Analytical Aspects of Complex Fluids

Matthias Geissert

Techn. Universität Darmstadt/Univ. of Hannover
Germany

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Outline

The problem (NS)

Consider

$$\begin{aligned} \rho(\partial_t u + (u \cdot \nabla)u) &= f + \operatorname{div} T(u) - \nabla\pi && \text{in } J \times \Omega, \\ \operatorname{div}\ u &= 0 && \text{in } J \times \Omega, \\ u &= 0, && \text{on } J \times \partial\Omega, \\ u(0) &= u_0 && \text{in } \Omega, \end{aligned}$$

Here:

- u velocity of the fluid, π pressure of the fluid,
- u_0 initial velocity of the fluid, f extra body force,
- ρ density, $J = (0, T)$, $\Omega \subset \mathbb{R}^n$ domain.
- $T(u)$ = extra stress tensor.

Newtonian Fluids/Navier-Stokes equations

We set

$$T(u) := T_N(u) = \mu Du$$

Then,

$$\text{div } T(u) = \Delta u.$$

Here:

- $\mu > 0$ viscosity,
- $Du = \frac{1}{2}\left(\nabla u + (\nabla u)^T\right)$.

Generalized Newtonian Fluids

We set

$$T(u) := T_{GN}(u) = \mu(|Du|_2^2)Du$$

Here,

- μ viscosity function.

Examples:

- *Power-Law*: $\mu(|D\tilde{u}|_2^2) = \mu_0(1 + |D\tilde{u}|_2^2)^{\frac{d}{2}-1}$, $\mu_0 > 0$, $d \geq 1$.

Generalized Newtonian Fluids

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Generalized Viscoelastic Fluids

We set

$$T(u) := T_{GN}(u) + \tau,$$

where

$$\begin{aligned} \partial_t \tau + (u \cdot \nabla)\tau + b\tau &= g(\nabla u, \tau) && \text{in } J \times \Omega, \\ \tau(0) &= \tau_0 && \text{in } \Omega. \end{aligned}$$

Here:

- τ: elastic part of the stress,

Examples

- *Oldroyd-B fluids*: $\mu > 0$,

$$g(\nabla u, \tau) = \beta Du - \tau Wu + Wu\tau + a(Du\tau + \tau Du)$$

  for $\beta > 0$, $-1 \le a \le 1$ and $Wu = \frac{1}{2}(\nabla u - \nabla u^T)$,
- *Generalized Oldroyd-B*: Replace constant β by $\beta(|Du|^2)$,
- *White-Metzner*: $\mu > 0$, $b = 0$, and

$$g(\nabla u, \tau) = \beta(|Du|^2)Du + \gamma(|Du|^2)\tau - \tau Wu + Wu\tau + a(Du\tau + \tau Du)$$

  for some functions β and γ.

The Helmholtz projection

- Let $1 < q < \infty$, $\Omega \subset \mathbb{R}^n$ be a domain.
- We say that the *Helmholtz decomposition* exists if

$$L^q(\Omega)^n = L^q_\sigma(\Omega) \oplus G_q(\Omega),$$

where

$$G_q(\Omega) := \{g \in L^q(\Omega)^n : \exists h \in \widehat{W}^{1,q}(\Omega) \text{ such that } g = \nabla h\},$$

$$L^q_\sigma(\Omega) := \{f \in L^q(\Omega)^n : \int_\Omega f \nabla \varphi = 0,\ \varphi \in \widehat{W}^{1,q'}(\Omega)\}$$

In this case there exists the *Helmholtz projection*

$$P_q : L^q(\Omega)^n \to L^q_\sigma(\Omega)$$

and (under mild assumption on $\partial\Omega$)

$$L^q_\sigma(\Omega) = \{\varphi \in C_c^\infty(\Omega)^n : div\, \varphi = 0\}^{\|\cdot\|_{L^q(\Omega)}}.$$

Existence of the Helmholtz projection

- Let $1 < q < \infty$. Then the Helmholtz projection exists on $L^q(\Omega)^n$, where
  - $\Omega = \mathbb{R}^n$, $\Omega = \mathbb{R}^n_+$,
  - Ω bounded with smooth boundary,
  - Ω exterior domain with smooth boundary,
  - Ω layer,
  - $\ldots$
- The Helmholtz projection exists $L^q(\Omega)^n \cap L^2(\Omega)^n$, $2 < q < \infty$, or $L^q(\Omega)^n + L^2(\Omega)^n$, $1 < q < 2$, Ω uniform C^1.

Contributors: Farwig, Fujiwara, Kozono, Miyakawa, Morimoto, Simader, Sohr, Thäter, von Wahl, Weyl, . . .

Existence of the Helmholtz projection II

The Helmholtz projection exists on $L^q(\Omega)^n$, where

- $\Omega \subset \mathbb{R}^n$, bounded Lipschitz domain and $q \in (\frac{3}{2} - \varepsilon, 3 + \varepsilon)$, Fabes, Mendez and Mitrea.
- $\Omega \subset \mathbb{R}^2$ 'unbounded wedge' (smooth and non smooth), q depends on angle, Bogovskii.

Remark

The results above are sharp.

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Remark

The results above are sharp.

The Stokes operator

Let $1 < q < \infty$ and $\Omega \subset \mathbb{R}^n$ be a domain such that the Helmholtz projection exists. Set

$$D(A_q) = W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) \cap L_\sigma^q(\Omega)$$

and define the *Stokes operator*

$$A_q : \begin{cases} D(A_q) & \rightarrow & L_\sigma^q(\Omega), \\ u & \mapsto & P_q \Delta u. \end{cases}$$

Maximal Regularity

We say that A_q has *maximal L^p-regularity in $L^q_\sigma(\Omega)$* if for

$$f \in L^p(J; L^q_\sigma(\Omega))$$

there exists a unique

$$u \in W^{1,p}(J; L^q_\sigma(\Omega)) \cap L^p(J; D(A_q))$$

satisfying

$$\begin{aligned} u'(t) - A_q u(t) &= f(t), \quad t \in J, \\ u(0) &= 0. \end{aligned}$$

Known results on the Stokes operator

- Let $1 < p, q < \infty$. Then A_q has maximal L^p-regularity in $L^q_\sigma(\Omega)$, where
  - $\Omega = \mathbb{R}^n$, $\Omega = \mathbb{R}^n_+$,
  - Ω bounded with smooth boundary,
  - Ω exterior domain with smooth boundary,
  - Ω layer,
  - ...
- A_q has maximal L^p-regularity on $L^q_\sigma(\Omega) \cap L^2_\sigma(\Omega)$, $2 < q < \infty$, or $L^q\sigma(\Omega) + L^2_\sigma(\Omega)$, $1 < q < 2$, uniform C^2.
- Helmholtz exists +(suitable decomposition of pressure) $\Rightarrow$ A_q has maximal L^p-regularity on $L^q_\sigma(\Omega)$

Contributors: Amann, Abels, Borchers, Desch, Farwig, Fujita, Fujiwara, Galdi, Giga, Grubb, Hieber, Hishida, Kato, Masuda, Miyakawa, Morimoto, Prüss, Shibata, Shimizu, Simader, Sohr, Solonnikov, Ukai, Varnhorn, Wiegner ...

Maximal L^p-Regularity

In this case, for $f \in L^p(J; L^q(\Omega)^n)$ and $u_0 \in (L^q_\sigma(\Omega), D(A_q))_{1-\frac{1}{p},p}$ there exists a unique

$$(u, \pi) \in X_u(T) \times X_\pi(T),$$

satisfying

$$\begin{aligned} \partial_t u - \Delta u + \nabla \pi &= f && \text{in } J \times \Omega, \\ \nabla \cdot u &= 0 && \text{in } J \times \Omega, \\ u &= 0 && \text{on } J \times \partial\Omega, \\ u(0, \cdot) &= u_0 && \text{in } \Omega. \end{aligned}$$

Here,

$$X_u(T) := W^{1,p}(J; L^q_\sigma(\Omega)) \cap L^p(J; W^{2,q}(\Omega)), \; X_\pi(T) := L^p(J; \widehat{W}^{1,q}(\Omega)).$$

Maximal L^p-Regularity

In this case, for $f \in L^p(J; L^q(\Omega)^n)$ and $u_0 \in (L^q_\sigma(\Omega), D(A_q))_{1-\frac{1}{p},p}$ there exists a unique

$$(u, \pi) \in X_u(T) \times X_\pi(T),$$

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Moreover,

$$\|u\|_{X_u(T)} + \|\pi\|_{Y_u(T)} \leq C \left(\|f\|_{T,p,q} + \|u_0\|_{(L^q_\sigma(\Omega), D(A_q))_{1-\frac{1}{p},p}} \right), \quad T \in (0, T_0).$$

Proposition (WMR)

Let

- $p, q, q' \in (1, <\infty)$ *with* $\frac{1}{q} + \frac{1}{q'} = 1$, $T \in (0, T_0)$,
- Ω *uniformly* C^2*-domain,*
- $P_r : L_r(\Omega) \to L_{r,\sigma}(\Omega)$ *exists for* $r \in \{q, q'\}$,
- $\lambda + A_r \in BIP(\theta)$, $\theta < \pi/2$ *for* $r \in \{q, q'\}$ *and for some* $\lambda \geq 0$

Then for $u_0 = 0$ *and* $f = \operatorname{div} F$, $F \in L_p(0, T; H^1_q(\Omega))$ *the unique solution* $(u, \pi) \in X_u \times X_\pi$ *satisfies*

$$\|u\|_{Y_u(T)} \leq C\|F\|_{T,p,q},$$

where $C > 0$ *is independent of* F *and* T, $T \in (0, T_0)$.

Here:

$$Y_u(T) := H^{1/2,p}(0, T; L^q(\Omega)) \cap L^p(0, T; H^{1,q}(\Omega)).$$

Generalized Newtonian Fluids

Proposition

Let

- $p > n+2$,
- $\Omega \subset \mathbb{R}^n$ *bounded, class* $C^{2,1}$,
- $\mu \in C^{1,1}(\mathbb{R}_+)$ *satisfying*

$$\mu(s) > 0, \qquad \mu(s) + 2s\mu'(s) > 0, \quad s \geq 0,$$

Then, for $f \in L^p(0,T,L^p(\Omega))$, $u_0 \in W^{2-2/p,p}(\Omega)$ *satisfying* $\operatorname{div} u_0 = 0$ *and* $u = 0$ *on* $\partial\Omega$, *there exists a unique solution* $(u,\pi) \in X_u(T) \times Y_u(T)$ *of (NS) with* $T(u) = T_{GN(u)}$.

Proposition

Let

- $p, q \in (1, \infty)$ *satisfy* $\frac{1}{p} + \frac{n}{2q} < \frac{1}{2}$, $T_0 > 0$,
- $\Omega \subset \mathbb{R}^n$ *uniform* C^2*-domain.*

Then for $T \in (0, T_0)$

$$X_u(T) \hookrightarrow L_\infty(0, T; W^1_\infty(\Omega)) \cap L_\infty(0, T; L_q(\Omega))$$
$$Y_u(T) \hookrightarrow L_\infty(0, T; L_\infty(\Omega)).$$

Moreover, if $u(0) = 0$

$$\|u\|_{L_\infty(0,T;W^1_\infty(\Omega))} + \|u\|_{T,\infty,q} \leq C\|u\|_{X_u(T)}, \quad T \in (0, T_0],\ u \in X_u(T),$$
$$\|u\|_{T,\infty,\infty} \leq C\|u\|_{Y_u(T)}, \quad T \in (0, T_0],\ u \in Y_u(T).$$

Consider

$$\begin{aligned} \partial_t \tau + (u \cdot \nabla)\tau + b\tau &= g && \text{in } J \times \Omega, \\ \tau(0) &= \tau_0 && \text{in } \Omega. \end{aligned} \tag{1}$$

for

- $1 < p < \infty$, $n < q < \infty$, $T \in (0, T_0)$,
- $u \in X_u(T)$ such that $u \cdot \nu = 0$ on $\partial\Omega$, $b \geq 0$,
- $\Omega \subset \mathbb{R}^n$ a uniform C^2-domain.

We set

$$X_\tau(T) := L^\infty(0, T; H^{1,q}(\Omega)),$$
$$Y_\tau(T) := L^\infty(0, T; L^q(\Omega)).$$

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$$\begin{aligned} \partial_t \tau + (u \cdot \nabla)\tau + b\tau &= g \qquad \text{in } J \times \Omega, \\ \tau(0) &= \tau_0 \qquad \text{in } \Omega. \end{aligned} \tag{1}$$

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Proposition

For

- $g \in L^1(0,T;H^{1,q}(\Omega)) \cap L^\infty(0,T;L^q(\Omega))$, $\tau_0 \in H^{1,q}(\Omega)$

there exists a unique solution $\tau \in X_\tau(T) \cap W^{1,\infty}(0,T;L^q(\Omega))$ *of* (1) *such that*

$$\|\tau\|_{X_\tau(T)} \leq C_1 \left(\|\tau_0\|_{H^{1,q}(\Omega)} + \|g\|_{L^1(0,T;H^{1,q}(\Omega))} \right) e^{C_1 T^{1-1/p} \|u\|_{L^p(0,T;H^{2,q}(\Omega))}}.$$

Moreover,

$$\|\tau\|_{Y_\tau(T)} \leq C_2 \left(\|\tau_0\|_q + \|g\|_{T,1,q} \right) e^{C_2 \|\operatorname{div} \tilde{u}\|_{L^1(0,T;H^{1,q}(\Omega))}}.$$

Here:

- C_1, C_2 independent of g, τ_0, u and $T \in (0, T_0)$,

Large initial data: Part I

Theorem

Let

- $p > n+2$, $\Omega \subset \mathbb{R}^n$ *bounded, class* $C^{2,1}$, $g \in C^1$,
- $\mu \in C^{1,1}(\mathbb{R}_+)$ *satisfying*

$$\mu(s) > 0, \qquad \mu(s) + 2s\mu'(s) > 0, \quad s \geq 0.$$

Then, for $f \in L^p(0,T,L^p(\Omega))$, $u_0 \in W^{2-2/p,p}(\Omega)$ *satisfying* $\operatorname{div} u_0 = 0$ *and* $u = 0$ *on* $\partial\Omega$, $\tau_0 \in W^{1,p}(\Omega)$ *there exists a unique solution*

$$(u,\pi,\tau) \in X_u(T) \times Y_u(T) \times (X_\tau(T) \cap W^{1,\infty}(0,T;L^q(\Omega))$$

of (NS) with $T(u) = T_V(u)$.

Idea of proof: Large data I

Rewrite as fixed point problem:

$$\begin{aligned} \partial_t u - \operatorname{div} T_{GN}(u) + \nabla\pi &= f - (\tilde{u}\cdot\nabla)\tilde{u}) + \operatorname{div} \tilde{\tau} && \text{in } J\times\Omega,\\ \operatorname{div}\ u &= 0 && \text{in } J\times\Omega,\\ \partial_t\tau + (\tilde{u}\cdot\nabla)\tau + b\tau &= g(\nabla\tilde{u},\tilde{\tau}) && \text{in } J\times\Omega,\\ u &= 0, && \text{on } J\times\partial\Omega,\\ u(0) &= u_0 && \text{in } \Omega,\\ \tau(0) &= \tau_0 && \text{in } \Omega. \end{aligned}$$

Define

$$\Phi : \begin{cases} X_u^A(T_0)\times X_\tau(T_0) & \to X_u^A(T_0)\times X_\tau(T_0)\\ (\tilde{u},\tilde{\tau}) & \mapsto (u,\tau) \end{cases}.$$

Here:

$$X_u^A(T) := X_u(T)\cap L^p(0,T;D(A_q))$$

Fixed point argument: Large data I

We set

$$\begin{aligned} K_u(T,R_1) &:= \{u \in X_u^A(T) : u(0) = u_0, \|u - u_*\|_{X_u(T)} \le R_1\} \\ K_\tau(T,R_2) &:= \{\tau \in X_\tau(T) : \tau(0) = \tau_0, \|\tau\|_{X_\tau(T)} \le R_2\} \\ K(T,R_1,R_2) &:= K_u(T_0,R_1) \times K_\tau(T,R_2) \end{aligned}$$

where

$$u_* = \mathrm{e}^{-A_q t} u_0.$$

Fixed point argument: Large data I

Lemma

Let

- $\frac{1}{p} + \frac{n}{2q} < \frac{1}{2}$

Then, there exists $T_0, R_0 > 0$ *such that for* $T \in (0, T_0)$, $R \in (0, R_0)$*:*

$$\Phi(K(T,R,R)) \subset K(T,R,R).$$

Moreover,

$$\|g(\nabla u, \tau)\|_{T,\infty,q} \le C, \quad (u,\tau) \in K(T,R,R).$$

Fixed point argument: Large data I

Apply Schauder's fixed point theorem in

$$Z(T) := C([0,T]; C^1(\overline{\Omega})) \times C([0,T], C(\overline{\Omega}))$$

We show

- $K(T, R_0, R_0)$ closed w.r.t. topology of $Z(T)$.
- Continuity of Φ in topology of $Z(T)$.

Uniqueness follows from an energy argument.

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- Continuity of Φ in topology of $Z(T)$.

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Large initial data: Part II

Theorem

Let

- $p, q \in (1, \infty)$ *satisfying* $\frac{1}{p} + \frac{n}{2q} \leq \frac{1}{2}$.
- *assumptions of Proposition WMR fulfilled,*
- $g \in C^1$, $g(0,0) = 0$
- $\mu > 0$

Then, for $f \in L^p(0,T,L^p(\Omega))$, $u_0 \in W^{2-2/p,p}(\Omega)$ *satisfying* $\operatorname{div} u_0 = 0$ *and* $u = 0$ *on* $\partial\Omega$, $\tau_0 \in W^{1,p}(\Omega)$ *there exists a unique solution*

$$(u, \pi, \tau) \in X_u(T) \times Y_u(T) \times (X_\tau(T) \cap W^{1,\infty}(0,T;L^q(\Omega))$$

of (NS) with $T(u) = T_V(u)$.

Idea of proof: Large data II

Lemma

Let

- $\frac{1}{p} + \frac{n}{2q} < \frac{1}{2}$

Then, there exists $T_0, R_0 > 0$ *such that for* $T \in (0, T_0)$, $R \in (0, R_0)$*:*

$$\Phi(K(T,R,R)) \subset K(T,R,R).$$

Moreover,

$$\|g(\nabla u, \tau)\|_{T,\infty,q} \leq C, \quad (u,\tau) \in K(T,R,R).$$

Fixed point argument: Large data II

Proposition

Let

- X *be a reflexive Banach space or* X *has a separable pre-dual,*
- K *be a convex, closed and bounded subset of* X*,*
- $X \hookrightarrow Y$*, where* Y *is a Banach space,*
- $\Phi: X \to X$ *map* K *into* K*,*
- *for some* $\eta \in (0,1)$*,*

$$\|\Phi(x) - \Phi(y)\|_Y \leq \eta \|x - y\|_Y, \quad x, y \in K.$$

Then there exists a unique fixed point of Φ *in* K*.*

Fixed point argument: Large data II

Lemma

Let

- $p, q \in (1, \infty)$ *such that* $\frac{1}{p} + \frac{n}{2q} < \frac{1}{2}$.

Then, there exists $T_0, R_0 > 0$ *such that for* $T \in (0, T_0)$, $R \in (0, R_0)$

$$\|\tilde{u}_2 \otimes \tilde{u}_2 - \tilde{u}_1 \otimes \tilde{u}_1\|_{T,p,q} \leq \varepsilon \|\tilde{u}_2 - \tilde{u}_1\|_{Y_u(T)},$$

$$\|\tilde{\tau}_2 - \tilde{\tau}_1\|_{T,p,q} \leq \varepsilon \|\tilde{\tau}_2 - \tilde{\tau}_1\|_{Y_\tau(T)},$$

$$\|g(\nabla \tilde{u}_2, \tilde{\tau}_2) - g(\nabla \tilde{u}_1, \tilde{\tau}_1)\|_{T,1,q} \leq \varepsilon \left(\|\tilde{u}_2 - \tilde{u}_1\|_{Y_u(T)} + \|\tilde{\tau}_2 - \tilde{\tau}_1\|_{Y_\tau(T)}\right),$$

$$\|(\tilde{u}_2 - \tilde{u}_1).\nabla \tau_1\|_{T,1,q} \leq \varepsilon \|\tilde{u}_2 - \tilde{u}_1\|_{Y_u(T)}$$

for $(\tilde{u}_i, \tilde{\tau}_i), (\tilde{u}, \tilde{\tau}), (u_i, \tau_i) \in K(T, R, R)$, $i = 1, 2$. *Here:*
$\varepsilon \to 0$ *for* $T \to 0$.

Fixed point arguemnt: Large data II

Combining the last to lemata and the last proposition concludes the argument.

Small data

Theorem

Let

- *assumptions of Proposition WMR fulfilled,* $\frac{1}{p} + \frac{n}{2q} < \frac{1}{2}$,
- $g \in C^1$, $g(0,0) = 0$, $\mu \in C^1$, $\mu(0) > 0$, $T_0 > 0$.

Then, there exists $\kappa > 0$ *such that for* $f \in L^p(0, T_0, L^p(\Omega))$, $u_0 \in W^{2-2/p,p}(\Omega)$ *satisfying* $\operatorname{div} u_0 = 0$ *and* $u = 0$ *on* $\partial\Omega$, $\tau_0 \in W^{1,p}(\Omega)$ *there exists a unique solution*

$$(u, \pi, \tau) \in X_u(T_0) \times Y_u(T_0) \times (X_\tau(T_0) \cap W^{1,\infty}(0, T_0; L^q(\Omega))$$

of (NS) with $T(u) = T_V(u)$ *provided*

$$\|f\|_{T_0,p,q} + \|u_0\|_{B_{qp}^{2-\frac{2}{p}}} + \|\tau_0\|_{H^{1,q}(\Omega)} + |\nabla g(0,0)| \leq \kappa.$$

Idea of Proof: Small data

Rewrite as fixed point problem:

$$\begin{aligned}
\partial_t u - \mu(0)\Delta u + \nabla\pi &= f - (\tilde{u}\cdot\nabla)\tilde{u}) + \operatorname{div}\tilde{\tau} \\
&\quad +\operatorname{div} T_{GN}(u) - \mu(0)\Delta u && \text{in } J\times\Omega, \\
\operatorname{div}\ u &= 0 && \text{in } J\times\Omega, \\
\partial_t\tau + (\tilde{u}\cdot\nabla)\tau + b\tau &= g(\nabla\tilde{u},\tilde{\tau}) && \text{in } J\times\Omega, \\
u &= 0, && \text{on } J\times\partial\Omega, \\
u(0) &= u_0 && \text{in } \Omega, \\
\tau(0) &= \tau_0 && \text{in } \Omega.
\end{aligned}$$

Define

$$\Phi : \begin{cases} X_u^A(T_0)\times X_\tau(T_0) & \to X_u^A(T_0)\times X_\tau(T_0) \\ (\tilde{u},\tilde{\tau}) & \mapsto (u,\tau) \end{cases}.$$

Here:

$$X_u^A(T) := X_u(T)\cap L^p(0,T;D(A_q))$$

Fixed point argument: Small data

Lemma

Let

- $\frac{1}{p} + \frac{n}{2q} < \frac{1}{2}$
- $T_0 > 0$

Then, there exists $\kappa_0, C, R_0 > 0$ *such that for* $R \in (0, R_0)$, $\kappa \in (0, \kappa_0)$*:*

$$\Phi(K(T_0, R, R/C)) \subset K(T_0, R, R/C).$$

Moreover,

$$\|g(\nabla u, \tau)\|_{T_0,\infty,q} \leq C, \quad (u, \tau) \in K(T_0, R, R/C).$$

Fixed point argument: Small data

Lemma

Let

- $p, q \in (1, \infty)$ *such that* $\frac{1}{p} + \frac{n}{2q} < \frac{1}{2}$, $T_0 > 0$.

Then, there exists $C, R_0 > 0$ *such that for,* $R \in (0, R_0)$

$$\|\tilde{u}_2 \otimes \tilde{u}_2 - \tilde{u}_1 \otimes \tilde{u}_1\|_{T,p,q} \leq \varepsilon \|\tilde{u}_2 - \tilde{u}_1\|_{Y_u(T)},$$

$$\|\tilde{\tau}_2 - \tilde{\tau}_1\|_{T,p,q} \leq T_0^{\frac{1}{p}} \|\tilde{\tau}_2 - \tilde{\tau}_1\|_{Y_\tau(T)},$$

$$\|g(\nabla \tilde{u}_2, \tilde{\tau}_2) - g(\nabla \tilde{u}_1, \tilde{\tau}_1)\|_{T,1,q} \leq \varepsilon \left(\|\tilde{u}_2 - \tilde{u}_1\|_{Y_u(T)} + \|\tilde{\tau}_2 - \tilde{\tau}_1\|_{Y_\tau(T)} \right),$$

$$\|(\tilde{u}_2 - \tilde{u}_1).\nabla \tau_1\|_{T,1,q} \leq \varepsilon \|\tilde{u}_2 - \tilde{u}_1\|_{Y_u(T)}$$

for $(\tilde{u}_i, \tilde{\tau}_i), (\tilde{u}, \tilde{\tau}), (u_i, \tau_i) \in K(T, R, R/C)$, $i = 1, 2$. *Here,* $\varepsilon \to 0$ *for* $\kappa + R \to 0$.

Small coupling

Remark

The condition

$$|\nabla g(0,0)| \leq \kappa$$

means small *coupling.*

Example (Oldroyd-B):

$$g(\nabla u, \tau) = \beta Du - \tau Wu + Wu\tau + a(Du\tau + \tau Du)$$

Global L^2-Solutions: Small coupling

Situation:

- $\Omega \subset \mathbb{R}^n$, $n = 2, 3$ bounded.
- Oldroyd-B fluid

Results:

- Global solutions for small data in

$$
\begin{aligned}
u &\in L^2(0,T;H^{3,2}(\Omega)) \cap C([0,T];D(A)) \\
\partial_t u &\in L^2(0,T;H^{1,2}(\Omega)) \cap C([0,T];L^2(\Omega)) \\
\pi &\in L^2(0,T;H^{2,2}(\Omega)) \\
\tau &\in C([0,T];H^{2,2}(\Omega))
\end{aligned}
$$

Another Approach: Lagrangian coordinates

We set

$$\Theta : \begin{cases} J \times \Omega_0 & \rightarrow \quad J \times \Omega \\ (t,x) & \mapsto \quad (t, X_u(t,\xi) = \xi + \int_0^t u(s, X_u(s,\xi))\,\mathrm{d}s) \end{cases}$$

and

$$u(t,x) := (\Theta^* \tilde{u})(t,x) := \tilde{u}(\Theta^{-1}(t,x)),$$
$$\pi(t,x) := (\Theta^* \tilde{\pi})(t,x) := \tilde{\pi}(\Theta^{-1}(t,x)),$$
$$\tau(t,x) := (\Theta^* \tau)(t,x) := \tilde{\tau}(\Theta^{-1}(t,x)).$$

Note: $(\partial_t \tau + u.\nabla\tau)(t, X_u(t,\xi)) = \partial_t \tilde{\tau}(t,\xi)$

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Functional Calculus

Let

- $A : D(A) \to X$ be sectorial,
- $\theta \in (0, \pi - \Phi_A)$.

We define

$$f(A)g := \frac{1}{2\pi i} \int_{\partial\Sigma_\psi} f(z)(z - A)^{-1} g \, \mathrm{d}z, \quad g \in X.$$

for

$$f \in \mathcal{H}_0^\infty(\overline{\Sigma_\theta}^c) := \{f : \overline{\Sigma_\theta}^c \to \mathbb{C} \text{ analytic}, \sup_{|\lambda|<1} |\lambda^{-\alpha} f(\lambda)| + \sup_{|\lambda|>1} |\lambda^\alpha f(\lambda)| < \infty \text{ for some } \alpha > 0\},$$

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and $\psi \in (\theta, \pi - \Phi_A)$

$\mathcal{H}^\infty$-calculus

Let

- $A : D(A) \to X$ be sectorial.

Then, A admits a *bounded* $\mathcal{H}^\infty$*-calculus* if for $\theta \in (0, \pi - \Phi_A)$

$$\|f(A)\|_{\mathcal{L}(X)} \leq C_\theta \|f\|_{L^\infty(\overline{\Sigma_\theta}^c)}, \quad f \in \mathcal{H}_0^\infty(\overline{\Sigma_\theta}^c).$$

In that case, $f(A)$ is well-defined for

$$f \in \mathcal{H}^\infty(\overline{\Sigma_\theta}^c) := \{f : \overline{\Sigma_\theta}^c \to \mathbb{C} \text{ analytic}, \|f\|_{L^\infty(\overline{\Sigma_\theta}^c)} < \infty\}.$$

The weak Neumann problem

Consider

$$\begin{cases} \Delta v &= \nabla \cdot g \quad \text{in } \Omega, \\ n \cdot \nabla v &= n \cdot g \quad \text{on } \partial\Omega. \end{cases} \qquad (\mathrm{WNP_q})$$

Does there exists a unique weak solution $v \in \widehat{W}^{1,q}(\Omega)$ of $(\mathrm{WNP_q})$, i.e.

$$\int_\Omega \nabla v \nabla \varphi = \int_\Omega g \nabla \varphi, \quad \varphi \in \widehat{W}^{1,q'}(\Omega),$$

for $g \in L^q(\Omega)^n$ satisfying $\|v\|_{\widehat{W}^{1,q}(\Omega)} \leq C\|g\|_{L^q(\Omega)^n}$.

Proposition

(WNP_q) is uniquely solvable $\Leftrightarrow$ P_q exists.

$\mathcal{H}^\infty$-Calculus

Theorem

Assume that

- $\Omega \subset \mathbb{R}^n$ *has uniform* C^3*-boundary,*
- (WNP_q) *is uniquely solvable for some* $q \in (1, \infty)$.

Then the Stokes operator $\lambda_0 - A_q$ *has a bounded* $\mathcal{H}^\infty$*-calculus for some* $\lambda_0 > 0$.

Idea of Proof: Key Tool

Proposition (N.J. Kalton, P. Kunstmann, L. Weis)

Assume that

- (X_0, X_1) *interpolation couple of reflexive and B-convex spaces,*
- $P_j : X_j \to Y_j$ *compatible surjections with compatible right inverses* $J_j : Y_j \to X_j$, $j = 0, 1$,
- A_j *has an* $\mathcal{H}^\infty$*-calculus in* X_j, B_j $\mathcal{R}$*-sectorial on* Y_j, *for* $\alpha < 0 < \beta$

$$P_0((X_0)^{\cdot}_{\alpha,A_0}) = (Y_0)^{\cdot}_{\alpha,B_0}, \quad P_1((X_1)^{\cdot}_{\beta,A_1}) = (Y_1)^{\cdot}_{\beta,B_1},$$
$$J_0((Y_0)^{\cdot}_{\alpha,B_0}) = (X_0)^{\cdot}_{\alpha,A_0}, \quad J_1((Y_1)^{\cdot}_{\beta,B_1}) = (X_1)^{\cdot}_{\beta,A_1},$$

Then, B_θ *has* $\mathcal{H}^\infty$*-calculus on* $Y_\theta = [Y_0, Y_1]_\theta$, $\theta \in (0, 1)$.

Idea of Proof

Apply the previous proposition to

$$X_0 := L^2(\Omega), Y_0 := L^2_\sigma(\Omega),$$
$$X_1 := L^p(\Omega), Y_1 := L^p_\sigma(\Omega)$$

and A_0/A_1 shifted Dirichlet-Laplacian, B_0/B_1 shifted Stokes-Operator.