mitrea.
- $\Omega \subset \mathbb{R}^2$  'unbounded wedge' (smooth and non smooth),  $q$  depends on angle, Bogovskii.

## Remark

*The results above are sharp.*

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## Remark

*The results above are sharp.*

# The Stokes operator

Let  $1 < q < \infty$  and  $\Omega \subset \mathbb{R}^n$  be a domain such that the Helmholtz projection exists. Set

$$D(A_q) = W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) \cap L_\sigma^q(\Omega)$$

and define the *Stokes operator*

$$A_q : \begin{cases} D(A_q) & \rightarrow L_\sigma^q(\Omega), \\ u & \mapsto P_q \Delta u. \end{cases}$$

# Maximal Regularity

We say that  $A_q$  has *maximal  $L^p$ -regularity in  $L^q_\sigma(\Omega)$*  if for

$$f \in L^p(J; L^q_\sigma(\Omega))$$

there exists a unique

$$u \in W^{1,p}(J; L^q_\sigma(\Omega)) \cap L^p(J; D(A_q))$$

satisfying

$$\begin{aligned} u'(t) - A_q u(t) &= f(t), & t \in J, \\ u(0) &= 0. \end{aligned}$$

# Known results on the Stokes operator

- Let  $1 < p, q < \infty$ . Then  $A_q$  has maximal  $L^p$ -regularity in  $L_\sigma^q(\Omega)$ , where
  - $\Omega = \mathbb{R}^n, \Omega = \mathbb{R}_+^n$ ,
  - $\Omega$  bounded with smooth boundary,
  - $\Omega$  exterior domain with smooth boundary,
  - $\Omega$  layer,
  - ...
- $A_q$  has maximal  $L^p$ -regularity on  $L_\sigma^q(\Omega) \cap L_\sigma^2(\Omega)$ ,  $2 < q < \infty$ , or  $L^q\sigma(\Omega) + L_\sigma^2(\Omega)$ ,  $1 < q < 2$ , uniform  $C^2$ .
- Helmholtz exists +( suitable decomposition of pressure )  $\Rightarrow$   $A_q$  has maximal  $L^p$ -regularity on  $L_\sigma^q(\Omega)$

Contributors: Amann, Abels, Borchers, Desch, Farwig, Fujita, Fujiwara, Galdi, Giga, Grubb, Hieber, Hishida, Kato, Masuda, Miyakawa, Morimoto, Prüss, Shibata, Shimizu, Simader, Sohr, Solonnikov, Ukai, Varnhorn, Wiegner ...

# Maximal $L^p$ -Regularity

In this case, for  $f \in L^p(J; L^q(\Omega)^n)$  and  $u_0 \in (L^q_\sigma(\Omega), D(A_q))_{1-\frac{1}{p}, p}$  there exists a unique

$$(u, \pi) \in X_u(T) \times X_\pi(T),$$

satisfying

$$\begin{aligned} \partial_t u - \Delta u + \nabla \pi &= f && \text{in } J \times \Omega, \\ \nabla \cdot u &= 0 && \text{in } J \times \Omega, \\ u &= 0 && \text{on } J \times \partial\Omega, \\ u(0, \cdot) &= u_0 && \text{in } \Omega. \end{aligned}$$

Here,

$$X_u(T) := W^{1,p}(J; L^q_\sigma(\Omega)) \cap L^p(J; W^{2,q}(\Omega)), \quad X_\pi(T) := L^p(J; \widehat{W}^{1,q}(\Omega)).$$



# Maximal $L^p$ -Regularity

In this case, for  $f \in L^p(J; L^q(\Omega)^n)$  and  $u_0 \in (L_\sigma^q(\Omega), D(A_q))_{1-\frac{1}{p}, p}$  there exists a unique

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Moreover,

$$\|u\|_{X_u(T)} + \|\pi\|_{Y_u(T)} \leq C \left( \|f\|_{T,p,q} + \|u_0\|_{(L_\sigma^q(\Omega), D(A_q))_{1-\frac{1}{p}, p}} \right), \quad T \in (0, T_0).$$

## Proposition (WMR)

Let

- $p, q, q' \in (1, < \infty)$  with  $\frac{1}{q} + \frac{1}{q'} = 1$ ,  $T \in (0, T_0)$ ,
- $\Omega$  uniformly  $C^2$ -domain,
- $P_r : L_r(\Omega) \rightarrow L_{r,\sigma}(\Omega)$  exists for  $r \in \{q, q'\}$ ,
- $\lambda + A_r \in BIP(\theta)$ ,  $\theta < \pi/2$  for  $r \in \{q, q'\}$  and for some  $\lambda \geq 0$

Then for  $u_0 = 0$  and  $f = \operatorname{div} F$ ,  $F \in L_p(0, T; H_q^1(\Omega))$  the unique solution  $(u, \pi) \in X_u \times X_\pi$  satisfies

$$\|u\|_{Y_u(T)} \leq C \|F\|_{T,p,q},$$

where  $C > 0$  is independent of  $F$  and  $T$ ,  $T \in (0, T_0)$ .

Here:

$$Y_u(T) := H^{1/2,p}(0, T; L^q(\Omega)) \cap L^p(0, T; H^{1,q}(\Omega)).$$

# Generalized Newtonian Fluids

## Proposition

*Let*

- $p > n + 2$ ,
- $\Omega \subset \mathbb{R}^n$  *bounded, class  $C^{2,1}$ ,*
- $\mu \in C^{1,1}(\mathbb{R}_+)$  *satisfying*

$$\mu(s) > 0, \quad \mu(s) + 2s\mu'(s) > 0, \quad s \geq 0,$$

*Then, for  $f \in L^p(0, T, L^p(\Omega))$ ,  $u_0 \in W^{2-2/p, p}(\Omega)$  satisfying  $\operatorname{div} u_0 = 0$  and  $u = 0$  on  $\partial\Omega$ , there exists a unique solution  $(u, \pi) \in X_u(T) \times Y_u(T)$  of (NS) with  $T(u) = T_{GN(u)}$ .*

## Proposition

Let

- $p, q \in (1, \infty)$  satisfy  $\frac{1}{p} + \frac{n}{2q} < \frac{1}{2}$ ,  $T_0 > 0$ ,
- $\Omega \subset \mathbb{R}^n$  uniform  $C^2$ -domain.

Then for  $T \in (0, T_0)$

$$X_u(T) \hookrightarrow L_\infty(0, T; W_\infty^1(\Omega)) \cap L_\infty(0, T; L_q(\Omega))$$

$$Y_u(T) \hookrightarrow L_\infty(0, T; L_\infty(\Omega)).$$

Moreover, if  $u(0) = 0$

$$\|u\|_{L_\infty(0, T; W_\infty^1(\Omega))} + \|u\|_{T, \infty, q} \leq C \|u\|_{X_u(T)}, \quad T \in (0, T_0], \quad u \in X_u(T),$$

$$\|u\|_{T, \infty, \infty} \leq C \|u\|_{Y_u(T)}, \quad T \in (0, T_0], \quad u \in Y_u(T).$$

Consider

$$\begin{aligned} \partial_t \tau + (u \cdot \nabla) \tau + b \tau &= g && \text{in } J \times \Omega, \\ \tau(0) &= \tau_0 && \text{in } \Omega. \end{aligned} \tag{1}$$

for

- $1 < p < \infty, n < q < \infty, T \in (0, T_0)$ ,
- $u \in X_u(T)$  such that  $u \cdot \nu = 0$  on  $\partial\Omega$ ,  $b \geq 0$ ,
- $\Omega \subset \mathbb{R}^n$  a uniform  $C^2$ -domain.

We set

$$\begin{aligned} X_\tau(T) &:= L^\infty(0, T; H^{1,q}(\Omega)), \\ Y_\tau(T) &:= L^\infty(0, T; L^q(\Omega)). \end{aligned}$$

Consider

$$\begin{aligned} \partial_t \tau + (u \cdot \nabla) \tau + b \tau &= g && \text{in } J \times \Omega, \\ \tau(0) &= \tau_0 && \text{in } \Omega. \end{aligned} \tag{1}$$

for

- $1 < p < \infty, n < q < \infty, T \in (0, T_0)$ ,
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We set

$$\begin{aligned} X_\tau(T) &:= L^\infty(0, T; H^{1,q}(\Omega)), \\ Y_\tau(T) &:= L^\infty(0, T; L^q(\Omega)). \end{aligned}$$

## Proposition

For

- $g \in L^1(0, T; H^{1,q}(\Omega)) \cap L^\infty(0, T; L^q(\Omega))$ ,  $\tau_0 \in H^{1,q}(\Omega)$

there exists a unique solution  $\tau \in X_\tau(T) \cap W^{1,\infty}(0, T; L^q(\Omega))$  of (1) such that

$$\|\tau\|_{X_\tau(T)} \leq C_1 \left( \|\tau_0\|_{H^{1,q}(\Omega)} + \|g\|_{L^1(0,T;H^{1,q}(\Omega))} \right) e^{C_1 T^{1-1/p} \|u\|_{L^p(0,T;H^{2,q}(\Omega))}}.$$

Moreover,

$$\|\tau\|_{Y_\tau(T)} \leq C_2 (\|\tau_0\|_q + \|g\|_{T,1,q}) e^{C_2 \|\operatorname{div} \tilde{u}\|_{L^1(0,T;H^{1,q}(\Omega))}}.$$

Here:

- $C_1, C_2$  independent of  $g, \tau_0, u$  and  $T \in (0, T_0)$ ,

# Large initial data: Part I

## Theorem

Let

- $p > n + 2$ ,  $\Omega \subset \mathbb{R}^n$  bounded, class  $C^{2,1}$ ,  $g \in C^1$ ,
- $\mu \in C^{1,1}(\mathbb{R}_+)$  satisfying

$$\mu(s) > 0, \quad \mu(s) + 2s\mu'(s) > 0, \quad s \geq 0.$$

Then, for  $f \in L^p(0, T, L^p(\Omega))$ ,  $u_0 \in W^{2-2/p,p}(\Omega)$  satisfying  $\operatorname{div} u_0 = 0$  and  $u = 0$  on  $\partial\Omega$ ,  $\tau_0 \in W^{1,p}(\Omega)$  there exists a unique solution

$$(u, \pi, \tau) \in X_u(T) \times Y_u(T) \times (X_\tau(T) \cap W^{1,\infty}(0, T; L^q(\Omega)))$$

of (NS) with  $T(u) = T_V(u)$ .



# Idea of Proof: Large data

Rewrite as fixed point problem:

$$\begin{aligned}
 \partial_t u - \operatorname{div} T_{GN}(u) + \nabla \pi &= f - (\tilde{u} \cdot \nabla) \tilde{u} + \operatorname{div} \tilde{\tau} && \text{in } J \times \Omega, \\
 \operatorname{div} u &= 0 && \text{in } J \times \Omega, \\
 \partial_t \tau + (\tilde{u} \cdot \nabla) \tau + b\tau &= g(\nabla \tilde{u}, \tilde{\tau}) && \text{in } J \times \Omega, \\
 u &= 0, && \text{on } J \times \partial\Omega, \\
 u(0) &= u_0 && \text{in } \Omega, \\
 \tau(0) &= \tau_0 && \text{in } \Omega.
 \end{aligned}$$

Define

$$\Phi : \begin{cases} X_u^A(T_0) \times X_\tau(T_0) & \rightarrow X_u^A(T_0) \times X_\tau(T_0) \\ (\tilde{u}, \tilde{\tau}) & \mapsto (u, \tau) \end{cases}.$$

Here:

$$X_u^A(T) := X_u(T) \cap L^p(0, T; D(A_q))$$

# Fixed point argument: Large data

We set

$$K_u(T, R_1) := \{u \in X_u^A(T) : u(0) = u_0, \|u - u_*\|_{X_u(T)} \leq R_1\}$$

$$K_\tau(T, R_2) := \{\tau \in X_\tau(T) : \tau(0) = \tau_0, \|\tau\|_{X_\tau(T)} \leq R_2\}$$

$$K(T, R_1, R_2) := K_u(T_0, R_1) \times K_\tau(T, R_2)$$

# Fixed point argument: Large data

## Lemma

Let

- $\frac{1}{p} + \frac{n}{2q} < \frac{1}{2}$

Then, there exists  $T_0, R_0 > 0$  such that for  $T \in (0, T_0)$ ,  
 $R \in (0, R_0)$ :

$$\Phi(K(T, R, R)) \subset K(T, R, R).$$

Moreover,

$$\|g(\nabla u, \tau)\|_{T, \infty, q} \leq C, \quad (u, \tau) \in K(T, R, R).$$

# Fixed point argument: Large data

Apply Schauder's fixed point theorem in

$$Z(T) := C([0, T]; C^1(\overline{\Omega})) \times C([0, T], C(\overline{\Omega}))$$

We show

- $K(T, R_0, R_0)$  closed w.r.t. topology of  $Z(T)$ .
- Continuity of  $\Phi$  in topology of  $Z(T)$ .

# Fixed point argument: Large data

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We show

- $K(T, R_0, R_0)$  closed w.r.t. topology of  $Z(T)$ .
- Continuity of  $\Phi$  in topology of  $Z(T)$ .

# Large initial data: Part II

## Theorem

Let

- $p, q \in (1, \infty)$  satisfying  $\frac{1}{p} + \frac{n}{2q} \leq \frac{1}{2}$ .
- assumptions of Proposition WMR fulfilled,
- $g \in C^1$ ,  $g(0, 0) = 0$
- $\mu > 0$

Then, for  $f \in L^p(0, T, L^p(\Omega))$ ,  $u_0 \in W^{2-2/p, p}(\Omega)$  satisfying  $\operatorname{div} u_0 = 0$  and  $u = 0$  on  $\partial\Omega$ ,  $\tau_0 \in W^{1, p}(\Omega)$  there exists a unique solution

$$(u, \pi, \tau) \in X_u(T) \times Y_u(T) \times (X_\tau(T) \cap W^{1, \infty}(0, T; L^q(\Omega)))$$

of (NS) with  $T(u) = T_V(u)$ .

# Fixed point argument: Large data

## Lemma

Let

- $\frac{1}{p} + \frac{n}{2q} < \frac{1}{2}$

Then, there exists  $T_0, R_0 > 0$  such that for  $T \in (0, T_0)$ ,  
 $R \in (0, R_0)$ :

$$\Phi(K(T, R, R)) \subset K(T, R, R).$$

Moreover,

$$\|g(\nabla u, \tau)\|_{T, \infty, q} \leq C, \quad (u, \tau) \in K(T, R, R).$$



# A fixed point argument

## Proposition

Let

- $X$  be a reflexive Banach space or  $X$  has a separable pre-dual,
- $K$  be a convex, closed and bounded subset of  $X$ ,
- $X \hookrightarrow Y$ , where  $Y$  is a Banach space,
- $\Phi: X \rightarrow X$  map  $K$  into  $K$ ,
- for some  $\eta \in (0, 1)$ ,

$$\|\Phi(x) - \Phi(y)\|_Y \leq \eta \|x - y\|_Y, \quad x, y \in K.$$

Then there exists a unique fixed point of  $\Phi$  in  $K$ .

# Fixed point argument: Large data

## Lemma

Let

- $p, q \in (1, \infty)$  such that  $\frac{1}{p} + \frac{n}{2q} < \frac{1}{2}$ .

Then, there exists  $T_0, R_0 > 0$  such that for  $T \in (0, T_0)$ ,  $R \in (0, R_0)$

$$\|\tilde{u}_2 \otimes \tilde{u}_2 - \tilde{u}_1 \otimes \tilde{u}_1\|_{T,p,q} \leq \varepsilon \|\tilde{u}_2 - \tilde{u}_1\|_{Y_u(T)},$$

$$\|\tilde{\tau}_2 - \tilde{\tau}_1\|_{T,p,q} \leq \varepsilon \|\tilde{\tau}_2 - \tilde{\tau}_1\|_{Y_\tau(T)},$$

$$\|g(\nabla \tilde{u}_2, \tilde{\tau}_2) - g(\nabla \tilde{u}_1, \tilde{\tau}_1)\|_{T,1,q} \leq \varepsilon (\|\tilde{u}_2 - \tilde{u}_1\|_{Y_u(T)} + \|\tilde{\tau}_2 - \tilde{\tau}_1\|_{Y_\tau(T)}),$$

$$\|(\tilde{u}_2 - \tilde{u}_1) \cdot \nabla \tau_1\|_{T,1,q} \leq \varepsilon \|\tilde{u}_2 - \tilde{u}_1\|_{Y_u(T)}$$

for  $(\tilde{u}_i, \tilde{\tau}_i), (\tilde{u}, \tilde{\tau}), (u_i, \tau_i) \in K(T, R, R)$ ,  $i = 1, 2$ . Here:

$\varepsilon \rightarrow 0$  for  $T \rightarrow 0$ .

# Small data

## Theorem

Let

- assumptions of Proposition WMR fulfilled,  $\frac{1}{p} + \frac{n}{2q} < \frac{1}{2}$ ,
- $g \in C^1$ ,  $g(0, 0) = 0$ ,  $\mu \in C^1$ ,  $\mu(0) > 0$ ,  $T_0 > 0$ .

Then, there exists  $\kappa > 0$  such that for  $f \in L^p(0, T_0, L^p(\Omega))$ ,  $u_0 \in W^{2-2/p, p}(\Omega)$  satisfying  $\operatorname{div} u_0 = 0$  and  $u = 0$  on  $\partial\Omega$ ,  $\tau_0 \in W^{1, p}(\Omega)$  there exists a unique solution

$$(u, \pi, \tau) \in X_u(T_0) \times Y_u(T_0) \times (X_\tau(T_0) \cap W^{1, \infty}(0, T_0; L^q(\Omega)))$$

of (NS) with  $T(u) = T_V(u)$  provided

$$\|f\|_{T_0, p, q} + \|u_0\|_{B_{qp}^{2-\frac{2}{p}}} + \|\tau_0\|_{H^{1, q}(\Omega)} + |\nabla g(0, 0)| \leq \kappa.$$

# Idea of Proof: Small data

Rewrite as fixed point problem:

$$\begin{aligned}
 \partial_t u - \mu(0)\Delta u + \nabla \pi &= f - (\tilde{u} \cdot \nabla)\tilde{u} + \operatorname{div} \tilde{\tau} \\
 &\quad + \operatorname{div} T_{GN}(u) - \mu(0)\Delta u && \text{in } J \times \Omega, \\
 \operatorname{div} u &= 0 && \text{in } J \times \Omega, \\
 \partial_t \tau + (\tilde{u} \cdot \nabla)\tau + b\tau &= g(\nabla \tilde{u}, \tilde{\tau}) && \text{in } J \times \Omega, \\
 u &= 0, && \text{on } J \times \partial\Omega, \\
 u(0) &= u_0 && \text{in } \Omega, \\
 \tau(0) &= \tau_0 && \text{in } \Omega.
 \end{aligned}$$

Define

$$\Phi : \begin{cases} X_u^A(T_0) \times X_\tau(T_0) & \rightarrow X_u^A(T_0) \times X_\tau(T_0) \\ (\tilde{u}, \tilde{\tau}) & \mapsto (u, \tau) \end{cases}.$$

Here:

$$X_u^A(T) := X_u(T) \cap L^p(0, T; D(A_q))$$

# Fixed point argument: Small data

## Lemma

Let

- $\frac{1}{p} + \frac{n}{2q} < \frac{1}{2}$
- $T_0 > 0$

Then, there exists  $\kappa_0, C, R_0 > 0$  such that for  $R \in (0, R_0)$ ,  $\kappa \in (0, \kappa_0)$ :

$$\Phi(K(T_0, R, R/C)) \subset K(T_0, R, R/C).$$

Moreover,

$$\|g(\nabla u, \tau)\|_{T_0, \infty, q} \leq C, \quad (u, \tau) \in K(T_0, R, R/C).$$

# Fixed point argument: Small data

## Lemma

Let

- $p, q \in (1, \infty)$  such that  $\frac{1}{p} + \frac{n}{2q} < \frac{1}{2}$ ,  $T_0 > 0$ .

Then, there exists  $C, R_0 > 0$  such that for,  $R \in (0, R_0)$

$$\|\tilde{u}_2 \otimes \tilde{u}_2 - \tilde{u}_1 \otimes \tilde{u}_1\|_{T,p,q} \leq \varepsilon \|\tilde{u}_2 - \tilde{u}_1\|_{Y_u(T)},$$

$$\|\tilde{\tau}_2 - \tilde{\tau}_1\|_{T,p,q} \leq T_0^{\frac{1}{p}} \|\tilde{\tau}_2 - \tilde{\tau}_1\|_{Y_\tau(T)},$$

$$\|g(\nabla \tilde{u}_2, \tilde{\tau}_2) - g(\nabla \tilde{u}_1, \tilde{\tau}_1)\|_{T,1,q} \leq \varepsilon (\|\tilde{u}_2 - \tilde{u}_1\|_{Y_u(T)} + \|\tilde{\tau}_2 - \tilde{\tau}_1\|_{Y_\tau(T)}),$$

$$\|(\tilde{u}_2 - \tilde{u}_1) \cdot \nabla \tau_1\|_{T,1,q} \leq \varepsilon \|\tilde{u}_2 - \tilde{u}_1\|_{Y_u(T)}$$

for  $(\tilde{u}_i, \tilde{\tau}_i), (\tilde{u}, \tilde{\tau}), (u_i, \tau_i) \in K(T, R, R/C)$ ,  $i = 1, 2$ . Here,  $\varepsilon \rightarrow 0$  for  $\kappa + R \rightarrow 0$ .

# Small coupling

## Remark

*The condition*

$$|\nabla g(0, 0)| \leq \kappa$$

*means small coupling.*

*Example (Oldroyd-B):*

$$g(\nabla u, \tau) = \beta Du - \tau Wu + Wu\tau + a(Du\tau + \tau Du)$$

# Summary

|            | bdd./N. | bdd./NN. | unbdd./N. | unbdd./NN. |
|------------|---------|----------|-----------|------------|
| small data | S/B(SC) | S/B(SC)  | B(SC)     | B(SC)      |
| large data | S/B     | S        | B         | ??         |



# Global $L^2$ -Solutions: Small coupling

Situation:

- $\Omega \subset \mathbb{R}^n$ ,  $n = 2, 3$  bounded.
- Oldroyd-B fluid

Results:

- Global solutions for small data in

$$u \in L^2(0, T; H^{3,2}(\Omega)) \cap C([0, T]; D(A))$$

$$\partial_t u \in L^2(0, T; H^{1,2}(\Omega)) \cap C([0, T]; L^2(\Omega))$$

$$\pi \in L^2(0, T; H^{2,2}(\Omega))$$

$$\tau \in C([0, T]; H^{2,2}(\Omega))$$

# Another Approach: Lagrangian coordinates

We set

$$\Theta : \begin{cases} J \times \Omega_0 & \rightarrow J \times \Omega \\ (t, x) & \mapsto (t, X_u(t, \xi) = \xi + \int_0^t u(s, X_u(s, \xi)) \, ds) \end{cases}$$

and

$$u(t, x) := (\Theta^* \tilde{u})(t, x) := \tilde{u}(\Theta^{-1}(t, x)),$$

$$\pi(t, x) := (\Theta^* \tilde{\pi})(t, x) := \tilde{\pi}(\Theta^{-1}(t, x)),$$

$$\tau(t, x) := (\Theta^* \tilde{\tau})(t, x) := \tilde{\tau}(\Theta^{-1}(t, x)).$$

Note:  $(\partial_t \tau + u \cdot \nabla \tau)(t, X_u(t, \xi)) = \partial_t \tilde{\tau}(t, \xi)$

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**Note:**  $(\partial_t \tau + u \cdot \nabla \tau)(t, X_u(t, \xi)) = \partial_t \tilde{\tau}(t, \xi)$

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- 1 Introduction
- 2 Preliminaries
  - The Helmholtz decomposition
  - The Stokes operator
  - Generalized Newtonian Fluids
  - Embeddings
  - Transport Equation
- 3 Viscoelastic Fluids
  - Large data
  - Small data
  - Furter/Related Results
- 4  $\mathcal{H}^\infty$ -calculus for the Stokes Operator

# The weak Neumann problem

Consider

$$\begin{cases} \Delta v = \nabla \cdot g & \text{in } \Omega, \\ n \cdot \nabla v = n \cdot g & \text{on } \partial\Omega. \end{cases} \quad (\text{WNP}_q)$$

Does there exist a unique weak solution  $v \in \widehat{W}^{1,q}(\Omega)$  of  $(\text{WNP}_q)$ , i.e.

$$\int_{\Omega} \nabla v \nabla \varphi = \int_{\Omega} g \nabla \varphi, \quad \varphi \in \widehat{W}^{1,q'}(\Omega),$$

for  $g \in L^q(\Omega)^n$  satisfying  $\|v\|_{\widehat{W}^{1,q}(\Omega)} \leq C \|g\|_{L^q(\Omega)^n}$ .

## Proposition

$(\text{WNP}_q)$  is uniquely solvable  $\Leftrightarrow P_q$  exists.

# Main Tool

## Proposition (N.J. Kalton, P. Kunstmann, L. Weis)

Assume that

- $(X_0, X_1)$  interpolation couple of reflexive and  $B$ -convex spaces,
- $P_j : X_j \rightarrow Y_j$  compatible surjections with compatible right inverses  $J_j : Y_j \rightarrow X_j, j = 0, 1,$
- $A_j$  has an  $\mathcal{H}^\infty$ -calculus in  $X_j, B_j$   $\mathcal{R}$ -sectorial on  $Y_j,$  for  $\alpha < 0 < \beta$

$$P_0((X_0)_{\alpha, A_0}) = (Y_0)_{\alpha, B_0}, \quad P_1((X_1)_{\beta, A_1}) = (Y_1)_{\beta, B_1},$$

$$J_0((Y_0)_{\alpha, B_0}) = (X_0)_{\alpha, A_0}, \quad J_1((Y_1)_{\beta, B_1}) = (X_1)_{\beta, A_1},$$

Then,  $B_\theta$  has  $\mathcal{H}^\infty$ -calculus on  $Y_\theta = [Y_0, Y_1]_\theta, \theta \in (0, 1).$

# $\mathcal{H}^\infty$ -Calculus

## Theorem (M.G., P. Kunstmann)

*Assume that*

- $\Omega \subset \mathbb{R}^n$  has uniform  $C^3$ -boundary,
- $(WNP_q)$  is uniquely solvable for some  $q \in (1, \infty)$ .

*Then the Stokes operator  $\lambda_0 - A_q$  has a bounded  $\mathcal{H}^\infty$ -calculus for some  $\lambda_0 > 0$ .*