Mathematical analysis on some coherent structures of vorticity fields for viscous incompressible flows

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1. Overview - introduction

There are several structures observed in viscous incompressible flows, which exhibit typical behaviors of flows or persist for a relatively long time. These structures are often described in terms of vorticity fields.

- Vortex tubes in 3D flows:

$\cdots$ tube-like structures of intense vorticity fields

- ‘Inverse cascade’ in 2D flows:

$\cdots$ tendency for small vortices to form larger vortices

- Boundary layer of flows at high Reynolds number:

$\cdots$ vortex sheets or lines attached to the boundary

1. Overview - vorticity field

In this lecture we are interested in the vorticity filed of the viscous incompressible flows.

$$u = (u_1, u_2, u_3)^\top \ : \ \text{velocity field}$$

$$\omega = \nabla \times u = \begin{pmatrix} \partial_2 u_3 - \partial_3 u_2 \\ \partial_3 u_1 - \partial_1 u_3 \\ \partial_1 u_2 - \partial_2 u_1 \end{pmatrix} \ : \ \text{vorticity field in } 3D$$

$$\omega = \mathrm{Rot}\, u = \partial_{x_1} u_2 - \partial_{x_2} u_1 \ : \ \text{vorticity field in } 2D$$

1. Overview - vorticity equations

- Navier-Stokes equations for viscous incompressible flows:

$$\partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla p = 0 \qquad \nabla \cdot u = 0. \qquad \text{(NS)}$$

The difficulty comes from nonlinearity and also from non-local nature due to the pressure term.

- Vorticity equations for viscous incompressible flows:

$$\partial_t \omega + u \cdot \nabla \omega - \omega \cdot \nabla u - \nu \Delta \omega = 0 \qquad \text{in } 3D. \qquad (\mathrm{V}_3)$$

$$\partial_t \omega + u \cdot \nabla \omega - \nu \Delta \omega = 0 \qquad \text{in } 2D. \qquad (\mathrm{V}_2)$$

1. Overview - contents of this lecture

In view of the vorticity equations the vorticity is a 'local' quantity, and useful to study local properties of the flow.

Topics:

I. Inviscid limit for viscous incompressible flows in $\mathbb{R}^2_+$
- approach from the vorticity formulation -

II. Stability of the Burgers vortex for 3D perturbations

III. Stability of the Lamb-Oseen vortex in 2D exterior domains

$*$ Firstly we will see overviews of each topic.
$*$ In this lecture the first topic will be more focused.

1-1. Overview of Topic I:
- behavior of NS flows at the inviscid limit -

By formally taking the limit $\nu \to 0$ of the NS equations

$$\partial_t u_{NS} + u_{NS} \cdot \nabla u_{NS} - \nu \Delta u_{NS} + \nabla p_{NS} = 0 \qquad \nabla \cdot u_{NS} = 0, \tag{NS$_\nu$}$$

we get the Euler equations

$$\partial_t u_E + u_E \cdot \nabla u_E + \nabla p_E = 0 \qquad \nabla \cdot u_E = 0. \tag{E}$$

Boundary condition:

$u_{NS} = 0$ (no-slip) $\quad \longleftrightarrow \quad n \cdot u_E = 0$ $\;$ n: exterior unit normal

$\Longrightarrow$ boundary layer appears

1-1. Overview of Topic I:
- behavior of NS flows at the inviscid limit -

· **Prandtl theory (1904)**

(i) Outer region (the region away from the boundary):

· The fluid motion will be described by the Euler equations.

(ii) Boundary layer region (the region where the viscosity effect essentially exists):

· The fluid motion will be described by the Prandtl equations. The formal boundary layer thickness is $O(\nu^{1/2})$.

1-1. Overview of Topic I:
- behavior of NS flows at the inviscid limit in $\mathbb{R}^2_+$ -

Mathematically the above description corresponds with the asymptotic expansion at $\nu \to 0$:

$$u_{NS_\nu}(t,x) \;=\; u_E(t,x) \;+\; u_P^{(\nu)}(t,x) \;+\; O(\nu^{\frac{1}{2}}) \qquad \text{in } \; L^\infty_{t,x}, \qquad (1)$$

$$u_P^{(\nu)}(t,x) \;=\; \Big(\, v_{P,1}(t,x_1,\frac{x_2}{\nu^{\frac{1}{2}}}), \;\, \nu^{\frac{1}{2}} v_{P,2}(t,x_1,\frac{x_2}{\nu^{\frac{1}{2}}}) \, \Big)^\top .$$

However, the rigorous verification of (1) is still widely open.

So far it is verified only for the initial data with analytic regularity: Asano ('88), Sammartino-Caflisch ('98).

1-1. Overview of Topic I:
- behavior of NS flows at the inviscid limit in $\mathbb{R}^2_+$ -

a: initial velocity $b = \mathrm{Rot}\, a$: initial vorticity

- Main result in Topic I (M. preprint) -

Assume that $a \in L^p_\sigma(\mathbb{R}^2_+)$ *for some* $1 < p < \infty$ *and* $b \in W^{4,1}(\mathbb{R}^2_+) \cap W^{4,2}(\mathbb{R}^2_+)$. *Assume also that* $a = 0$ *on* $\partial\mathbb{R}^2_+$ *and*

$$d_0 = \mathrm{dist}\,(\partial\mathbb{R}^2_+,\ \mathrm{supp}\, b) > 0. \tag{2}$$

Then the asymptotic expansion (1) *holds at least for a short time* $T > 0$, *where* T *satisfies* $T \geq c \min\{d_0, 1\}$ *with* $c > 0$ *depending only on* $\|b\|_{W^{4,1}\cap W^{4,2}}$.

- For the proof we use the vorticity formulation.

1-2. Overview of Topic II:
- NS flows with a linear strain and the Burgers vortex -

It is believed that vortex tubes take place due to the interplay of two mechanism:

(i) amplification of vorticity due to stretching

(ii) diffusion through the action of viscosity

To study their typical interaction we consider the NS velocity field V of the form

$$V = Mx + U \qquad\qquad M = \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$U = (U_1, U_2, U_3)$: unknown perturbation velocity

1-2. Overview of Topic II:
- NS flows with a linear strain and the Burgers vortex -

Let $x = (x_h, x_3) \in \mathbb{R}^2 \times \mathbb{R}$ and set

$$G = g\,\mathbf{e}_3 \qquad g(x_h) = \frac{1}{4\pi} e^{-\frac{|x_h|^2}{4}} \qquad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

$$U^G(x_h) = \begin{pmatrix} -x_2\, u^g(|x_h|^2) \\ x_1\, u^g(|x_h|^2) \\ 0 \end{pmatrix} \qquad u^g(r) = \frac{1}{2\pi r}(1 - e^{-\frac{r}{4}}).$$

Burgers ('48): *For each* $\alpha \in \mathbb{R}$ *the velocity field* $Mx + \alpha U^G$ *is a stationary solution to* (NS).

Problem: Stability of the Burgers vortex αG for a given circulation number α?

1-2. Overview of Topic II:
- NS flows with a linear strain and the Burgers vortex -

Consider the initial vorticity of the form

$$\Omega_0 = \alpha G + \omega_0 \qquad \qquad \int_{\mathbb{R}^2} \omega_{0,3} \, \mathrm{d}x_h = 0.$$

$$\omega_0(x) = \begin{pmatrix} 0 \\ 0 \\ \omega_{0,3}(x_h) \end{pmatrix} \qquad \Longleftrightarrow \qquad \text{Two dimensional perturbation}$$

$$\omega_0(x) = \begin{pmatrix} \omega_{0,1}(x) \\ \omega_{0,2}(x) \\ \omega_{0,3}(x) \end{pmatrix} \qquad \Longleftrightarrow \qquad \text{Three dimensional perturbation}$$

1-2. Overview of Topic II:
- NS flows with a linear strain and the Burgers vortex -

Theorem (stability in 2D). *For any $\alpha \in \mathbb{R}$ the Burgers vortex αG is asymptotically stable with respect to 2D perturbations.*

Giga-Kambe ('88): $\|\omega_{0,3}\|_{L^1} + |\alpha| \ll 1$
Carpio ('94): $|\alpha| \ll 1$
Gallay-Wayne ('05): without smallness on $\|\omega_{0,3}\|_{L^1}$, α

See also a book by Giga-Giga-Saal ('10).

- Main result in Topic II (Gallay-M. ('11)) -

For any $\alpha \in \mathbb{R}$ the Burgers vortex αG is asymptotically stable with respect to small 3D perturbations.

cf. Gallay-Wayne ('06): smallness on both $|\alpha|$ and 3D perturbations

1-3. Overview of Topic III: - stability of the Lamb-Oseen vortex

Through the self-similar transformation

$$\xi = \frac{x}{t^{\frac{1}{2}}} \qquad \tau = \log t,$$

the 2D stability of the Burgers vortex αU^G is equivalent with the asymptotic convergence at $t \to \infty$ of 2D Navier-Stokes solution $u(t, x_h)$ to the self-similar solution $\alpha t^{-\frac{1}{2}} U^G(\frac{x_h}{t^{\frac{1}{2}}})$, called the Lamb-Oseen vortex.

The aim of Topic III is to extend the above result in $\mathbb{R}^2$ to the case of the exterior domains Ω in $\mathbb{R}^2$.

1-3. Overview of Topic III: - stability of the Lamb-Oseen vortex

• For finite kinetic energy solutions of (NS) in 2D exterior domains the temporal decay of their norms at $t \to \infty$ has been well studied.

Masuda ('84), Borchers-Miyakawa ('92), Maremonti ('92), Kozono-Ogawa ('93), Dan-Shibata ('99), Bae-Jin ('06)

• The object of research here is a special case of the $L^{2,\infty}_{\sigma}(\Omega)$ solutions to (NS) (with no-slip B.C.).

· Global solvability of (NS) in 2D exterior domains is proved by Kozono-Yamazaki ('95) under the smallness condition on the local singularity in $L^{2,\infty}_{\sigma}(\Omega)$.

1-3. Overview of Topic III: - stability of the Lamb-Oseen vortex

u_0: initial velocity $\qquad \omega_0 = \mathrm{Rot}\, u_0$: initial vorticity

$\alpha = \displaystyle\int_\Omega \omega_0(x)\,\mathrm{d}x$: circulation number

- Main result in Topic III (Gallay-M. preprint) -

Let $u_0 \in \dot{W}^{1,p}_{0,\sigma}(\Omega)$ *for some* $p \in [1,2)$ *and* ω_0 *satisfies*

$$\int_\Omega (1+|x|^2)^m |\omega_0(x)|^2\,\mathrm{d}x < \infty \qquad \text{for some } m > 1.$$

Then there is $\epsilon > 0$ *such that if* $|\alpha| \le \epsilon$ *then the solution* u *to* (NS) *satisfies*

$$\lim_{t\to\infty} \|u(t) - \frac{\alpha}{(1+t)^{\frac{1}{2}}} U^G\Big(\frac{\cdot}{(1+t)^{\frac{1}{2}}}\Big)\|_{L^2} = 0.$$

cf. Iftimie-Karch-Lacave (preprint): smallness on $|\alpha|$ and $\|u_0 - \alpha U^G\|_{L^2}$.

Topic I

Inviscid limit for viscous incompressible flows in $\mathbb{R}^2_+$
- approach from the vorticity formulation -

- Vorticity boundary condition and vorticity formulation
- Inviscid limit when the initial vorticity is located away from the boundary

I-1. Navier-Stokes equations in the half plane

$$\begin{cases} \partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla p = 0 & t > 0 \quad x \in \mathbb{R}^2_+, \\ \nabla \cdot u = 0 & t \geq 0 \quad x \in \mathbb{R}^2_+, \\ u = 0 & t \geq 0 \quad x \in \partial\mathbb{R}^2_+, \\ u|_{t=0} = a & \qquad\quad\; x \in \mathbb{R}^2_+. \end{cases} \tag{NS}$$

$u = u(t, x) = (u_1(t, x), u_2(t, x))$; velocity field

$p = p(t, x)$; pressure field

$\nu > 0$; kinematic viscosity coefficient

$\mathbb{R}^2_+ = \{ x = (x_1, x_2) \in \mathbb{R}^2 \mid x_2 > 0 \}$

I-2. Vorticity boundary conditions

In the case of the half plane the vorticity ω is subject to the following boundary condition:

$$\nu\,(\partial_2 + (-\partial_1^2)^{\frac{1}{2}})\,\omega \;=\; -\,\partial_2(-\Delta_D)^{-1}(u\cdot\nabla\omega) \qquad t>0 \quad x\in\partial\mathbb{R}^2_+. \tag{3}$$

Here $f = (-\Delta_D)^{-1}h$ denote the solution to the Poisson equations:

$$\begin{cases} -\Delta f \;=\; h & \text{in } \mathbb{R}^2_+, \\ \quad\; f \;=\; 0 & \text{on } \partial\mathbb{R}^2_+. \end{cases}$$

The operator $(-\partial_1^2)^{1/2}$ is defined in terms of the Fourier transform $\mathcal{F}$ in the tangential (x_1) direction:

$$(-\partial_1^2)^{\frac{1}{2}} f\,(x_1) = \mathcal{F}^{-1}[\;|\xi_1|\,\mathcal{F}[f](\xi_1)\;](x_1). \tag{4}$$

I-2. Vorticity boundary condition: derivation

The condition (3) is derived from a simple mathematical consideration using the Biot-Savart law in $\mathbb{R}^2_+$:

$$u(t) = \nabla^{\perp}(-\Delta_D)^{-1}\omega(t) \qquad\qquad \nabla^{\perp} = (\partial_2, -\partial_1). \tag{5}$$

By (5) the condition $u_2(t) = 0$ on $\partial\mathbb{R}^2_+$ is automatically satisfied. We need $u_1(t) = 0$ on $\partial\mathbb{R}^2_+$, that is,

$$u_1(t, x_1, 0) = \frac{1}{2\pi}\int_{\mathbb{R}^2_+} \frac{y_2}{(x_1 - y_1)^2 + y_2^2}\omega(t, y)\,\mathrm{d}y = 0. \tag{6}$$

However, (6) is highly non-local and difficult to adopt as a boundary condition on the vorticity. Thus we impose the condition so that (6) is preserved under the evolution of the vorticity in $\mathbb{R}^2_+$, i.e., the vorticity equations: $\partial_t\omega + u\cdot\nabla\omega - \nu\Delta\omega = 0$.

I-3. Initial boundary value problem for vorticity equations

The IBP for the vorticity equations in $\mathbb{R}^2_+$ is described as

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega - \nu \Delta \omega = 0 & t > 0 \quad x \in \mathbb{R}^2_+, \\ u = \nabla^\perp (-\Delta_D)^{-1} \omega & t \geq 0 \quad x \in \mathbb{R}^2_+ \\ \omega|_{t=0} = b := \mathrm{Rot}\, a & x \in \mathbb{R}^2_+, \end{cases} \tag{V}$$

with the boundary conditions

$$\nu \left(\partial_2 + (-\partial_1^2)^{\frac{1}{2}} \right) \omega = -\partial_2 (-\Delta_D)^{-1} (u \cdot \nabla \omega) \qquad t > 0 \quad x \in \partial \mathbb{R}^2_+. \tag{BC}$$

I-4. Solution formula for linearized problem

Let us consider the linear problem

$$\begin{cases} \partial_t \omega - \nu \Delta \omega = f & t > 0 \quad x \in \mathbb{R}^2_+, \\ \omega|_{t=0} = b & x \in \mathbb{R}^2_+, \end{cases} \tag{LV}$$

subject to the boundary conditions

$$\nu \left(\partial_2 + (-\partial_1^2)^{\frac{1}{2}}\right)\omega = g \qquad t > 0 \quad x \in \partial\mathbb{R}^2_+. \tag{LBC}$$

I-4. Solution formula for linearized problem

Let $G(t, x)$ be the Gauss kernel in $\mathbb{R}^2$, and $E(x)$ be the Newton potential in $\mathbb{R}^2$, i.e.,

$$G(t, x) = \frac{1}{4\pi t} \exp\left(-\frac{|x|^2}{4t}\right) \qquad E(x) = -\frac{1}{2\pi} \log|x|.$$

The following notations will be used:

$$(h_1 * h_2)(x) = \int_{\mathbb{R}^2_+} h_1(x - y) h_2(y)\, \mathrm{d}y,$$

$$(h_1 \star h_2)(x) = \int_{\mathbb{R}^2_+} h_1(x - y^*) h_2(y)\, \mathrm{d}y \qquad y^* = (y_1, -y_2),$$

$$h * (g\mathcal{H}^1_{\{\partial\mathbb{R}^2_+\}})(x) = h \star (g\mathcal{H}^1_{\{\partial\mathbb{R}^2_+\}})(x) = \int_{\mathbb{R}} h(x_1 - y_1, x_2) g(y_1)\, \mathrm{d}y_1.$$

I-4. Solution formula for linearized problem

Then we set

$$\Gamma(t,x) \; = \; 2\big(\, \partial_1^2 \; + \; (-\partial_1^2)^{\frac{1}{2}}\partial_2 \,\big)(E * G(t))(x),$$

$$e^{t\Delta_N} f \; = \; G(t) * f \; + \; G(t) \star f.$$

Remark. (i) $\theta(t) = e^{t\Delta_N} f$ defines the solution to the heat equations in $\mathbb{R}^2_+$ subject to the homogeneous Neumann B. C. with the initial data f, i.e.,

$$\begin{cases} \partial_t\theta \; - \; \Delta\theta \; = \; 0 & t > 0 \quad x \in \mathbb{R}^2_+, \\ \partial_2\theta \; = \; 0 & t > 0 \quad x \in \partial\mathbb{R}^2_+, \\ \theta|_{t=0} \; = \; f & \qquad\quad\;\; x \in \mathbb{R}^2_+. \end{cases}$$

(ii) $\Gamma(0) \star f \; := \; \lim_{\epsilon \downarrow 0} \Gamma(\epsilon) \star f \; = \; 2\big(\, \partial_1^2 + (-\partial_1^2)^{\frac{1}{2}}\partial_2 \,\big) E \star f$ satisfies

$$(\partial_2 + (-\partial_1^2)^{\frac{1}{2}})\Gamma(0) \star f \; = \; 0 \qquad \text{in} \;\; \mathbb{R}^2_+.$$

I-4. Solution formula for linearized problem

Theorem. *The integral representation for solutions to (LV)-(LBC) is given by*

$$\omega(t) = e^{\nu t B} b - \Gamma(0) \star b + \int_0^t e^{\nu(t-s)B}\big(f(s) - g(s)\mathcal{H}^1_{\{x_2=0\}}\big)\,\mathrm{d}s$$
$$- \int_0^t \Gamma(0) \star \big(f(s) - g(s)\mathcal{H}^1_{\{x_2=0\}}\big)\,\mathrm{d}s.$$

Here $e^{\nu t B}$ *is defined by*

$$e^{\nu t B} f = e^{\nu t \Delta_N} f + \Gamma(\nu t) \star f\ .$$

Remark. For (NS) the solution formula is obtained by Solonnikov ('68) and Ukai ('87).

I-5. Note on the operator $\Gamma(0)\star$

$$\Gamma(0)\star f = 2(\partial_1^2 + (-\partial_1^2)^{\frac{1}{2}}\partial_2)E\star f \qquad E(x) = -\frac{1}{2\pi}\log|x|.$$

Thus the operator $\Gamma(0)\star$ does not possess a smoothing effect near the boundary. But this term does not appear in the vorticity equations, due to the following cancellation property.

Proposition. *If* $g = \partial_2(-\Delta_D)^{-1} f\,|_{x_2=0}$ *then* $\Gamma(0)\star(f - g\mathcal{H}^1_{\{x_2=0\}}) = 0$ *in* $\mathbb{R}^2_+$. *In particular, we have* $\Gamma(0)\star b = 0$ *in* $\mathbb{R}^2_+$ *if* $\partial_2(-\Delta_D)^{-1}b = 0$ *on* $\partial\mathbb{R}^2_+$.

Note that the condition $\partial_2(-\Delta_D)^{-1}b = 0$ on $\partial\mathbb{R}^2_+$ is nothing but the compatibility condition: $a_1 = 0$ on $\partial\mathbb{R}^2_+$.

I-6. Note on the generator of $\{e^{tB}\}_{t\geq 0}$

$$\dot{W}^{1,q}_{0,\sigma}(\mathbb{R}^2_+) = \text{the completion on } \|\nabla\cdot\|_{L^q} \text{ of the space of smooth and divergence free vector fields with compact support in } \mathbb{R}^2_+,$$

$$X_q = \{\ \mathrm{Rot}\, u \in L^q(\mathbb{R}^2_+) \,\big|\, u \in \dot{W}^{1,q}_{0,\sigma}(\mathbb{R}^2_+)\ \}.$$

Proposition. *Let* $q \in (1,\infty)$. *Then the one-parameter family* $\{e^{tB}\}_{t\geq 0}$ *defines a* C_0*-analytic semigroup in* X_q. *Moreover, the generator* B_q *of* $\{e^{tB}\}_{t\geq 0}$ *in* X_q *is given by*

$$\begin{aligned} D(B_q) &= \{f \in X_q \cap W^{2,q}(\mathbb{R}^2_+) \,\big|\, (\partial_2 + (-\partial_1^2)^{\frac{1}{2}})f = 0 \quad \text{on} \quad \partial\mathbb{R}^2_+\}, \\ B_q f &= \Delta f \qquad\qquad f \in D(B_q), \end{aligned}$$

and it follows that $\|\nabla^2 f\|_{L^q} \leq C\|B_q f\|_{L^q}$ *for all* $f \in D(B_q)$.

I-7. $L^p - L^q$ estimates for e^{tB}

Lemma. (i) *Let* $1 \le q < p \le \infty$ *or* $1 < q \le p < \infty$. *Then*

$$\|e^{tB} f\|_{L^p} \le C t^{-\frac{1}{q}+\frac{1}{p}} \|f\|_{L^q} \qquad t > 0.$$

(ii) *Let* $1 \le q \le p \le \infty$ *and* $k \in \mathbb{N}$. *Then*

$$\|\nabla^k e^{tB} f\|_{L^p} \le C t^{-\frac{1}{q}+\frac{1}{p}-\frac{k}{2}} \|f\|_{L^q} \qquad t > 0.$$

(iii) *Let* $1 \le q \le p \le \infty$ *and* $g = \partial_2(-\Delta)^{-1} f\,|_{x_2=0}$. *Then*

$$\|e^{tB}(f - g\mathcal{H}^1_{\{x_2=0\}})\|_{L^p} \le C t^{-\frac{1}{q}+\frac{1}{p}-\frac{1}{2}} \|\nabla^\perp(-\Delta_D)^{-1} f\|_{L^q} \qquad t > 0.$$

I-8. Solvability of IBP for the vorticity equations in $\mathbb{R}^2_+$

Theorem. *Assume that* $b \in L^p(\mathbb{R}^2_+)$, $\exists p \in (1,2)$, *and that* b *satisfies* $\partial_2(-\Delta_D)^{-1}b = 0$ *on* $\partial\mathbb{R}^2_+$. *Then there is* $T_\nu > 0$ *such that (V)-(BC) has a unique mild solution* $\omega \in C([0,T_\nu); L^p)$ *satisfying*

$$\sup_{0<t<T_\nu} t^{1/p-1/4}\|\omega(t)\|_{L^4} < \infty.$$

Furthermore, the solution is smooth in positive time.

Remark. (i) When $\Omega = \mathbb{R}^2$ the solvability of the vorticity equations is classical; Giga-Miyakawa-Osada ('88), Ben-Artzi ('94), Kato ('94).

(ii) In view of the solvability of the Navier-Stokes equations, the above theorem does not give a new result. For (NS) the L^p theory is already well developed; e.g. Solonnikov ('77), Weissler ('80).

I-9. Analysis of vorticity at inviscid limit

$$
\begin{cases}
\partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla p = 0 & t > 0 \quad x \in \mathbb{R}^2_+, \\
\nabla \cdot u = 0 & t \geq 0 \quad x \in \mathbb{R}^2_+, \\
u = 0 & t \geq 0 \quad x \in \partial\mathbb{R}^2_+, \\
u|_{t=0} = a & \quad\quad\quad x \in \mathbb{R}^2_+.
\end{cases}
\tag{NS}
$$

The behavior of solutions to (NS) at the inviscid limit: $\nu \to 0$

cf. Without boundaries the convergence to the Euler solutions is proved in various settings.

Ebin-Marsden ('70), Swann ('71), Kato ('72) Constantin-Wu ('94, '96), Chemin ('96), Danchin ('97,'99), Taniuchi ('04), Hmidi ('05, '06),Caflisch-Sammartino ('06) Masmoudi ('07), Sueur ('08), $\cdots$

I-10. Recall: Formal asymptotics - the Euler equations

Formally, by tending $\nu \to 0$ in (NS) we get the Euler equations for the ideal incompressible flows:

$$
\begin{cases}
\partial_t u_E + u_E \cdot \nabla u_E + \nabla p_E = 0 & t > 0, \quad x \in \mathbb{R}^2_+, \\
\operatorname{div} u_E = 0 & t \geq 0, \quad x \in \mathbb{R}^2_+, \\
u_{E,2} = 0 & t \geq 0, \quad x \in \partial\mathbb{R}^2_+, \\
u_E|_{t=0} = a & \qquad\quad x \in \mathbb{R}^2_+.
\end{cases}
\tag{E}
$$

Recall that $u_{NS} = 0$ on $\partial\mathbb{R}^2_+$, while $u_{E,1} \not\equiv 0$ on $\partial\mathbb{R}^2_+$ in general.
$\Rightarrow$ The boundary layer appears.

I-11. Recall: Formal asymptotics - the Prandtl equations

By assuming the expansion at $\nu \to 0$ as

$$u_{NS}(t,x) = u_E(t,x) \; + \; u_P^{(\nu)}(t,x) \; + \; \mathcal{O}(\nu^{\frac{1}{2}}), \tag{7}$$

$$u_P^{(\nu)}(t,x) = \Big(\, v_{P,1}\big(t, x_1, \frac{x_2}{\nu^{\frac{1}{2}}}\big), \; \nu^{\frac{1}{2}} \, v_{P,2}\big(t, x_1, \frac{x_2}{\nu^{\frac{1}{2}}}\big) \, \Big),$$

we get the Prandtl equations for $v_P(t, x_1, X_2)$:

$$\begin{cases} (\partial_t - \partial_{X_2}^2) v_{P,1} + v_{P,1}\partial_{x_1} v_{P,1} + v_{P,2}\partial_{X_2} v_{P,1} + \partial_{x_1}\pi_P = 0, \\ \partial_{x_1} v_{P,1} + \partial_{X_2} v_{P,2} = 0, \qquad \partial_{X_2}\pi_P = 0, \\ v_P \,|_{X_2=0} = 0, \\ \lim\limits_{X_2\to\infty} v_{P,1}(t,x_1,X_2) = u_{E,1}(t,x_1,0), \\ \lim\limits_{X_2\to\infty} \pi_P(t,x_1,X_2) = p_E(t,x_1,0). \end{cases} \tag{P}$$

I-12. Known mathematical results

So far the solvability of the Prandtl equations and the verification of the expansion (7) are established under particular situations only.

(1) Mathematical analysis of the Prandtl equations

- Solvability

Oleinik ('66), Matsui-Shirota ('84), Xin-Zhang ('04); Monotonic data
Asano ('88), Sammartino-Caflisch ('98); Analytic data
Lombardo et al. ('03); Analytic (in x_1 direction) data

- Ill-posedness of linearized Prandtl equations in Sobolev class

GérardVaret-Dormy ('10)

(2) Verification of the asymptotic expansion (7)

Asano ('88), Sammartino-Caflisch ('98); Analytic initial data

I-12. Known mathematical results

(3) Counter example for (7)

Grenier ('00); Invalidity of (7) around the linearly unstable shear layer profile $u_{stat}(x_2)$ for the stationary Euler equations

Roughly speaking, he considered the initial data of the form

$$u_0^{(\nu)}(x) \approx u_{stat}\left(\frac{x_2}{\nu^{\frac{1}{2}}}\right) + \nu^n w_0\left(\frac{x}{\nu^{\frac{1}{2}}}\right) \qquad n \gg 1,$$

and then, with a suitable choice of w_0, established the estimate for the corresponding NS solution u_{NS_ν} such as

$$\|u_{NS_\nu}(T_\nu) - u_{stat}(T_\nu, \frac{\cdot}{\nu^{\frac{1}{2}}})\|_{L^\infty} \geq c\nu^{\frac{1}{4}} \qquad \text{for some } \; T_\nu = O(\nu^{\frac{1}{2}} \log \nu^{-1}).$$

Here $u_{stat}(t, X_2)$ is the solution to the heat equations with the homogeneous Dirichlet B.C. and with the initial data $u_{stat}(X_2)$.

I-13. Known mathematical results

• L^2 **convergence:**

$$\lim_{\nu\to 0} \|u_{NS_\nu}(t) - u_E(t)\|_{L^2(\Omega)} = 0. \tag{8}$$

(1) Under the radial symmetry of the domain and the solution the convergence (8) is verified.

Matsui ('94), LopesFilho et al ('08), Kelliher ('09), $\cdots$

(2) Criterion on the convergence (8):

Kato ('84), Temam-Wang ('97), Kelliher ('08,'09), $\cdots$

$$\lim_{\nu\to 0} \int_0^T \nu \|\nabla u(t)\|^2_{L^2(\Omega_\nu)} \, \mathrm{d}t = 0 \qquad \Omega_\nu = \{x \in \Omega \mid \mathrm{dist}\,(x, \partial\Omega) \le \nu\}.$$

I-14. Inviscid limit for initial vorticity located away from the boundary

Goal: Establish the asymptotic expansion (for a short time) when the initial vorticity is located away from the boundary.

This class of initial data includes a dipole-type vortex as a typical example, which is used as a benchmark in the study (numerical or experimental one) of the interaction between the vorticity originated from the initial one and the vorticity created on the boundary; cf. Orlandi ('90).

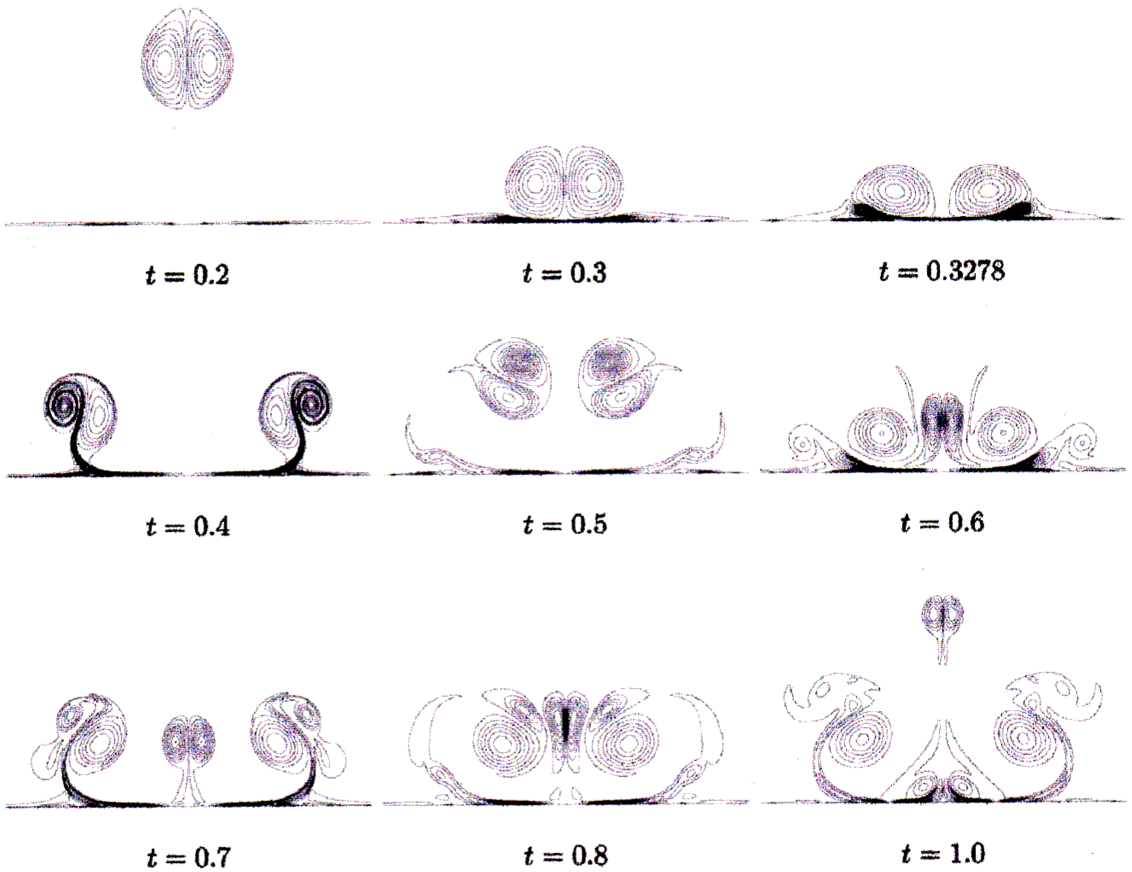

FIG. 2. Sequence of vorticity contour plots showing the flow evolution of a dipole colliding with a no-slip wall for integral-scale Reynolds number $\mathrm{Re} = 2500$. The contour levels are drawn for $\ldots, -100, -60, -20, 20, 60, 100, \ldots$

Extract from Nguyen-Farge-Schneider (Phys.Rev.Lett. 2011):
(the horizontal direction is $\longrightarrow$)

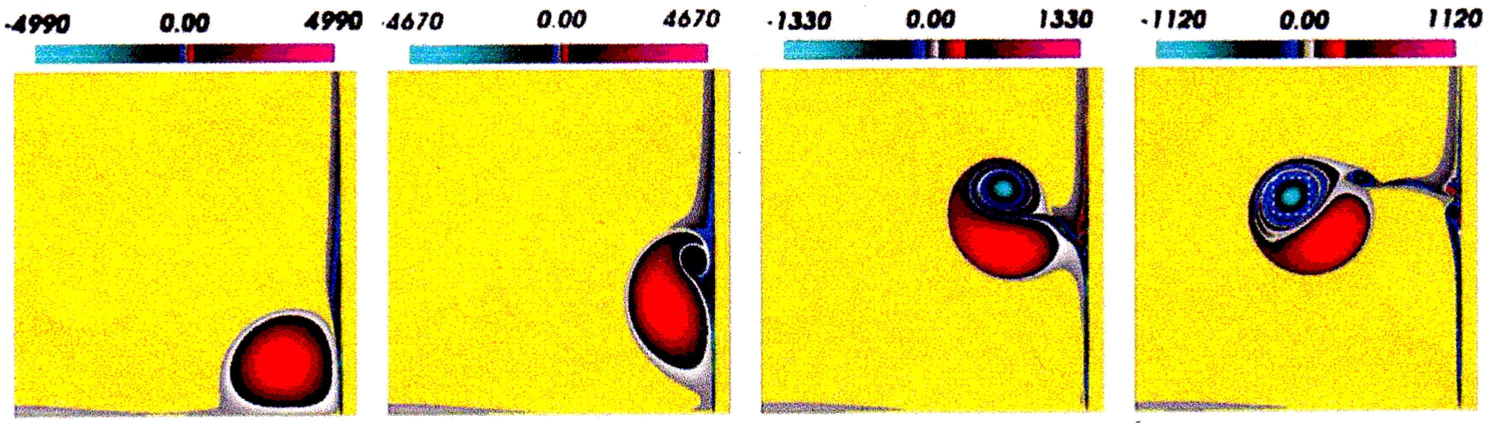

FIG. 1 (color online). Vorticity in the subdomain $[0.708, 0.962] \times [0.5, 0.754]$ at $t = 0.36$, 0.4, 0.45, and 0.495 (left to right) for $\mathrm{Re} = 7880$. The white dotted box at $t = 0.495$ frames region B (see text). Black pixels correspond to $\omega = \pm 300$ in all pictures.

Rough description of the behavior of vorticity with a dipole initial vortex

1. The dipole vortex approaches to the boundary, while the high vorticity is immediately created along the boundary (vortex line).

2. The dipole vortex collides into the boundary and the produced vortex line starts to roll up.

3. The produced vortex pairs with the original one and forms a secondary dipole.

4. The secondary dipole bounces back to the boundary.

Note. For the Euler flows the vorticity is independent of time along the trajectory flows (the Lagrange theorem) and the rebound of the secondary dipole vortex is not observed; Saffman ('79).

Formal asymptotic expansion by Prandtl will be verified (only) for Step **1**.

I-15. Main result

Theorem. *Let* $a \in L^p_\sigma(\mathbb{R}^2_+)$ *for some* $1 < p < \infty$ *and* $b \in W^{4,1}(\mathbb{R}^2_+) \cap W^{4,2}(\mathbb{R}^2_+)$. *Assume also that* $a = 0$ *on* $\partial\mathbb{R}^2_+$ *and that*

$$d_0 = \mathrm{dist}\,(\partial\mathbb{R}^2_+, \mathrm{supp}\ \mathrm{Rot}\ a) > 0. \tag{9}$$

Then there is $T > 0$ *such that the solution* $u^{(\nu)}_{NS}$ *to (NS) satisfies*

$$\|u^{(\nu)}_{NS}(t) - u_E(t) - \tilde{u}^{(\nu)}_P(t)\|_{L^\infty(\mathbb{R}^2_+)} \leq C\nu^{\frac{1}{2}} \qquad 0 < t \leq T. \tag{10}$$

The time T *is estimated from below as* $T \geq c\min\{d_0, 1\}$, *where* $c > 0$ *depends only on* $\|b\|_{W^{4,1}\cap W^{4,2}}$. *Here* $\tilde{u}^{(\nu)}_P$ *is the solution to the (modified) Prandtl equations.*

I-16. Remark on main theorem

(1) Sammartino-Caflisch ('98) proved (10) for the analytic initial data directly from the Navier-Stokes equations.

Our solution is not analytic in the region away from the boundary, and we use the vorticity formulation.

(2) The lower bound of T gives an information of the time period such that the vortex line remains stable and does not separate beyond the classical boundary layer thickness.

I-17. Formal asymptotics - vorticity for the Euler flows

By taking the formal limit $\nu \to 0$ in (V) we get

$$\partial_t \omega_E + u_E \cdot \nabla \omega_E = 0, \qquad u_E = \nabla^{\perp}(-\Delta_D)^{-1}\omega_E, \qquad \omega_E|_{t=0} = b. \quad (\mathrm{V_E})$$

cf.) The solvability of the two-dimensional Euler equations for the incompressible flows is well known,

Wolibner ('33), Yudovich ('63), Kato ('67), Bardos ('72), $\cdots$

I-18. Formal asymptotics - vorticity for boundary layer

Assume : $\omega^{(\nu)}(t,x) = \omega_E(t,x) \; + \; \nu^{-\frac{1}{2}} w_P(t, x_1, x_2/\nu^{\frac{1}{2}}) \; + \;$ remainder.

$$
\begin{cases}
\partial_t w_P - \partial_{X_2}^2 w_P \;=\; -(v_{E,1} + v_{P,1})\partial_1 w_P - (v_{E,2} + v_{P,2})\partial_2 w_P, \\
v_{E,1}(t, x_1, X_2) \;=\; u_{E,1}(t, x_1, 0), \\
v_{E,2}(t, x_1, X_2) \;=\; X_2 \partial_2 u_{E,2}(t, x_1, 0), \\
v_{P,1}(t, x_1, X_2) \;=\; \int_{X_2}^{\infty} w_P(t, x_1, Y_2)\, \mathrm{d}Y_2, \\
v_{P,2}(t, x_1, X_2) \;=\; -\partial_1 \Big(\int_0^{X_2} Y_2 w_P(t, x_1, Y_2)\, \mathrm{d}Y_2 \\
\qquad\qquad + X_2 \int_{X_2}^{\infty} w_P(t, x_1, Y_2)\, \mathrm{d}Y_2 \Big), \\
w_P|_{t=0} \;=\; 0,
\end{cases}
$$

subject to the boundary condition

$$
\partial_2 w_P \;=\; -\int_0^{\infty} (v_E + v_P) \cdot \nabla w_P \, \mathrm{d}Y_2 \;-\; \partial_2(-\Delta_D)^{-1}(u_E \cdot \nabla \omega_E).
$$

I-19. Key observation for the proof of main theorem

(1) Since the vorticity field ω_E of the Euler flows solves the transport equation we have

$$\cup_{0<t<T'} \ \mathrm{supp}\ \omega_E(t) \subset \{x \in \mathbb{R}^2_+ \mid x_2 \geq \frac{\tilde{d}_0}{2} > 0\}, \qquad (11)$$

for some $T' \geq c\tilde{d}_0$. In particular, all data in the vorticity equations are analytic near the boundary layer region.

$\Longrightarrow$ The Prandtl type equations should be solved (construction of the boundary layer).

But ω loses the analyticity as it leaves the boundary.

I-20. Key observation for the proof of main theorem

(2) We have to work in the Sobolev class away from the boundary.

But in the region away from the boundary, the arguments for the heat-convection equations in $\mathbb{R}^2$ will be applied to some extent.

(3) Due to the support property of ω_E, there should be the region

$$D_{small} = \{x \in \mathbb{R}^2_+ \mid c_1\tilde{d}_0 \leq x_2 \leq c_2\tilde{d}_0\} \qquad \exists c_2 > c_1 > 0,$$

where ω is exponentially small, say,

$$|\omega(t,x)| \leq C \exp\left(-\frac{c}{\nu t}\right) \quad \text{in} \quad D_{small}.$$

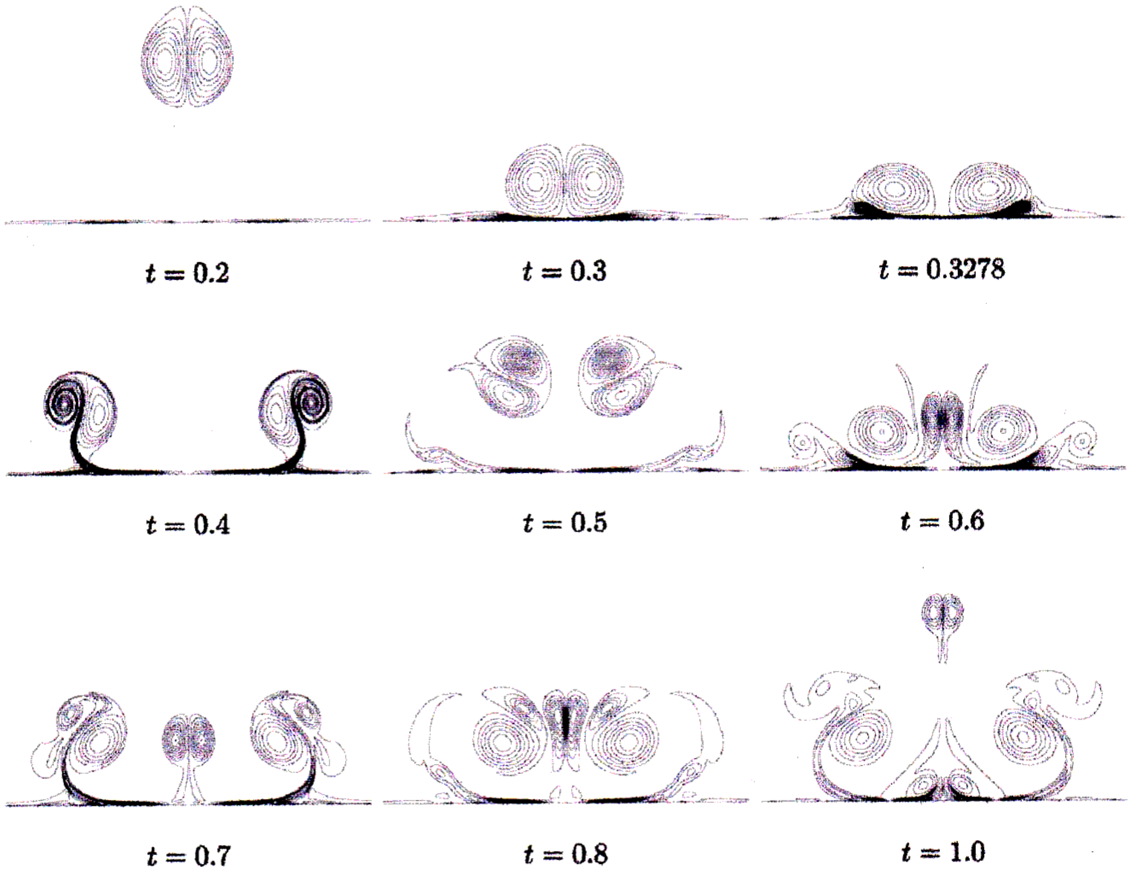

FIG. 2. Sequence of vorticity contour plots showing the flow evolution of a dipole colliding with a no-slip wall for integral-scale Reynolds number Re $=2500$. The contour levels are drawn for $\ldots,-100,-60,-20,20,60,100,\ldots$

I-21. Difficulty and Key idea

Difficulty: How to capture the properties (1) - (3) rigorously by taking into account the interaction between the vorticities inside and outside the boundary layer.

Key idea

(1) Decompose ω as

$$\omega = \omega_E + \omega_{B_\nu} + \omega_{I_\nu}, \qquad \omega_{B_\nu} = R_{\frac{1}{\nu}} w_{B_\nu}, \quad \omega_{I_\nu} = R_{\frac{1}{\nu}} w_{IB_\nu} + w_{II_\nu}.$$

$$(R_s f)(x) = s^{\frac{1}{2}} f(x_1, s^{\frac{1}{2}} x_2).$$

· The profile w_{B_ν} is taken so that converges to w_P (the Prandtl flows).

· The profiles w_{IB_ν} and w_{II} will be estimated as the order $O(\nu^{1/2})$ in suitable norms.

(2) Introduce suitable weighted function spaces for w_{B_ν}, w_{IB_ν}, w_{II_ν}.

(3) In order to estimate w_{II_ν} (the remainder of the Euler part) we appeal to the optimal pointwise estimate by Carlen-Loss ('95) for fundamental solutions to

$$\partial_t \theta - \nu \Delta \theta + u \cdot \nabla \theta = 0 \qquad \nabla \cdot u = 0 \qquad t > 0 \quad x \in \mathbb{R}^2.$$

$$P_u^{(\nu)}(t, x; s, y) \leq \frac{1}{4\pi\nu(t-s)} \exp\left(- \frac{\left(|x-y| - \int_s^t \|u(\tau)\|_{L^\infty} \, d\tau\right)_+^2}{4\nu(t-s)} \right).$$

Here $(\alpha)_+ = \max\{\alpha, 0\}$.

(4) Construct ω_{B_ν} and (w_{IB_ν}, w_{II_ν}) by the iteration scheme (with the aid of the solution formula) and use the abstract Cauchy-Kowalewski theorem (ACK).

cf.) ACK theorem; Nirenberg ('72), Nishida ('77), Kano-Nishida ('79), Safonov ('95)

I-22. Equation for ω_{B_ν}: $\omega = \omega_E + \omega_{B_\nu} + \omega_{I_\nu}$

$$\begin{cases} \partial_t \omega_{B_\nu} - \nu \Delta \omega_{B_\nu} + M(\omega_E + \omega_{B_\nu}, \omega_{B_\nu}) = 0 \\ \omega_{B_\nu}|_{t=0} = 0 \end{cases}$$

subject to the boundary condition

$$\nu(\partial_2 \omega_{B_\nu} + (-\partial_1^2)^{\frac{1}{2}} \omega_{B_\nu}) = N(\omega_E + \omega_{B_\nu}, \omega_{B_\nu}) + N(\omega_E, \omega_E)$$

$$M(f,g) = J(f) \cdot \nabla g \qquad\qquad J(f) = \nabla^\perp (-\Delta_D)^{-1} f$$
$$N(f,g) = -J_1(\ M(f,g)\ )|_{x_2=0}.$$

The solution is constructed in the form $\omega_{B_\nu} = R_{1/\nu} w_{B_\nu}$,

$$\sup_{0<t<T} \|w_{B_\nu}(t) - w_P(t)\|_{X_{B_\nu}^{(\mu, \frac{\rho}{t})}} \leq C \nu^{\frac{1}{2}} \qquad \text{for some } \mu, \rho > 0.$$

Here w_P is the vorticity of the Prandtl flows.

I-23. Equation for ω_{I_ν}: $\omega = \omega_E + \omega_{B_\nu} + \omega_{I_\nu}$

$$\begin{cases} \partial_t \omega_{I_\nu} - \nu\Delta\omega_{I_\nu} + J(\omega)\cdot\nabla\omega_{I_\nu} = -M(\omega_{I_\nu}, \omega_E + \omega_{B_\nu}) + F_\nu, \\ F_\nu = -M(\omega_{B_\nu}, \omega_E) + \nu\Delta\omega_E, \\ \omega_{I_\nu}|_{t=0} = 0. \end{cases}$$

subject to the boundary condition

$$\nu(\partial_2\omega_{I_\nu} + (-\partial_1^2)^{\frac{1}{2}}\omega_{I_\nu}) = N(\omega, \omega_{I_\nu}) + N(\omega_{I_\nu}, \omega_E + \omega_{B_\nu}) + N(\omega_{B_\nu}, \omega_E) + \nu J_1(\Delta\omega_E)\,|_{x_2=0}.$$

The solution ω_{I_ν} is constructed in the form $\omega_{I_\nu} = R_{1/\nu} w_{IB_\nu} + w_{II_\nu}$.

$\Longrightarrow$ Solve a suitable system for (w_{IB_ν}, w_{II_ν}).

I-24. Equation for w_{II_ν}**:** $\omega = \omega_E + \omega_{B_\nu} + \omega_{I_\nu}$, $\omega_{I_\nu} = R_{\frac{1}{\nu}} w_{IB_\nu} + w_{II_\nu}$

$$\begin{cases} \partial_t w_{II} - \nu\Delta w_{II} + J(\omega)\cdot\nabla w_{II} = -J(\omega_{I_\nu})\cdot\nabla\omega_E + F_\nu, \\ F_\nu = -J(\omega_{B_\nu})\cdot\nabla\omega_E + \nu\Delta\omega_E, \\ \partial_2 w_{II}\,|_{x_2=0} = 0, \\ w_{II}\,|_{t=0} = 0. \end{cases}$$

Remark. (1) We have

$\cup_{0<t<T'}\,\mathrm{supp}_x\left(-J(\omega_{I_\nu})\cdot\nabla\omega_E + F_\nu\right) \subset \{x \in \mathbb{R}^2_+ \mid x_2 \geq 32 d_E > 0\}$.

(2) The equation for w_{II_ν} is the heat-convection equations with the homogeneous Neumann boundary condition.

$\Longrightarrow$ By using the reflection we can use the estimates in $\mathbb{R}^2$ for the heat equation with a divergence free drift.

Recall: We construct the vorticity field ω of the form

$$\begin{aligned} \omega_{NS} &= \omega_E + R_{\frac{1}{\nu}} w_{B_\nu} + R_{\frac{1}{\nu}} w_{IB_\nu} + w_{II_\nu} \\ (R_{\frac{1}{\nu}} f)(x) &= \nu^{-\frac{1}{2}} f(x_1, \frac{x_2}{\nu^{\frac{1}{2}}}) \end{aligned}$$

- $R_{\frac{1}{\nu}} w_{B_\nu}$, $R_{\frac{1}{\nu}} w_{IB_\nu}$: flows with boundary layer structure
- ω_E, w_{II_ν} : flows without boundary layer structure

(i) Our solution loses the analyticity as it leaves the boundary.
Then, how?

(ii) The remainder term w_{II_ν} should be estimated as

$$w_{II_\nu}(t,x) = \begin{cases} \mathcal{O}(e^{-\frac{c}{\nu}}) & 0 \le x_2 \le c_1 d_0 \\ \mathcal{O}(\nu^{\frac{1}{2}}) & x_2 \ge c_2 d_0 \end{cases}$$

How to describe this drastic transition?

I-25. Function space for $w_{B_\nu}(t,x_1,X_2)$, $w_{IB_\nu}(t,x_1,X_2)$, $w_{II_\nu}(t,x)$

$$\varphi_{B_\nu}^{(\mu,\rho)}(\xi_1,X_2) = \exp\left(\frac{(\mu-\nu^{\frac{1}{2}}X_2)_+}{4}|\xi_1| + \rho X_2^2\right),$$
$$\varphi_{I_\nu}^{(\mu,\theta)}(\xi_1,x_2) = \exp\left(\frac{(\mu-x_2)_+}{4}|\xi_1| + \frac{\theta}{\nu}(6d_E - x_2)_+^2\right).$$

$$\|f\|_{X_{B_\nu}^{(\mu,\rho)}} = \sum_{0\le j,k\le 1} \left\|\varphi_{B_\nu}^{(\mu,\rho)}\langle\xi_1\rangle^{2-j}X_2^{\frac{k}{2}+j}\partial_{X_2}^j \hat{f}(\xi_1,X_2)\right\|_{L^2_{\xi_1}L^{1+k}_{X_2}},$$
$$\|f\|_{X_{IB_\nu}^{(\mu,\rho)}} = \sum_{0\le j,k\le 1} \left\|\varphi_{B_\nu}^{(\mu,\rho)}\langle\xi_1\rangle^{1-j}X_2^{\frac{k}{2}+j}\partial_{X_2}^j \hat{f}(\xi_1,X_2)\right\|_{L^2_{\xi_1}L^{1+k}_{X_2}},$$
$$\|f\|_{X_{II_\nu}^{(\mu,\theta)}} = \sum_{0\le j\le 1} \left\|\varphi_{I_\nu}^{(\mu,\theta)}\langle\xi_1\rangle^{j}\partial_2^{1-j} \hat{f}(\xi_1,x_2)\right\|_{L^2_{\xi_1}L^2_{x_2}} + \|\varphi_{I_\nu}^{(0,\theta)}f\|_{L^1_x}$$

$\hat{f}(\xi_1,X_2) = \mathcal{F}[f(\cdot,X_2)](\xi_1)$ $\quad (\alpha)_+ = \max\{\alpha,0\}$ $\quad d_E = 2^{-6}\min\{d_0,1\}$

I-26. Note on the invariant property of $X_{B_\nu}^{(\mu,\rho)}, X_{IB_\nu}^{(\mu,\theta)}, X_{II_\nu}^{(\mu,\theta)}$

Lemma. *Let* $t > s \geq 0,\ \mu \geq 0,\ 0 \leq \rho \leq 2^{-4},$ *and* $0 \leq \theta \leq 2^{-4}$. *Then it follows that*

$$\|\varphi_{B_\nu}^{(\mu,\frac{\rho}{t})}\mathcal{F}(\ R_\nu e^{\nu(t-s)\Delta_N} R_{\frac{1}{\nu}}\ f\ )\|_{L^2_{\xi_1}L^1_{X_2}} \ \leq\ C\ \|\varphi_{B_\nu}^{(\mu,\frac{\rho}{s})}\mathcal{F}(f)\|_{L^2_{\xi_1}L^1_{X_2}},$$

$$\|\varphi_{I_\nu}^{(\mu,\frac{\theta}{t})}\mathcal{F}(\ e^{\nu(t-s)\Delta_N}\ f\ )\|_{L^2_{\xi_1}L^2_{x_2}} \leq C\|\varphi_{I_\nu}^{(\mu,\frac{\theta}{s})}\mathcal{F}(f)\|_{L^2_{\xi_1}L^2_{x_2}}.$$

$$\varphi_{B_\nu}^{(\mu,\rho)}(\xi_1, X_2) = \exp\left(\frac{(\mu-\nu^{\frac{1}{2}}X_2)_+}{4}|\xi_1| + \rho X_2^2\right),$$

$$\varphi_{I_\nu}^{(\mu,\theta)}(\xi_1, x_2) = \exp\left(\frac{(\mu-x_2)_+}{4}|\xi_1| + \frac{\theta}{\nu}(6d_E - x_2)_+^2\right).$$

Proof. Set $g(t, X_2) = (4\pi t)^{-1/2} \exp\big(-\frac{X_2^2}{4t}\big)$. Then

$$\mathcal{F}(R_\nu e^{\nu(t-s)\Delta} R_{\frac{1}{\nu}} f)(\xi_1, X_2) = e^{-\nu(t-s)\xi_1^2} \int_{\mathbb{R}_+} g(t-s, X_2 - Y_2) \hat{f}(\xi_1, Y_2)\, \mathrm{d}Y_2.$$

From the inequalities

$$\begin{aligned}
(\mu - \nu^{\frac{1}{2}} X_2)_+ |\xi_1| \ &\le\ (\mu - \nu^{\frac{1}{2}} Y_2)_+ |\xi_1| \ + \ \nu^{\frac{1}{2}} |X_2 - Y_2|\, |\xi_1|, \\
\nu^{\frac{1}{2}} |X_2 - Y_2|\, |\xi_1| \ &\le\ \nu(t-s)\xi_1^2 \ + \ \frac{|X_2 - Y_2|^2}{4(t-s)},
\end{aligned}$$

we have

$$\begin{aligned}
|\mathcal{F}(R_\nu e^{\nu(t-s)\Delta} R_{\frac{1}{\nu}} f)(\xi_1, X_2)| &\lesssim e^{-\frac{3}{4}\nu(t-s)\xi_1^2 - \frac{1}{4}(\mu - \nu^{\frac{1}{2}} X_2)_+ |\xi_1|} \\
&\quad \cdot \int_{\mathbb{R}_+} g(2(t-s), X_2 - Y_2) e^{-\frac{\rho}{s} Y_2^2} |(\varphi_B^{(\mu, \frac{\rho}{s})} \hat{f})(\xi_1, Y_2)|\, \mathrm{d}Y_2.
\end{aligned}$$

Thus the desired estimate follows from

$$\|e^{\frac{\beta}{t} X_2^2} g(t-s) * h\, (X_2)\|_{L^1_{X_2}} \lesssim \|e^{\frac{\beta}{s} X_2^2} h(X_2)\|_{L^1_{X_2}} \qquad 0 < \beta < \frac{1}{4}.$$

I-27. Open problem

1. Qualitative or quantitative estimates of vorticity fields when the instability of the boundary layer occurs.

2. L^∞ bound of velocity fields such as

$$\sup_{0<\nu\ll 1} \int_0^T \|u_{NS}^{(\nu)}(t)\|_{L^\infty}\, \mathrm{d}t < \infty.$$

for sufficiently general class of initial data.

cf.) So far we have only $\displaystyle \sup_{0<\nu\ll 1} \sup_{0<t<c\nu^{\frac{1}{3}}} \|u_{NS}^{(\nu)}(t)\|_{L^\infty} < \infty.$

Topic II

Stability of the Burgers vortex for 3D perturbations

Topic II: Stability of the Burgers vortex
II-1. Flows with a background strain

$V = (V_1, V_2, V_3)^\top$: velocity field, $\qquad$ P: pressure field

$$
\text{(NS)} \quad \begin{cases} \partial_t V - \Delta V + (V, \nabla)V + \nabla P = 0 & t > 0 \quad x \in \mathbb{R}^3, \\ \nabla \cdot V = 0 & t > 0 \quad x \in \mathbb{R}^3, \\ V|_{t=0} = V_0 & \qquad\quad x \in \mathbb{R}^3. \end{cases}
$$

We consider the solution V of the form

$$
V = Mx + U \qquad\qquad M = \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.
$$

$U = (U_1, U_2, U_3)$: unknown perturbation velocity

II-2. Burgers vortex

Let $x = (x_h, x_3) \in \mathbb{R}^2 \times \mathbb{R}$ and set

$$G = g\,\mathbf{e}_3 \qquad g(x_h) = \frac{1}{4\pi} e^{-\frac{|x_h|^2}{4}} \qquad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

$$U^G(x_h) = \begin{pmatrix} -x_2\, v^g(|x_h|^2) \\ x_1\, v^g(|x_h|^2) \\ 0 \end{pmatrix} \qquad v^g(r) = \frac{1}{2\pi r}(1 - e^{-\frac{r}{4}}).$$

Burgers ('48): *For each* $\alpha \in \mathbb{R}$ *the velocity field* $Mx + \alpha U^G$ *is a stationary solution to* (NS).

- Main result in Topic II (Gallay-M. ('11)) -

For any $\alpha \in \mathbb{R}$ *the Burgers vortex* αG *is asymptotically stable with respect to small 3D perturbations.*

II-3. Vorticity equations

$$\Omega \;=\; \nabla\times V \;=\; \nabla\times U \;=\; \alpha G+\omega : \quad \text{expansion around } \alpha G$$

$$U \;=\; \nabla\times(-\Delta_{\mathbb{R}^3})^{-1}\Omega \;=\; \alpha U^G + K_{3D}*\omega : \quad \text{Biot-Savart law}$$

$$(V')\;\begin{cases} \partial_t\omega \;-\; (L-\alpha\Lambda)\omega \;+\; B(\omega,\omega) \;=0 & t>0 \quad x\in\mathbb{R}^3, \\ \nabla\cdot\omega \;=\; 0 & t>0 \quad x\in\mathbb{R}^3, \\ \omega|_{t=0} \;=\; \omega_0 & x\in\mathbb{R}^3. \end{cases}$$

$$L\omega = \Delta\omega - (Mx,\nabla)\omega + M\omega \qquad M = \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\Lambda\omega = (U^G,\nabla)\omega - (\omega,\nabla)U^G + (K_{3D}*\omega,\nabla)G - (G,\nabla)K_{3D}*\omega,$$

$$B(\omega,\omega) = (K_{3D}*\omega,\nabla)\omega - (\omega,\nabla)K_{3D}*\omega.$$

II-4. Function setting

$$L^2_g(\mathbb{R}^2) = \{ f \in L^2(\mathbb{R}^2) \mid \int_{\mathbb{R}^2} |f(x_h)|^2 \frac{\mathrm{d}x_h}{g(x_h)} < \infty \}$$

$$L^2_{g,0}(\mathbb{R}^2) = \{ f \in L^2_g(\mathbb{R}^2) \mid \int_{\mathbb{R}^2} f(x_h)\, \mathrm{d}x_h = 0 \}$$

$$X = BC(\mathbb{R};\ L^2_g(\mathbb{R}^2)) \qquad X_0 = BC(\mathbb{R};\ L^2_{g,0}(\mathbb{R}^2))$$

$$\|f\|_X = \|f\|_{X_0} = \sup_{x_3 \in \mathbb{R}} \|f(\cdot, x_3)\|_{L^2_g(\mathbb{R}^2)}$$

The perturbation vorticity ω is taken from the function space

$$\mathbb{X} = X^2 \times X_0$$

II-5. Key linear estimate

Theorem (Gallay-M. ('11)).

Let $f \in \mathbb{X}$ *satisfy* $\nabla \cdot f = 0$. *Then it follows that*

$$\|e^{t(L-\alpha\Lambda)} f\|_{\mathbb{X}} \le C_\alpha e^{-\frac{t}{2}} \|f\|_{\mathbb{X}} \qquad t > 0.$$

- The above result implies that the spectral bound of $L - \alpha\Lambda$ is estimated uniformly in the circulation number α.

- The Gaussian weight can be relaxed to a polynomial weight.

II-6. Decomposition of $L - \alpha\Lambda$ into 2D parts and 3D parts

$$L_{2D,\alpha}\,\omega \;=\; \begin{pmatrix} L_{2D,\alpha,h}\;\omega_h \\ L_{2D,\alpha,3}\;\omega_3 \end{pmatrix}$$

$$= \begin{pmatrix} \mathcal{L}_h\omega_h \;-\; \frac{3}{2}\omega_h \;-\; \alpha(U_h^G, \nabla_h)\omega_h \;+\; \alpha(\omega_h, \nabla_h)U_h^G \\ \\ \mathcal{L}_h\omega_3 \;-\; \alpha(U_h^G, \nabla_h)\omega_3 \;-\; \alpha(K_{2D} * \omega_3, \nabla_h)g \end{pmatrix}.$$

Here $\mathcal{L}_h \;=\; \Delta_h + \dfrac{x_h}{2}\cdot\nabla_h + 1$ and $K_{2D}{*}\omega_3 \;=\; \nabla_h^{\perp}(-\Delta_{\mathbb{R}^2})^{-1}\omega_3$.

II-6. Decomposition of $L - \alpha\Lambda$ into 2D parts and 3D parts

$$L_{2D,\alpha}\,\omega \;=\; \begin{pmatrix} L_{2D,\alpha,h}\,\omega_h \\ L_{2D,\alpha,3}\,\omega_3 \end{pmatrix}$$

$$= \begin{pmatrix} \mathcal{L}_h\omega_h - \frac{3}{2}\omega_h - \alpha(U_h^G,\nabla_h)\omega_h + \alpha(\omega_h,\nabla_h)U_h^G \\ \\ \mathcal{L}_h\omega_3 - \alpha(U_h^G,\nabla_h)\omega_3 - \alpha(K_{2D}*\omega_3,\nabla_h)g \end{pmatrix}.$$

Here $\mathcal{L}_h = \Delta_h + \dfrac{x_h}{2}\cdot\nabla_h + 1$ and $K_{2D}*\omega_3 = \nabla_h^{\perp}(-\Delta_{\mathbb{R}^2})^{-1}\omega_3$.

Then $L - \alpha\Lambda = L_{2D,\alpha} + \partial_3^2 - x_3\partial_3 - \alpha H$, where

$$H\omega = (K_{3D}*\omega,\nabla)G - (K_{2D}*\omega_3,\nabla)G - (G,\nabla)K_{3D}*\omega.$$

Note: If $\omega = (0,0,\omega_3(x_h))^\top$ then $H\omega = 0$.

II-7. Decomposition of $e^{t(L-\alpha\Lambda)}$ into 2D parts and 3D parts

The operator $L_{2D,\alpha}$ is vectorial but two-dimensional.

The semigroup $R_\alpha(t)$ of $L_{2D,\alpha} + \partial_3^2 - x_3\partial_3$ is

$$(R_\alpha(t)f)(x) \;=\; \frac{1}{\sqrt{2\pi(1-e^{-2t})}}\int_{\mathbb{R}} e^{-\frac{|x_3e^{-t}-y_3|^2}{2(1-e^{-2t})}}(e^{tL_{2D,\alpha}}f(\cdot,y_3))(x_h)\,\mathrm{d}y_3.$$

Then the original solution $\omega(t) = e^{t(L-\alpha\Lambda)}f$ is given by

$$\omega(t) \;=\; e^{t(L-\alpha\Lambda)}f \;=\; R_\alpha(t)f \;-\; \alpha\int_0^t R_\alpha(t-s)H\omega(s)\,\mathrm{d}s.$$

II-7. Key structure of $e^{t(L-\alpha\Lambda)}$

• From a rough estimate $\|e^{t(L-\alpha\Lambda)}\| \le C_\alpha e^{C_\alpha t}$ and a stretching effect $\partial_3^k e^{t(L-\alpha\Lambda)} = e^{-kt} e^{t(L-\alpha\Lambda)} \partial_3^k$, we have an exponential temporal decay of $\partial_3^k e^{t(L-\alpha\Lambda)}$ at least for $k \gg 1$.

$$L - \alpha\Lambda = L_{2D,\alpha} + \partial_3^2 - x_3\partial_3 - \alpha H$$

$$H\omega = (K_{3D} * \omega, \nabla)G - (K_{2D} * \omega_3, \nabla)G - (G, \nabla)K_{3D} * \omega.$$

II-7. Key structure of $e^{t(L-\alpha\Lambda)}$

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• Since $H\omega$ is estimated in terms of $\partial_3\omega$, the inhomogeneous term $\int_0^t R_\alpha(t-s)H\omega(s)\,\mathrm{d}s$ is negligible for $t \gg 1$. Hence it suffices to focus on $R_\alpha(t)$, that is, the spectrum of the 2D operator $L_{2D,\alpha}$ in $(L^2_g(\mathbb{R}^2))^2 \times L^2_{g,0}(\mathbb{R}^2)$.

$$(R_\alpha(t)f)(x) = \frac{1}{\sqrt{2\pi(1-e^{-2t})}} \int_{\mathbb{R}} e^{-\frac{|x_3 e^{-t} - y_3|^2}{2(1-e^{-2t})}} (e^{tL_{2D,\alpha}} f(\cdot, y_3))(x_h)\,\mathrm{d}y_3.$$

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$$L_{2D,\alpha}\,\omega = \begin{pmatrix} \mathcal{L}_h\omega_h - \frac{3}{2}\omega_h - \alpha(U_h^G, \nabla_h)\omega_h + \alpha(\omega_h, \nabla_h)U_h^G \\ \\ \mathcal{L}_h\omega_3 - \alpha(U_h^G, \nabla_h)\omega_3 - \alpha(K_{2D} * \omega_3, \nabla_h)g \end{pmatrix}.$$

Here $\mathcal{L}_h = \Delta_h + \dfrac{x_h}{2}\cdot\nabla_h + 1$, and $-\mathcal{L}_h \ge 0$ in $L^2_g(\mathbb{R}^2)$.

II-7. Key structure of $e^{t(L-\alpha\Lambda)}$

• From a rough estimate $\|e^{t(L-\alpha\Lambda)}\| \le C_\alpha e^{C_\alpha t}$ and a stretching effect $\partial_3^k e^{t(L-\alpha\Lambda)} = e^{-kt} e^{t(L-\alpha\Lambda)} \partial_3^k$, we have an exponential temporal decay of $\partial_3^k e^{t(L-\alpha\Lambda)}$ at least for $k \gg 1$.

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From G.-W. ('05) we already have $\sigma(L_{2D,\alpha,3}) \subset \{\lambda \in \mathbb{C} \mid \mathrm{Re}\,\lambda \le -1/2\}$. So we consider the eigenvalue problem

$$L_{2D,\alpha,h}\omega_h := \mathcal{L}_h \omega_h - \frac{3}{2}\omega_h - \alpha(U_h^G, \nabla_h)\omega_h + \alpha(\omega_h, \nabla_h) U_h^G = \lambda \omega_h. \tag{12}$$

By taking the inner product of $(L^2_g(\mathbb{R}^2))^2$ with ω_h we have

$$\begin{aligned} & \mathrm{Re}\lambda\, \|\omega_h\|^2 \\ = \; & \mathrm{Re}\,\langle(\mathcal{L}_h - \frac{3}{2}I)\omega_h, \omega_h\rangle \; - \; \alpha \mathrm{Re}\,\langle(U_h^G, \nabla_h)\omega_h, \omega_h\rangle \; + \; \alpha \mathrm{Re}\,\langle(\omega_h, \nabla_h)U_h^G, \omega_h\rangle. \end{aligned}$$

By taking the inner product of $(L^2_g(\mathbb{R}^2))^2$ with ω_h we have

$$
\begin{aligned}
&\mathrm{Re}\lambda\, \|\omega_h\|^2 \\
&= \mathrm{Re}\,\langle(\mathcal{L}_h - \frac{3}{2}I)\omega_h, \omega_h\rangle \;-\; \alpha\mathrm{Re}\,\langle(U^G_h, \nabla_h)\omega_h, \omega_h\rangle \;+\; \alpha\mathrm{Re}\,\langle(\omega_h, \nabla_h)U^G_h, \omega_h\rangle \\
&\le -\frac{3}{2}\|\omega_h\|^2 \;+\; \alpha\mathrm{Re}\,\langle(\omega_h, \nabla_h)U^G_h, \omega_h\rangle \\
&= -\frac{3}{2}\|\omega_h\|^2 \;+\; 2\alpha\mathrm{Re}\int_{\mathbb{R}^2} x_h \cdot \omega_h\; x_h^{\perp} \cdot \bar{\omega}_h\; f(|x_h|^2)\, \frac{\mathrm{d}x_h}{g(x_h)},
\end{aligned}
\tag{13}
$$

where $f(r) = \frac{1}{2\pi}\frac{\mathrm{d}}{\mathrm{d}r}(1 - e^{-\frac{r}{4}})/r$.

By taking the inner product of $(L^2_g(\mathbb{R}^2))^2$ with ω_h we have

$$
\begin{aligned}
&\mathrm{Re}\lambda\, \|\omega_h\|^2 \\
=\ & \mathrm{Re}\,\langle(\mathcal{L}_h - \frac{3}{2}I)\omega_h, \omega_h\rangle \ - \ \alpha \mathrm{Re}\,\langle(U_h^G, \nabla_h)\omega_h, \omega_h\rangle \ + \ \alpha \mathrm{Re}\,\langle(\omega_h, \nabla_h)U_h^G, \omega_h\rangle \\
\leq\ & -\frac{3}{2}\|\omega_h\|^2 \ + \ \alpha \mathrm{Re}\,\langle(\omega_h, \nabla_h)U_h^G, \omega_h\rangle \\
=\ & -\frac{3}{2}\|\omega_h\|^2 \ + \ 2\alpha \mathrm{Re} \int_{\mathbb{R}^2} x_h \cdot \omega_h \ x_h^{\perp} \cdot \bar{\omega}_h \ f(|x_h|^2) \, \frac{\mathrm{d}x_h}{g(x_h)}.
\end{aligned}
\tag{14}
$$

where $f(r) = \frac{1}{2\pi}\frac{\mathrm{d}}{\mathrm{d}r}(1 - e^{-\frac{r}{4}})/r$. The function $x_h \cdot \omega_h$ satisfies

$$
\lambda x_h \cdot \omega_h \ = \ (\mathcal{L}_h - 2I)\, x_h \cdot \omega_h \ - \ \alpha(U_h^G, \nabla)\, x_h \cdot \omega_h \ - \ 2\nabla_h \cdot \omega_h,
$$

thus we have

$$
\mathrm{Re}\lambda\, \|x_h \cdot \omega_h\|^2 \ \leq \ -2\|x_h \cdot \omega_h\|^2 \ - \ 2\mathrm{Re}\,\langle \nabla_h \cdot \omega_h, x_h \cdot \omega_h\rangle. \tag{15}
$$

By taking the inner product of $(L^2_g(\mathbb{R}^2))^2$ with ω_h we have

$$\begin{aligned}
&\mathrm{Re}\lambda\,\|\omega_h\|^2\\
&= \mathrm{Re}\,\langle(\mathcal{L}_h - \frac{3}{2}I)\omega_h, \omega_h\rangle - \alpha\mathrm{Re}\,\langle(U^G_h, \nabla_h)\omega_h, \omega_h\rangle + \alpha\mathrm{Re}\,\langle(\omega_h, \nabla_h)U^G_h, \omega_h\rangle\\
&\le -\frac{3}{2}\|\omega_h\|^2 + \alpha\mathrm{Re}\,\langle(\omega_h, \nabla_h)U^G_h, \omega_h\rangle\\
&= -\frac{3}{2}\|\omega_h\|^2 + 2\alpha\mathrm{Re}\int_{\mathbb{R}^2} x_h\cdot\omega_h\; x_h^{\perp}\cdot\bar{\omega}_h\; f(|x_h|^2)\,\frac{\mathrm{d}x_h}{g(x_h)}. \qquad (16)
\end{aligned}$$

where $f(r) = \frac{1}{2\pi}\frac{\mathrm{d}}{\mathrm{d}r}(1 - e^{-\frac{r}{4}})/r$. The function $x_h\cdot\omega_h$ satisfies

$$\lambda x_h\cdot\omega_h = (\mathcal{L}_h - 2I)x_h\cdot\omega_h - \alpha(U^G_h, \nabla)x_h\cdot\omega_h - 2\nabla_h\cdot\omega_h,$$

thus we have

$$\mathrm{Re}\lambda\,\|x_h\cdot\omega_h\|^2 \le -2\|x_h\cdot\omega_h\|^2 - 2\mathrm{Re}\,\langle\nabla_h\cdot\omega_h, x_h\cdot\omega_h\rangle. \qquad (17)$$

The function $\nabla_h\cdot\omega_h$ satisfies

$$\lambda\nabla_h\cdot\omega_h = (\mathcal{L}_h - I)\nabla_h\cdot\omega_h - \alpha(U^G_h, \nabla_h)\nabla_h\cdot\omega_h,$$

which leads to

$$\mathrm{Re}\lambda\,\|\nabla_h\cdot\omega_h\|^2 \le -\frac{3}{2}\|\nabla_h\cdot\omega_h\|^2. \qquad (18)$$

Topic III

Stability of the Lamb-Oseen vortex in 2D exterior domains

II-0. Recall: Main result

$\Omega \subset \mathbb{R}^2$: exterior domain
u_0: initial velocity $\qquad \omega_0 = \mathrm{Rot}\, u_0$: initial vorticity
$\alpha = \displaystyle\int_\Omega \omega_0(x)\,\mathrm{d}x$: circulation number

- Main Theorem (Gallay-M. preprint) -

Let $u_0 \in \dot{W}^{1,p}_{0,\sigma}(\Omega)$ *for some* $p \in [1,2)$ *and* ω_0 *satisfies*

$\int_\Omega (1+|x|^2)^m |\omega_0(x)|^2\,\mathrm{d}x < \infty$ *for some* $m > 1$.

Then there is $\epsilon > 0$ *such that if* $|\alpha| \le \epsilon$ *then the solution* u *to* (NS) *satisfies*

$$\lim_{t\to\infty} \|u(t) - \frac{\alpha}{(1+t)^{\frac12}} u^g\Big(\frac{\cdot}{(1+t)^{\frac12}}\Big)\|_{L^2} = 0. \qquad (19)$$

III-1. Decomposition of initial velocity

Consider the Navier-Stokes equations in 2D exterior domains

$$\partial_t u + u \cdot \nabla u - \Delta u + \nabla p = 0 \qquad \nabla \cdot u = 0 \qquad \text{(NS)}$$

subject to the no-slip boundary condition. Under the assumption of the theorem we can express the initial velocity as

$$u(0) = \alpha \chi u^g(1) + v(0).$$

Here χ is a radial cut-off such that $\chi = 1$ for $|x| \gg 1$, $\alpha u^g(1)$ is the Lamb-Oseen vortex at time $t = 1$, and $v(0) \in L^2_\sigma(\Omega) \cap L^q_\sigma(\Omega)$ with some $q \in (1, 2)$ is an initial perturbation.

III-2. Logarithmic energy estimate for $v(t) = u(t) - \alpha\chi u^g(1+t)$

$$\partial_t v + \alpha \mathbf{P}\Big(u^\chi \cdot \nabla v + v \cdot \nabla u^\chi + v \cdot \nabla v\Big) = -Av + \alpha R^\chi. \qquad (20)$$

$\mathbf{P}$: the Helmholtz projection in $(L^2(\Omega))^2$
A : the Stokes operator in $L^2_\sigma(\Omega)$,
$R^\chi = (\Delta\chi)u^g + 2\nabla\chi \cdot \nabla u^g$: circular flow with compact support

Proposition (Gallay-M.). *There is $K > 0$ such that for any α and $v(0) \in L^2_\sigma(\Omega)$ the solution to (20) with the initial data $v(0)$ satisfies*

$$\|v(t)\|_{L^2}^2 + \int_0^t \|\nabla v(s)\|_{L^2}^2 \, \mathrm{d}s \le K_1\Big(\|v_0\|_{L^2(\Omega)}^2 + \alpha^2 \log(1+t) + D_{\alpha,\rho}\Big),$$

where $D_{\alpha,\rho} = \alpha^2 \log(1+|\alpha|) + \alpha^2\rho^2$ and ρ is the diameter of $\mathbb{R}^2 \backslash \Omega$.

III-3. Fractional primitive of v

Lemma (Borchers-Miyakawa ('92), Kozono-Ogawa ('93)).
Let $q \in (1,2)$ *and* $\mu = 1/q - 1/2$. *For all* $v \in L^2_\sigma(\Omega) \cap L^q(\Omega)^2$, *there exists a unique* $w \in D(A^\mu) \subset L^2_\sigma(\Omega)$ *such that* $v = A^\mu w$. *Moreover, there exists a constant* $C = C(q) > 0$ *(independent of* v *and* Ω*) such that* $\|w\|_{L^2(\Omega)} \le C\|v\|_{L^q(\Omega)}$.

Then we combine the argument of Kozono-Ogawa ('93) with the logarithmic energy estimate to get the estimate of the fractional primitive $w(t) = A^{-\mu}v(t)$ such that

$$\|w(t)\|_{L^2}^2 + \int_0^t \|\nabla w(s)\|_{L^2}^2 \, \mathrm{d}s$$
$$\le K'(1+t)^{c\alpha^2} \exp\Big(K'(\|v_0\|_{L^2}^2 + D_{\alpha,\rho})\Big)(\|v_0\|_{L^q}^2 + \rho^2\alpha^2),$$

which leads to the temporal decay of $\|\nabla w(t)\|_{L^2}$ if $c\alpha^2 < 1$.