Waseda University

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Lecture Notes on

Stokes and Navier-Stokes initial boundary value problem

with nondecaying data

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1. Introduction

In the ambit of the mini course, to which I have been kindly invited to participate, I would like to develop some results concerning the Stokes and Navier-Stokes initial boundary value problems with nondecaying data.

To better explain the aims, we split the case of the Stokes problem from the one of the Navier-Stokes.

The Stokes initial boundary value problem with nondecaying data is essentially meant as the well posedeness of the IBVP in exterior domain with an initial data in L^∞.

The question concerns the existence and the properties of a possible analytic semigroup in L^∞. Usually, in literature this topic is known as the Maximum Modulus theorem. The problems related to the topic, which is one of the classical in partial differential equations, have been open for long time for the equations of the hydrodynamics. Just in the last ten years, especially for a bounded domain, the problem has had contributes of some authors: [Solonnikov $(2002)_1$]-[Solonnikov $(2006)_1$], [Abe & Giga (2011), Abe & Giga (2012)]). I will discuss the case of the initial boundary value problem in exterior domains. I would like to point out that the result has to be read in the following sense. Given any maximum modulus theorem for solutions to the Stokes IBVP in bounded domains, then, the theorem proves a maximum modulus theorem for the solutions of the Stokes IBVP in exterior domains with an initial data just belonging to $L^\infty(\Omega)$.

For the Navier-Stokes initial boundary value problem with non decaying data also it is meant the IBVP in exterior domains with a data in L^∞.

I start saying that this topic had contributes concerning the uniqueness several years ago, but it has had a systematic study of the well posedeness essentially in the last 15 years. Indeed, if we put aside the pioneer papers by [Leray (1934)] and [Knightly (1972)], we can consider the first contribute dates back to [Giga, Inui, & Matsui (1999)]. This result concerns the Cauchy problem. Concerning the 2-D nondacaying solutions, of a special interest are the papers [Giga, Matsui and Sawada (2001)], [Sawada & Taniuchi (2007)], where the authors prove existence of solutions global (in time).

Although the exterior domain is the interesting case (in the special assumption of the initial data in $L^\infty \cap C^{0,\alpha}$ a recent contribute is due to [Galdi, Maremonti & Zhou (2011)]) here, for the sake of brevity, we restrict ourselves to discuss the Cauchy problem in the terms proposed in [Maremonti $(2008)_2$]. Hence we analyze the Cauchy problem by means of a different approach with respect to the one employed in the quoted paper by [Giga, Inui, & Matsui (1999)]. Indeed, we are essentially interested in giving pointwise estimates for the pressure field.

Nevertheless we are able to give a uniqueness theorem which in some sense becomes a sort of structure theorem for the solutions given by other authors.

Finally, we recall that there is a wide and interesting literature concerning the Navier-Stokes Cauchy problem with a initial data non bounded like a linear function of x and the Navier-Stokes flows in the exterior of rotating obstacle. We do not consider these problems and we refer the reader to the papers [Hieber & Sawada] and [Hishida & Shibata], respectively, and the quoted references in them.

In summary, we consider the Stokes problem

$$\begin{aligned} & u_t - \Delta u = -\nabla \pi_u\,, \text{ in } (0,T)\times\Omega, \\ & \nabla\cdot u = 0\,, \text{ in } (0,T)\times\Omega, \\ & u(t,x) = 0\,, \text{ on } (0,T)\times\partial\Omega,\ u(0,x) = u_\circ(x), \text{ on } \{0\}\times\Omega, \end{aligned} \tag{1.1}$$

where $u_t = \frac{\partial u}{\partial t}$ and $u_0 \in L^\infty(\Omega)$ (here and in the sequel by the same symbol we denote the space of the vector functions and the one of the scalar functions) with

$$(u_\circ, \nabla\varphi) = 0, \text{ for all } \varphi \in \widehat{W}^{1,1}(\Omega), \tag{1.2}$$

where $\widehat{W}^{1,1}(\Omega) = \{\varphi \in L^1_{\ell oc}(\Omega) : \nabla\varphi \in L^1(\Omega)\}$. We call (1.2) the null divergence in weak form for elements belonging to $L^\infty(\Omega)$. The condition (1.2) of null divergence has been given by Abe and Giga in [Abe & Giga (2011)].

The same IVP with nondecaying data will be considered for the Navier-Stokes equations:

$$\begin{aligned} & v_t + v\cdot\nabla v + \nabla\pi = \Delta v,\ \nabla\cdot v = 0, \quad \text{in } (0,T)\times\mathbb{R}^n, \\ & v(0,x) = v_0(x) \text{ on } \{0\}\times\mathbb{R}^n. \end{aligned} \tag{1.3}$$

We initially discuss the problem (1.3). Hence sections 2. - 5. are devoted to this problem. Subsequently, in sections 6.-10. we discuss problem (1.1).

I would like to conclude the introduction by giving my special thanks to Professor Y. Shibata, for his kind invitation to give these lectures, for the stimulating and interesting discussions on related topics, and, last, but not least, for the warm hospitality.

The Navier-Stokes Cauchy problem with nondecaying initial data

2. The Giraud theorem

The statement and related proof of the following Giraud's theorem (1934) are due to Ladyzhenskaya and Uralceva, see Lemma 2.2 of [Ladyzhenskaya & Uralceva]. Also it is a special case of Lemma 3.5 of [Maremonti $(2008)_2$].

By $K(z)$ we denote a C^2 smooth function on $\mathbb{R}^n - \{0\}$. We assume that

(a) $K(z)$ is homogeneous of degree $1-n$ on $\mathbb{R}^n - \{0\}$, namely, $K(\mu z) = \mu^{1-n}K(z)$ for any $\mu > 0$ and $z \in \mathbb{R}^n - \{0\}$;

which implies $\displaystyle\int_{|z|=1} \frac{\partial K(z)}{\partial z_j} d\sigma = 0$, for any $j = 1, \dots, n$.

Let us consider the integral transform, $i \in (1, \cdots, n)$:

$$T(g)(x) := D_{x_i} \int_{\mathbb{R}^n} K(x-y)g(y)dy. \tag{2.1}$$

We set

$$\int_{\mathbb{R}^n}^{*} F(y)dy \equiv \text{P.V.} \int_{\mathbb{R}^n} F(y)dy := \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n - B(x,\varepsilon)} F(y)dy \,,$$

meaning the principal value singular integral in the Cauchy sense.

The following result holds:

Lemma 2.1 *Let $g(x) \in C^{0,\mu}(\mathbb{R}^n)$, $\mu \in (0,1)$, with compact support. Then, the transformation $T(g) \in C^{0,\mu}(\mathbb{R}^n)$ with*

$$[T(g)]^\mu \le c[g]^\mu \,, \tag{2.2}$$

where the constant c depends only on the Euclidean dimension n [1].

Proof. We begin proving that

$$T(g)(x) = -k_i g(x) + \int_{\mathbb{R}^n}^{*} D_{y_i} K(x-y)g(y)dy, \tag{2.3}$$

where $k_i = -\displaystyle\int_{|z|=1} K(z)z_i d\sigma$. We consider a sequence $\{g^k\} \subset C_0^1(\mathbb{R}^n)$ converging to g uniformly in $x \in \mathbb{R}^n$ and having $[g^k]^\alpha \le [g]^\alpha$ [2]. Hence we get

$$\int_{\mathbb{R}^n} K(x-y)g(y)dy = \lim_k \int_{\mathbb{R}^n} K(x-y)g^k(y)dy, \text{ uniformly in } x \in \mathbb{R}^n. \tag{2.4}$$

[1] We denote by the symbol $[\![\cdot]\!]^\mu$ the Hölder seminorm, by the symbol $|\cdot|^{m,\mu}$ the norm in the $C^{m,\mu}(\overline{\Omega})$ space.

[2] For this task it is enough to consider the mollification of g by means of a mollifier $J^k[\cdot]$.

Moreover we get

$$T(g^k)(x) = -\int_{\mathbb{R}^n} K(z) D_{z_i} g^k(x-z)dz$$
$$= \lim_{\eta\to 0}\Big[\int_{\mathbb{R}^n - B(0,\eta)} D_{z_i}K(z)g^k(x-z)dz + \int_{|z|=\eta} K(z)\frac{z_i}{\eta}(g^k(x-z)-g^k(x)d\sigma_\eta - k_i g^k(x)\Big].$$

The last formula implies:

1) $T(g^k)(x) = -k_i g^k(x) + \overset{*}{\int_{\mathbb{R}^n}} D_{y_i}K(x-y)g^k(y)dy.$

2) $T(g^k)(x)$ converges to $-k_i g(x) + \overset{*}{\int_{\mathbb{R}^n}} K(x-y)g(y)dy$, uniformly in $x\in\mathbb{R}^n$,

which proves (2.4) [3].

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[3] We prove the uniform convergence. We set

$$I_1 = \int_{\mathbb{R}^n - B(0,1)} D_{z_i}K(z)g(x-z)dz,\ I_2 = \int_{B(0,1)-B(0,\eta)} D_{z_i}K(z)(g(x-z)-g(x))dz.$$

Hence, with obvious meaning of the symbol, we get

$$\int_{\mathbb{R}^n - B(0,\eta)} D_{z_i}K(z)g^k(x-z)dz = I_1^k + I_2^k\,.$$

Hence we deduce

$$\big|I_2^k - I_2\big| \le \int_{B(0,1)-B(0,\eta)} |D_{z_i}K(z)|g^k(x-z) - g(x-z) + g(x) - g^k(x)|dz.$$

Since we have

$$|g^k(x-z) - g(x-z) + g(x) - g^k(x)| \le 2^a|z|^{a\mu}([\![g]\!]^\mu)^a\big(|g^k(x-z)-g(x-z)|^{1-a} + |g(x)-g^k(x)|^{1-a}\big),$$

trivially follows the uniform convergence in $x\in\mathbb{R}^n$. The uniform convergence of $I_2^k(x)$ is immediate.

Now, let us prove (2.2). By virtue of (2.3) we get for $\rho = 3|x-\overline{x}|$

$$\int^{*}_{\mathbb{R}^n} D_{y_i}K(x-y)g(y))dy = \int_{\mathbb{R}^n - B(x,\rho)} D_{y_i}K(x-y)g(y))dy + \int_{B(x,\rho)} D_{y_i}K(x-y)(g(y)-g(x))dy,$$

$$\int^{*}_{\mathbb{R}^n} D_{y_i}K(\overline{x}-y)g(y)dy = \int_{\mathbb{R}^n - B(\overline{x},\frac{\rho}{3})} D_{y_i}K(\overline{x}-y)g(y)dy + \int_{B(\overline{x},\frac{\rho}{3})} D_{y_i}K(\overline{x}-y)(g(y)-g(\overline{x}))dy$$

$$= \int_{\mathbb{R}^n - B(x,\rho)} D_{y_i}K(\overline{x}-y)g(y))dy + \int_{B(x,\rho)-B(\overline{x},\frac{\rho}{3})} D_{y_i}K(\overline{x}-y)g(y)dy + \int_{B(\overline{x},\frac{\rho}{3})} D_{y_i}K(\overline{x}-y)(g(y)-g(\overline{x}))dy$$

$$= \int_{\mathbb{R}^n - B(x,\rho)} D_{y_i}K(\overline{x}-y)g(y)dy + \int_{B(x,\rho)-B(\overline{x},\frac{\rho}{3})} D_{y_i}K(\overline{x}-y)(g(y)-g(\overline{x}))dy + \int_{B(\overline{x},\frac{\rho}{3})} D_{y_i}K(\overline{x}-y)(g(y)-g(\overline{x}))dy$$

$$+ g(\overline{x})\big(\int_{|x-y|=\rho} K(\overline{x}-y)\frac{(y_i-x_i)}{\rho}d\sigma_y + k_i\big)$$

$$= \int_{\mathbb{R}^n - B(x,\rho)} D_{y_i}K(\overline{x}-y)g(y)dy + \int_{B(x,\rho)} D_{y_i}K(\overline{x}-y)(g(y)-g(\overline{x}))dy + g(\overline{x})\big(\int_{|x-y|=\rho} K(\overline{x}-y)\frac{(y_i-x_i)}{\rho}d\sigma_y + k_i\big).$$

Hence we get

$$T(g)(x) - T(g)(\overline{x}) = \int_{\mathbb{R}^n - B(x,\rho)} \big(D_{y_i}K(x-y) - D_{y_i}K(\overline{x}-y)\big)g(y)dy + \int_{B(x,\rho)} D_{y_i}K(x-y)(g(y)-g(x))dy$$

$$- \int_{B(x,\rho)} D_{y_i}K(\overline{x}-y)(g(y)-g(\overline{x}))dy - g(\overline{x})\big(\int_{|x-y|=\rho} K(\overline{x}-y)\frac{(y_i-x_i)}{\rho}d\sigma_y + k_i\big) - k_i(g(x)-g(\overline{x}))\,.$$

Since

$$\int_{\mathbb{R}^n - B(x,\rho)} \big(D_{y_i}K(x-y) - D_{y_i}K(\overline{x}-y)\big)g(y)dy = \int_{|x-y|=\rho} K(\overline{x}-y)\frac{(y_i-x_i)}{\rho}d\sigma + k_i,$$

we also deduce

$$T(g)(x) - T(g)(\overline{x}) = \int_{\mathbb{R}^n - B(x,\rho)} \big(D_{y_i}K(x-y) - D_{y_i}K(\overline{x}-y)\big)(g(y)-g(\overline{x}))dy$$

$$+ \int_{B(x,\rho)} D_{y_i}K(x-y)(g(y)-g(x))dy - \int_{B(x,\rho)} D_{y_i}K(\overline{x}-y)(g(y)-g(\overline{x}))dy - k_i(g(x)-g(\overline{x})) = \sum_{i=1}^{4} I_i\,.$$

The following estimates easily follow [4]:

$$
\begin{aligned}
&|I_1| \le c|x-\overline{x}|[g]^\mu \int\limits_{|x-y|>\rho} \frac{|\overline{x}-y|^\mu}{|x-y|^{n+1}} dy \le c[g]^\mu |x-\overline{x}|^\mu,\\
&|I_2| \le c[g]^\mu \int\limits_{|x-y|<\rho} \frac{1}{|x-y|^{n-\mu}} dy \le c[g]^\mu |x-\overline{x}|^\mu,\\
&|I_3| \le c[g]^\mu \int\limits_{|\overline{x}-y|<2\rho} \frac{1}{|\overline{x}-y|^{n-\mu}} dy \le c[g]^\mu |x-\overline{x}|^\mu,\\
&|I_4| \le |k_i|[g]^\mu |x-\overline{x}|^\mu,
\end{aligned}
$$

which implies (2.2). *q.e.d.*

3. Solutions of a special Poisson equation

Let $\mathfrak{h}(x)$ be a smooth nonnegative cut-off function equal to 1 if $|x| \le R_0$, and equal to 0 for $|x| \ge R_0$, for some $R_0 > 0$, and consider the Poisson equation

$$\Delta \pi_{\mathfrak{h}} = -\nabla \cdot (U \cdot \nabla(\mathfrak{h} U)), \quad \text{in } \mathbb{R}^n. \tag{3.1}$$

In [Maremonti (2008)$_2$] Lemma 3.5 and Lemma 6.2, related to equation (3.1) is proved the following result:

Lemma 3.1 *Let $U \in C^{1,\mu}(\overline{\mathbb{R}}^n)$ with $\nabla \cdot U = 0$. Then there exists a unique smooth solution $\pi_{\mathfrak{h}} \in C^{2,\mu}(\mathbb{R}^n)$ of the equation (3.1) such that for each $\gamma \in (0,1)$*

$$
\begin{aligned}
&|\pi_{\mathfrak{h}}(x)| \le c\,|x|^\gamma \left(|U|^{0,\gamma}\right)^2, \; x \in \mathbb{R}^n;\\
&|\nabla \pi_{\mathfrak{h}}|^{1,\mu} \le c\left(|U|^{1,\mu}\right)^2,
\end{aligned} \tag{3.2}
$$

with c independent of the support of $\mathfrak{h}$ and depending on the $C^{1,\mu}$-norm of $\mathfrak{h}$.

Proof. We begin remarking that by our assumptions the right and side of (3.1) is a μ-Hölder function on $\mathbb{R}^n$. We set

$$\widetilde{\pi}_{\mathfrak{h}}(x) := \int\limits_{\mathbb{R}^n} \mathscr{E}(x-y) \nabla \cdot (U \cdot \nabla(\mathfrak{h} U)) dy, \tag{3.3}$$

[4] Taking the Hölder assumption on g, in order to discuss the first integral is enough to recall the following estimates:

- for $y \notin B(x,\rho)$ and $s \in (0,1)$ holds

$$|x-y-s(x-\overline{x})| \ge |x-y| - |x-\overline{x}| \ge \tfrac{2}{3}|x-y|,$$

- from the Lagrange theorem, we get

$$|D_{y_i} K(x-y) - D_{y_i} K(\overline{x}-y)| \le c \frac{c|x-\overline{x}|}{|x-y-s(x-\overline{x})|}{}^{n+1} \le c \frac{|x-\overline{x}|}{|x-y|}{}^{n+1},$$

- subsequently, the inequality $|\overline{x}-y| \le \frac{5}{3}|x-y|$.

For the third integral we recall that $B(x,\rho) \subset B(\overline{x}, 2\rho)$.

where $\mathscr{E}$ is the fundamental solution of the Laplace equation. An integration by parts furnishes

$$\widetilde{\pi}_{\mathfrak{h}}(x) := -\int\limits_{\mathbb{R}^n} \nabla_y \mathscr{E}(x-y) \cdot (U \cdot \nabla(\mathfrak{h}U))dy. \tag{3.4}$$

Employing Lemma 2.1, from (3.3) we get

$$[\nabla\nabla\widetilde{\pi}_{\mathfrak{h}}]^{\mu} \le c|\nabla \cdot (U \cdot \nabla(\mathfrak{h}U))|^{0,\mu},$$

and from (3.4) we get

$$|\nabla\widetilde{\pi}_{\mathfrak{h}}|^{0,\mu} \le c|U \otimes (\mathfrak{h}U)|^{1,\mu},$$

where in both estimates the constant c is independent of the support of $\mathfrak{h}$. We define the function $\pi_{\mathfrak{h}}(x)$ by means of the line integral of $\nabla\widetilde{\pi}_{\mathfrak{h}}(x)$ with end points x and o. Of course, $\nabla\pi_{\mathfrak{h}} = \nabla\widetilde{\pi}_{\mathfrak{h}}$, which implies $\pi_{\mathfrak{h}}(x) - \pi_{\mathfrak{h}}(y) = \widetilde{\pi}_{\mathfrak{h}}(x) - \widetilde{\pi}_{\mathfrak{h}}(y)$. By formula (3.3) we get

$$\widetilde{\pi}_{\mathfrak{h}}(x) = -\nabla_x \int\limits_{\mathbb{R}^n} \nabla_y \mathscr{E}(x-y) \cdot U(y) \otimes \mathfrak{h}(y)U(y)dy\,.$$

Hence, by employing again Lemma 2.1, for each $\gamma \in (0,1)$, we get

$$[\pi_{\mathfrak{h}}]^{\gamma} = [\widetilde{\pi}_{\mathfrak{h}}]^{\gamma} \le c|U \otimes \mathfrak{h}U|^{0,\gamma}\,,$$

with a constant c independent of the support of $\mathfrak{h}$. Since by definition $\pi_{\mathfrak{h}}(0) = 0$, we have proved the existence and the validity of (3.2). The uniqueness is a classical result. *q.e.d.*

4. The Cauchy problem with nondecaying data

The aim of this section is to prove

Theorem 4.1 *Let $U_0 \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ with $\nabla \cdot U_0 = 0$ in the weak sense. Then, there exists $T \ge C/|U_0|^2_\infty > 0$, such that problem (1.3) admits a unique classical solution (U, π_U) in $(0,T)$ satisfying the following properties*

$$\begin{aligned} &t^{\frac12}|\nabla U(t,x)| + |U(t,x)| \le c\,|U_0|_\infty(1 - c\,t|U_0|^2_\infty)^{-\frac12},\ t \in [0,T);\\ &|\pi_U(t,x)| \le c(U_0, t-T, \gamma)|x|^{\gamma} t^{-\frac{\gamma}{2}}|U_0|_\infty\,,\quad \gamma \in (0,1), \end{aligned} \tag{4.1}$$

with c independent of U_0 and (t,x). Finally, $\lim\limits_{t\to0} U(t,x) - U_0(x) = 0$

Theorem 4.1 is a special result of the ones proved in [Maremonti $(2008)_2$] sect. 7.

The proof of the theorem is achieved by means of a suitable approximation of the Navier-Stokes Cauchy problem. Hence we start with the following special initial value problem:

$$\begin{aligned} &U_{\mathfrak{h}_t} - \Delta U_{\mathfrak{h}} = -U_{\mathfrak{h}} \cdot \nabla(\mathfrak{h}U_{\mathfrak{h}}) - \nabla\pi_{U_{\mathfrak{h}}},\ \nabla \cdot U_{\mathfrak{h}} = 0,\ \text{in } (0,T) \times \mathbb{R}^n,\\ &U_{\mathfrak{h}}(0,x) = U_0(x),\ \text{on } \mathbb{R}^n, \end{aligned} \tag{4.2}$$

where $\mathfrak{h}$ is the same cut-off function previously introduced. The following result holds.

Lemma 4.1 *Let $U_0(x)\in C(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$, with $\nabla\cdot U_0=0$ in the weak sense. Then, there exists $T\ge c\,|U_0|_\infty^{-2}>0$, and a unique solution (U,π_U) on $(0,T)$ of the problem (4.2) such that uniformly with respect to $t,\bar t,\bar{\bar t}\in(0,T)$, and $x,\overline{x},\overline{\overline{x}}\in\mathbb{R}^n$,*

$$\begin{aligned}&t^{\frac12}|\nabla U_{\mathfrak{h}}(t,x)|+|U_{\mathfrak{h}}(t,x)|\le c\,|U_0|_\infty(1-c\,t|U_0|_\infty^2)^{-\frac12}\,,\ ;\\&|\pi(t,x)|\le c(U_0,T-t,\gamma)\,|x|^\gamma t^{-\frac{\gamma}{2}}\,,\quad\gamma\in(0,1),\\&|D^2U_{\mathfrak{h}}(\bar t,\overline{x})-D^2U_{\mathfrak{h}}(\bar{\bar t},\overline{\overline{x}})|+|U_{\mathfrak{h}\,t}(\overline{x},\bar t)-U_{\mathfrak{h}\,t}(\bar{\bar t},\overline{\overline{x}})|\\&\qquad\qquad\le c(|U_0|_0,T-t,\gamma)t_\circ^{-\mu-1}\big(|\overline{x}-\overline{\overline{x}}|^2+|\bar t-\bar{\bar t}|\big)^{\frac{\mu}{2}},t_\circ=\min\{\bar t,\bar{\bar t}\}\end{aligned}\tag{4.3}$$

where c is independent of the support of $\mathfrak{h}$ and depends on $C^{1,\mu}$-norm of $\mathfrak{h}$. Finally, $\lim\limits_{t\to0}U(t,x)=U_0(x)$.

Proof. We look for a solution to problem (4.2) in the form

$$U_{\mathfrak{h}}(t,x)=\int_{\mathbb{R}^n}H(x-y,t)U_0(y)dy+\int_0^t\int_{\mathbb{R}^n}E(x-y,t-\tau)\cdot U_{\mathfrak{h}}\cdot\nabla(\mathfrak{h}U_{\mathfrak{h}})(\tau,y)dyd\tau,\tag{4.4}$$

where H is the heat kernel and E is the Oseen fundamental tensor for the Stokes problem. We recall the well-known estimates, $k\ge0,|\alpha|\ge0$,

$$|D_s^kD_z^\alpha H(s,z)|+|D_s^kD_z^\alpha E(s,z)|\le c\big(|z|^2+s\big)^{-\frac{n}{2}-\frac{|\alpha|}{2}-k}.\tag{4.5}$$

Applying the method of successive approximations, and using estimates (4.5) along with the assumption $U_0\in C^0(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$, we get a smooth solution to the integral equation (4.4), such that for $t\in(0,T)$, with $T\ge c|U_0|_\infty^2$,

$$\begin{aligned}&t^{\frac12}|\nabla U_{\mathfrak{h}}(t)|_0+|U_{\mathfrak{h}}(t)|_0\le c|U_0|_0(1-ct|U_0|_0^2)^{-\frac12},\\&t\big(|D^2U_{\mathfrak{h}}(t)|_0+|U_{\mathfrak{h}\,t}(t)|_0\big)+t^{\frac12}|\nabla U_{\mathfrak{h}}(t)|_0\le c(|U_0|_0)\\&|D^2U_{\mathfrak{h}}(\bar t,\overline{x})-D^2U_{\mathfrak{h}}(\bar{\bar t},\overline{\overline{x}})|+|U_{\mathfrak{h}\,t}(\overline{x},\bar t)-U_{\mathfrak{h}\,t}(\bar{\bar t},\overline{\overline{x}})|\\&\qquad\qquad\le c(|U_0|_0)t_\circ^{-\frac{\mu}{2}-1}\big(|\overline{x}-\overline{\overline{x}}|^2+|\bar t-\bar{\bar t}|\big)^{\frac{\mu}{2}},\quad t_\circ=\min\{\bar t,\bar{\bar t}\}\,,\end{aligned}$$

where the constant c is independent of the support of $\mathfrak{h}$. By virtue of Lemma 3.1, we associate to the kinetic field $U_{\mathfrak{h}}$ a pressure field $\pi_{U_{\mathfrak{h}}}\in C^{1,\gamma}(\mathbb{R}^n)$ that satisfies estimates (3.2). Hence $(4.3)_2$ easily folows. Finally, it is easy to prove that the pair $(U_{\mathfrak{h}},\pi_{U_{\mathfrak{h}}})$ is a solution and it is unique.

q.e.d.

Proof of Theorem 4.1. Let $\{\mathfrak{h}^k\}$ be a sequence of smooth cut-off functions such that $\mathfrak{h}^k(x)\in[0,1]$, with $\mathfrak{h}^k(x)=1$ for $|x|\le k$ and $\mathfrak{h}^k(x)=0$ for $|x|\ge2k$. We can assume $\sum\limits_{|\alpha|=0}^{2}\max\limits_{\mathbb{R}^n}|D^\alpha\mathfrak{h}^k(x)|\le M$ uniformly in $k\in\mathbb{N}$. We consider the sequence of modified Navier-Stokes Cauchy problems of the kind (4.2) with $U^k\cdot\nabla(\mathfrak{h}^kU^k)$ for $U_{\mathfrak{h}}\cdot\nabla(\mathfrak{h}U_{\mathfrak{h}})$. By virtue of Lemma 4.1, we obtain a sequence of solutions (U^k,π_{U^k}) satisfying estimates (4.3) uniformly with respect to k. Let $\{\mathbb{B}^j\}$ be a sequence of open balls such that $\overline{\mathbb{B}}^j\subset\mathbb{B}^{j+1}$ and $\mathbb{R}^n=\bigcup\limits_{j\in\mathbb{N}}\mathbb{B}^j$ and $(0,T)=\bigcup\limits_{j\in\mathbb{N}}[\frac1j,T-\frac1j]$. Then, Lemma 4.1 ensures that the sequence $\{U^k\}$ is equi-continuous and equi-bounded on $(0,T-\frac{1}{j})\times\mathbb{R}^n$ and, for $|\alpha|\le 2$, the sequences $\{D^\alpha U^k\}$ and $\{U^k_t\}$ are equi-continuous and equi-bounded in $C^{0,\frac{\mu}{2}}([\frac{1}{j},T-\frac{1}{j}];C^{0,\mu}(\mathbb{R}^n))$. Moreover, for all $t\in[\frac{1}{j},T-\frac{1}{j}]$, for $|\alpha|\le 2$, $\{D^\alpha U^k\}$ and $\{U^k_t\}$ are relatively compact in $C^{0,\mu'}(\mathbb{B}_j)$, with $\mu'<\mu$ independent of $j\in\mathbb{N}$. Hence, for all $j\in\mathbb{N}$, the pair $\{(U^k,\pi_{U^k}\}$ admits an extract, again labelled by k, converging in $C(0,T;C(\mathbb{B}_j))\cap C^{0,\frac{\mu'}{2}}([\frac{1}{j},T-\frac{1}{j}];C^{2,\mu'}(\mathbb{B}_j))\times C^{0,\frac{\mu'}{2}}([\frac{1}{j},T-\frac{1}{j}];C^{1,\mu'}(\mathbb{B}_j))$ with $\{u^k_t\}$ converging in $C^{0,\frac{\mu'}{2}}([\frac{1}{j},T-\frac{1}{j}];C^{0,\mu'}(\mathbb{B}_j))$. The diagonal sequence trick ensures, for all $\eta>0$, the existence of a limit $(U,\pi_U)\in C((0,T)\times\mathbb{R}^n)\cap C^{0,\frac{\mu'}{2}}(\eta,T;C^{2,\mu'}(\mathbb{R}^n))\times C^{0,\frac{\mu'}{2}}(\eta,T;C^{1,\mu'}(\mathbb{R}^n))$ with $U_t\in C^{0,\frac{\mu'}{2}}(\eta,T;C^{0,\mu'}(\mathbb{R}^n))$ and, finally, (U,π_U) is a solution to the Navier-Stokes Cauchy problem. *q.e.d.*

5. A uniqueness result for non decaying solutions to the Navier-Stokes Cauchy problem

In this section we give a uniqueness theorem for nondecaying solutions. In this connection it is worth to stress that the result that we reproduce follows in part an approach given in [Maremonti (2009)] and in part employs a uniqueness theorem contained in [Kato (2003)]. However independently of the paper [Maremonti (2009)] there is the paper [Kukavica & Vicol (2008)] which deals an analogous uniqueness question.

We give a formulation for a bounded weak solution.

Definition 5.1 *By a bounded weak solution we mean a distribution (u,π) with*

$$u(x,t)\in L^\infty((0,T)\times\mathbb{R}^n),\ \pi\in L^1_{\ell oc}([0,T);L^1_{\ell oc}(\mathbb{R}^n))$$

such that

$$\int_0^T (u,\varphi_\tau)+(u,\Delta\varphi)+(u\otimes u,\nabla\varphi)+(\pi,\nabla\cdot\varphi)=-(u_0,\varphi(0)),$$

for each $\varphi\in C^1([0,T]\times\mathbb{R}^n)$ with $\varphi(x,t)=0$ in a neighborhood of T, and, for $t\in[0,T]$, $\varphi(t,x)\in C^\infty_0(\mathbb{R}^n)$. The symbol u_0 denotes the initial data.

Problem - *Find sufficient conditions for the uniqueness of solutions (u,π_u) to the above Navier-Stokes Cauchy problem (in weak form).*

The literature on this topic is wide and go back to 30 years ago. Starting from the paper by [Giga, Inui, & Matsui (1999)], the authors prove uniqueness in the set of distributional solutions $(u,\pi)_{[GIM]}$

$$u\in C([0,T);C(\overline{\mathbb{R}}^n))\text{ and }\pi(x,t)=R_iR_j(u^iu^j),$$

where R_i is the Riesz transform from $L^\infty(\mathbb{R}^n)$ into $\mathrm{BMO}(\mathbb{R}^n)$.

In the later paper [Giga, Inui, Kato & Matsui] the authors prove uniqueness assuming $(u,\pi)_{[GIKM]}$

$$u(x,t)\in C(0,T;C(\overline{\mathbb{R}}^n)),\ \pi=R_iR_j\Pi^{ij}+\pi_0(t),$$

with $\Pi^{ij}(t,x),\pi_0(t)\in L^1_{\ell oc}([0,T);L^\infty(\mathbb{R}^n))$.

At [Kato (2003)] the author furnishes a generalization of the above theorems, (bounded very weak solution) $(u,\pi)_{[K]}$ is meant as tempered distribution, that is

$$u(x,t)\in L^\infty((0,T)\times\mathbb{R}^n),\ \pi\in L^1_{\ell oc}([0,T);\ \mathrm{BMO}(\mathbb{R}^n)),$$

and, in the Navier-Stokes integral equation (very weak form), the test functions φ are of the kind $\varphi\in C^1([0,T]\times\mathbb{R}^n)$ with $\varphi(x,t)=0$ in a neighborhood of T, and, for $t\in[0,T]$, $\varphi(t,x)\in\mathscr{S}(\mathbb{R}^n)$.

Finally in [Sawada & Taniuchi(2004)], as far as we known, the Giga's school furnishes the best result. The authors consider very weak bounded solution $(u,\pi)_{[ST]}$ in distributional sense with

$$u\in C_w(0,T;L^\infty(\mathbb{R}^n))\ \text{ and }\ \pi\in L^1(0,T;\dot B^{-k}_{\infty,\infty}(\mathbb{R}^n)+\dot{\mathscr{A}}^1_{\infty,1}(\mathbb{R}^n)),$$

here $\dot B^{-k}_{\infty,\infty}, k>0$, and $\dot{\mathscr{A}}^1_{1,\infty}$ are homogeneous Besov spaces.

$$\begin{aligned}\{\pi=R_iR_ju^iu^j,\ R_i:L^\infty(\mathbb{R}^n)\to\ \mathrm{BMO}(\mathbb{R}^n)\}&\\ \subset\{\pi=R_iR_j\Pi^{ij},\ R_i:L^\infty(\mathbb{R}^n)\to\ \mathrm{BMO}(\mathbb{R}^n)\}&\\ \subset\{\pi\in\ \mathrm{BMO}(\mathbb{R}^n)\}\,,&\end{aligned}$$

$$\mathrm{BMO}(\mathbb{R}^n)\subset B^{-k}_{\infty,\infty}(\mathbb{R}^n)+\dot{\mathscr{A}}^1_{\infty,1}(\mathbb{R}^n).$$

There exists another set of uniqueness results. We just recall [Galdi & Maremonti (1986)] (see the references of the quoted paper for the contributes of other authors), where the uniqueness is proved for nondeacying solutions satisfying the assumption $(u,\pi)_{[GM]}$

$$\beta>0,\ u(t,x)\in L^\infty((0,T)\times\Omega)\text{ and }|\pi(t,x)|=O((1+|x|)^{1-\beta}).$$

This result is not comparable with the ones quoted above, in the sense that **is not proved** that a pressure field of a solution (u,π) is such that

$$\pi(t,x):=R_iR_ju^iu^j\in\mathrm{BMO}\subset B^{-k}_{\infty,\infty}(\mathbb{R}^n)+\dot{\mathscr{A}}^1_{\infty,1}(\mathbb{R}^n)\Rightarrow|\pi(t,x)|=O(|x|^{1-\beta})$$

and converse

$$|\pi(t,x)|=O(|x|^{1-\beta})\Rightarrow\pi(t,x)\in B^{-k}_{\infty,\infty}(\mathbb{R}^n)+\dot{\mathscr{A}}^1_{\infty,1}(\mathbb{R}^n).$$

The last question achieves special interest why the set of solutions $(u,\pi)_{[GIM]}$ and the one $(u,\pi)_{[GM]}$ are not empty, as proved by GIM and myself by the above theorem of existence. Actually, a priori, we need to consider the two sets of solutions as not comparable.

Theorem 5.1 (Uniqueness) *Let $u_0\in L^\infty(\mathbb{R}^n)$. Let (u,π) be a distribution solution belonging to $L^\infty((0,T)\times\mathbb{R}^n)\times L^1_{\ell oc}([0,T);L^1_{\ell oc}(\mathbb{R}^n))$. Assume that*

$$\lim_{R\to\infty}\frac{1}{R^{n+1}}\int\limits_{R<|x|<2R}|\pi(x,t)|dx=0,\quad \textit{a. e. in } t\in(0,T).$$

Then,

$$\pi(x,t)=R_iR_ju^iu^j+c(t),\quad \textit{with } c(t)\in L^1([0,T)),$$

and (u,π) is the unique bounded weak solution corresponding to u_0.

Just for the sake of completeness, I stress that both

$$\pi \in \mathrm{BMO} \qquad \text{and, } \beta > 0,\ |\pi(t,x)| \le O(|x|^{1-\beta}) \text{ a.e. } t \in (0,T),$$

imply

$$\lim_{R\to\infty} \frac{1}{R}^{n+1} \int\limits_{R<|x|<2R} |\pi(x,t)|dx = 0.$$

The converse is not true [5].

Thanks to the Kato uniqueness theorem, the proof is achieved by proving that in our hypotheses $\pi = R_i R_j u^i u^j + c(t)$.

Lemma 5.1 *Let (u,π) be a bounded weak solution. Then, there exists a Lebesgue measurable set $T_u \subseteq (0,T)$ such that $meas((0,T) - T_u) = 0$ and for each $t \in T_u$*

$$\begin{aligned}\int_0^t &[(u,\varphi_\tau) + (u,\Delta\varphi) + (u\otimes u,\nabla\varphi) + (\pi,\nabla\cdot\varphi)]\, d\tau \\ &= (u(t),\varphi(t)) - (u_0,\varphi(0)),\end{aligned} \tag{5.1}$$

for any $\varphi \in C^1([0,T)\times\mathbb{R}^n)$ and, for any $t\in(0,T)$, $\varphi(t,x) \in C_0^\infty(\mathbb{R}^n)$.

Proof. See [Maremonti (2009)]. □

Now, we prove

Lemma 5.2 *If (u,π) is a bounded weak solution, then almost everywhere in $t\in(0,T)$, π satisfies the following equation*

$$(\pi,\Delta g) = (u\otimes u,\nabla\nabla g), \quad \textit{for any } g \in C_0^\infty(\mathbb{R}^n).$$

Proof. From the integral equation (5.1) we deduce for any $\delta > 0$ and almost everywhere in $s \in (0,T)$ the equation

$$\begin{aligned}\int_s^{s+\delta} &[(u,\varphi_\tau) + (u,\Delta\varphi) + (u\otimes u,\nabla\varphi) + (\pi,\nabla\cdot\varphi)]\, d\tau \\ &= (u(s+\delta),\varphi(s+\delta)) - (u(s),\varphi(s)).\end{aligned} \tag{5.2}$$

Let us consider $\varphi = h(t)\nabla g(x)$ with $h(t)$ smooth and $g(x)\in C_0^\infty(\mathbb{R}^n)$. Therefore equation (5.2) becomes

$$\int_s^{s+\delta} h(\tau)\,[(u\otimes u,\nabla\nabla g) + (\pi,\Delta g)]\, d\tau = 0. \tag{5.3}$$

[5] We refer to [Stein (1993)] for the theory of the BMO space and related properties. In particular, at p. 178 6.3, it is furnished a sufficient condition for the implication: π satisfies the weighted integrability property, then $\pi \in$BMO.

Since $\pi \in L^1_{\ell oc}([0,T);L^1_{\ell oc}(\mathbb{R}^n))$, by virtue of Lebesgue theorem, there exists an interval T_π, with $meas((0,T)-T_\pi)=0$, such that for $t\in T_\pi\cap T_u$ we get

$$\begin{aligned} 0 &= \lim_{\delta\to 0}\frac{1}{\delta}\int\limits_t^{t+\delta} h(\tau)\,[(u\otimes u,\nabla\nabla g)+(\pi,\Delta g)]\,d\tau \\ &= (u\otimes u,\nabla\nabla g)+(\pi,\Delta g). \end{aligned} \tag{5.4}$$

q.e.d.

Lemma 5.3 *Let be $F(x):=D_{x_j}f(x)$ be, for some $i=1,\dots,n$ and $f\in C_0^\infty(\mathbb{R}^n)$. Then, there exists a unique smooth solution H to the equation*

$$\Delta H(x)=F(x)$$

such that, $0\le|\alpha|$,

$$|D^\alpha H(x)|\le c(1+|x|)^{-|\alpha|-n+1},\forall x\in\mathbb{R}^n.$$

Lemma 5.4 *Let π be a solution to the equation*

$$(u\otimes u,\nabla\nabla g)+(\pi,\Delta g)=0.\ \text{ for all } g\in C_0^\infty(\mathbb{R}^n),$$

with $u\in L^\infty(\mathbb{R}^n)$. If $\displaystyle\lim_{R\to\infty}\frac{1}{R^{n+1}}\int\limits_{R<|x|<2R}|\pi(x,t)|dx=0$, then,

$$\pi=-R_iR_j(u^iu^j)+\overline{c},\quad \text{a. e. in } x\in\mathbb{R}^n.$$

Proof. Let us consider $\overline{\pi}=-R_iR_ju^iu^j$. We known that $\overline{\pi}\in BMO(\mathbb{R}^n)$ and solve the equation

$$(\overline{\pi},\Delta g)=-(u\otimes u,\nabla\nabla g),\forall g\in C_0^\infty(\mathbb{R}^n).$$

Hence

$$(\overline{\pi}-\pi,\Delta g)=0,\forall g\in C_0^\infty(\mathbb{R}^n).$$

we prove that $\overline{\pi}-\pi=\overline{c}$ almost everywhere in $x\in\mathbb{R}^n$. We set $g=Hk_R$, where H is the solution of the problem $\Delta H=F=D_{x_i}f$ and k_R is a smooth cut-off function with $k_R(x)=1$ for $|x|\le R$, $k_R(x)=0$ for $|x|\ge 2R$ and $|D^\alpha k_R(x)|\le cR^{-|\alpha|}$. We assume $R>diam(\text{supp}F)$. Hence we get

$$\begin{aligned} &\Delta(Hk_R)=F+2\nabla H\cdot\nabla k_R+H\Delta k_R,\\ &\int\limits_{\mathbb{R}^n}\Delta(Hk_R)dx=0\ \text{ and } \Delta(Hk_R)\in C_0^\infty(\mathbb{R}^n). \end{aligned}$$

Substituting Hk_R in the equation $(\overline{\pi}-\pi,\Delta g)=0$, we get

$$(\overline{\pi}-\pi,F)=-(\overline{\pi}-\pi,2\nabla H\cdot\nabla k_R)-(\overline{\pi}-\pi,H\Delta k_R)=I_1+I_2.$$

We estimate the right hand side:

$$\begin{aligned} |I_1+I_2| &\le 2\int\limits_{\mathbb{R}^n}|\overline{\pi}-\pi|\big[|\nabla k_R|\nabla H|+|\Delta k_R||H|\big]dx\\ &\le c\int\limits_{R<|x|<2R}\big[R^{-1}\frac{|\overline{\pi}-\pi|}{(1+|x|)}^n+R^{-2}\frac{|\overline{\pi}-\pi|}{(1+|x|)}^{n-1}\big]dx. \end{aligned}$$

From the assumption on π :

$$\lim_{R\to\infty}\frac{1}{R^{n+1}}\int\limits_{R<|x|2R}|\pi|dx = 0$$

and from the properties of $\overline{\pi}$ (see [Stein (1993)]):

$$\overline{\pi} = R_iR_ju_iu_j \in BMO \Rightarrow (1+|x|)^{-n-1}\overline{\pi} \in L^1(\mathbb{R}^n)$$

we deduce in the limit for $R \to \infty$

$$(\overline{\pi}-\pi, F) = (\overline{\pi}-\pi, D_{x_i}f) = 0, \forall f \in C_0^\infty(\mathbb{R}^n), \forall i = 1,\dots,n$$

which implies $\overline{\pi}-\pi = \overline{c}$. Now, we are in a position to prove the theorem. By the previous Lemma we have a.e. in $t \in (0,T)$

$$\pi(t,x) - R_iR_ju^iu^j = \overline{c}(t), \text{ a.e. in } x \in \mathbb{R}^n.$$

Setting $c(t) = \overline{c}(t) + \overline{\pi}_{B_1}(t)$, since the right hand side is in $L^1_{\ell oc}([0,T);L^1_{\ell oc}(\mathbb{R}^n))$, then $c(t) \in L^1_{\ell oc}([0,T))$.

q.e.d.

The Stokes initial boundary value problem with nondecaying initial data

6. Estimates for the Stokes resolving operator with data in $C_0^1(\Omega)$

Let us consider

$$\begin{aligned}&\vartheta_t-\Delta\vartheta=-\nabla\pi_\vartheta+f,\ \nabla\cdot\vartheta=0,\ \text{in } (0,T)\times\Omega,\\&\vartheta(t,x)=0 \text{ on } (0,T)\times\partial\Omega,\\&(\vartheta(0),\varphi)=(w_0,\varphi) \text{ for any } \varphi\in\mathscr{C}_0(\Omega).\end{aligned}\tag{6.1}$$

In problem (6.1) the initial condition is given in the weak form $(\vartheta(0),\varphi)=(w_0,\varphi)$, $\varphi\in\mathscr{C}_0(\Omega)$ [6], in order to state the initial boundary value problem with an initial data w_0 belonging to the weaker Lebesgue space $L^p(\Omega)$, $p\geq 1$. Of course, if the data is an element of $J^p(\Omega)\subset L^p(\Omega)$, $p>1$, then, the problem is just the classical one. For our aims the case $w_0\in L^1(\Omega)\cap C_0^1(\Omega)$ has a special interest. This special study allows us to furnish estimates in the uniform norm ($L^\infty(\Omega)$).

We recall the following classical result on the Stokes problem[7]:

Theorem 6.1 *Let be $f=0$ in (6.1). Let $\vartheta(0,x)=\vartheta_0(x)\in J^p(\Omega)$, $p\in(1,\infty)$. Then, the Stokes operator forms a continuous semigroup in $J^p(\Omega)$. The following semigroup properties hold, for $q\geq p\in(1,\infty)$,*

$$\begin{aligned}&\|\vartheta(t)\|_q\leq c\|\vartheta_0\|_p t^{-\mu}, && \mu=\tfrac{n}{2}\big(\tfrac{1}{p}-\tfrac{1}{q}\big),\ t>0;\\&\|\nabla\vartheta(t)\|_q\leq c\|\vartheta_0\|_p t^{-\mu_1}, && \mu_1=\begin{cases}\frac12+\mu & \text{if } t\in(0,1],\\ \frac12+\mu & \text{if } t>0 \text{ and } q\in[p,n],\\ \frac12\frac{n}{p} & \text{if } t>1 \text{ and } q>n;\end{cases}\\&\|\vartheta_t(t)\|_q\leq c\|\vartheta_0\|_p t^{-\mu_2}, && \mu_2=1+\mu,\ t>0;\end{aligned}\tag{6.2}$$

where the constant c is independent of ϑ_0, and the exponent μ_1 is sharp. Moreover, if $\theta_0\in\bigcap\limits_{p>1}J^p(\Omega)$, then, $u\in\bigcap\limits_{p>1}C([0,T);J^p(\Omega))$.

For a proof of the above theorem see [Giga & Sohr (1991)$_1$, Giga & Sohr (1991)$_2$],[Iwashita], [Maremonti & Solonnikov (1997)], [Dan & Shibata (1999)$_1$, Dan & Shibata (1999)$_2$] (2-dimensional case).

The following result is a particular result of the ones proved in [Maremonti (2010)]

Theorem 6.2 *Let be $f=0$ in (6.1). Let $w_0\in C_0^1(\Omega)$. Then, to the data w_0 it corresponds a solution (ψ,π_ψ) of problem (6.1) such that, for $\eta>0$, $\psi\in\bigcap\limits_{q>1}C([0,T);J^q(\Omega))$, $\psi\in\bigcap\limits_{q>1}L^q(\eta,T;W^{2,q}(\Omega)\cap J^{1,q}(\Omega))$ and $\nabla\pi_\psi,\psi_t\in\bigcap\limits_{q>1}L^q(\eta,T;L^q(\Omega))$. Moreover, for*

[6] We set $\mathscr{C}_0(\Omega)$ as the set of all vector functions infinitely differentiable with compact support and which have divergence free.

[7] In order to state Theorem 6.1 we need to assume $\Omega\subseteq\mathbb{R}^n$, $\partial\Omega$ C^m-smooth with $2m>n$.

$q \in (1, \infty]$,

$$\begin{aligned} &\|\psi(t)\|_q \le c\|w_0\|_1 t^{-\mu}, && \mu = \tfrac{n}{2}\big(1-\tfrac{1}{q}\big),\ t>0;\\ &\|\nabla\psi(t)\|_q \le c\|w_0\|_1 t^{-\mu_1}, && \mu_1 = \begin{cases} \frac{1}{2}+\mu & \text{if } t\in(0,1],\\ \frac{1}{2}+\mu & \text{if } t>0 \text{ and } q\in(1,n],\\ \frac{n}{2} & \text{if } t>1 \text{ and } q>n; \end{cases}\\ &\|\psi_t(t)\|_q \le c\|w_0\|_1 t^{-\mu_2}, && \mu_2 = 1+\mu,\ t>0; \end{aligned} \tag{6.3}$$

where the constant c is independent of w_0. Finally, $\lim_{t\to 0}(\psi(t), \varphi) = (w_0, \varphi)$ for any $\varphi \in \mathscr{C}_0(\Omega)$.

Proof. Since $w_0 \in C_0^1(\Omega)$, then, for all $q \in (1, \infty)$, we get $P_q(w_0) = w_0 - \nabla h$, where h is a solution to the Neumann problem $\Delta h = \nabla \cdot w_0$, $\frac{dh}{dn} = 0$ and $\|\nabla h\|_q \le c\|w_0\|_q$. By virtue of Theorem 6.1, we get the existence and uniqueness of a solution (ψ, π_ψ) with $\psi \in \bigcap_{q>1} C([0,T); J^q(\Omega))$, where the last property is meant in the following sense: $\lim_{t\to 0} \|\psi(t) - P_q(w_0)\|_q = 0$. Now, let us consider the solution $(\hat\varphi, \pi_{\hat\varphi})$ with initial data $\varphi_0 \in \mathscr{C}_0(\Omega)$, whose existence is again ensured by Theorem6.1. We define $\varphi(\tau, x) = \widehat\varphi(t-\tau, x)$, $\tau \in (0,t)$. Multiplying the equation of (ψ, π_ψ) by φ, integrating by parts on $(0,t)\times\Omega$, we get

$$(\psi(t), \varphi_0) = (P(w_0), \varphi(t)) = (w_0 - \nabla h, \varphi(t)) = (w_0, \varphi(t))\,.$$

Employing the Hölder inequality and the semigroup properties, we get

$$|(\psi(t), \varphi_0)| \le \|w_0\|_1 \|\varphi(t)\|_\infty \le c\|w_0\|_1 \|\varphi_0\|_{q'} t^{-\frac{n}{2}\frac{1}{q}}, \text{ for all } \varphi_0 \in \mathscr{C}_0(\Omega).$$

Since, for $t > 0$, $\psi \in J^q(\Omega)$, we deduce

$$\|\psi(t)\|_q \le c\|w_0\|_1 t^{-\frac{n}{2}\frac{1}{q}}, \text{ for all } t > 0.$$

We also get

$$\|\nabla\psi(t)\|_q \le c\|\psi(\tfrac{t}{2})\|_q t^{-\mu_1(q)} \le c\|w_0\|_1 t^{-\mu_1(1)}, \text{ for all } t > 0.$$

A pair (ψ, π_ψ) with $\psi \in C([0,T); J^q(\Omega)) \cap L^q(\eta, T; W^{2,q}(\Omega) \cap J^{1,q}(\Omega))$ and $\nabla\pi_\psi, \psi_t \in L^q(\eta, T; L^q(\Omega))$ is the null solution if and only if $(w_0, \varphi) = 0$ for all $\varphi \in \mathscr{C}_0(\Omega)$. This claim, which we do not prove, ensures the uniqueness of (ψ, π_ψ) in its class of existence.

q.e.d.

Now we give an application of the above estimate, which gives a partial answer to a well known problem posed by Heywood.

Theorem 6.3 *For $n \ge 2$, let Ω be a C^m-smooth ($2m > n$) bounded or exterior domain of $\mathbb{R}^n$. For some $q \in (1, \infty)$ and $p \in (\frac{n}{2}, \infty)$, let u be in $J^q(\Omega)$ and $P\Delta u \in J^p(\Omega)$. Then there exists a constant c independent of u such that*

$$\|u\|_\infty \le c\|P\Delta u\|_p^a \|u\|_q^{1-a},$$

provided that $0 = a(\frac{1}{p} - \frac{2}{n}) + (1-a)\frac{1}{q}$.

Proof. Via our assumption on u and Sobolev embedding theorem, one easily deduces that $u \in L^\infty(\Omega)$. Now, we consider a solution (ψ, π_ψ) of problem (6.1), whose existence is ensured by Theorem 6.2. Since for $t > 0$, $\psi \in J^{p'}(\Omega) \cap J^{q'}(\Omega)$, then, an integration by parts furnishes

$$(P\Delta u, \psi(t)) = (u, P\Delta\psi(t)) = (u, \psi_t(t)), \text{ for all } t > 0.$$

Hence integrating on (s, t), we get

$$(u, \psi(s)) = (u, \psi(t)) - \int_s^t (P\Delta u, \psi(\tau))d\tau. \tag{6.4}$$

Since $u \in J^q(\Omega)$, there exists $\{u^k\} \subset \mathscr{C}_0(\Omega)$ converging to $u \in J^q(\Omega)$, hence

$$(u, \psi(s)) = (u - u^k, \psi(s)) + (u^k, \psi(s)).$$

Since $\psi \in C(0, T; J^{q'}(\Omega))$, we get $\|\psi(s)\|_{q'} \leq M$, for all $s \in [0, t]$, hence

$$|(u - u^k, \psi(s))| \leq \|u - u^k\|_q M, \text{ for all } s \in [0, t],$$
$$\lim_{s\to 0}(u^k, \psi(s)) = (u^k, w_0), \text{ for all } k \in \mathbb{N}.$$

Therefore we get

$$\lim_{s\to 0}(u, \psi(s)) = (u, w_0).$$

The above limit property and (6.4) imply

$$|(u, w_0)| \leq \|u\|_q\|\psi(t)\|_{q'} + \|P\Delta u\|_p \int_0^t \|\psi(\tau)\|_{q'} d\tau.$$

By applying the semigroup properties of ψ we also deduce

$$|(u, w_0)| \leq c(\|u\|_q t^{-\frac{n}{2}\frac{1}{q}} + \|P\Delta u\|_p t^{1-\frac{n}{2}\frac{1}{p}})\|w_0\|_1, \text{ for all } w_0 \in C_0^1(\Omega).$$

By density also we get

$$|(u, w_0)| \leq c(\|u\|_q t^{-\frac{n}{2}\frac{1}{q}} + \|P\Delta u\|_p t^{1-\frac{n}{2}\frac{1}{p}})\|w_0\|_1, \text{ for all } w_0 \in L^1(\Omega).$$

We have proved the estimate

$$\|u\|_\infty \leq c(\|u\|_q t^{-\frac{n}{2}\frac{1}{q}} + \|P\Delta u\|_p t^{1-\frac{n}{2}\frac{1}{p}}).$$

Hence setting $t = \|u\|_q^\alpha \|P\Delta u\|_p^{-\alpha}$, we get the result for $1 - \alpha\frac{n}{2}\frac{1}{q} = \alpha(1 - \frac{n}{2}\frac{1}{p})$, that is $\alpha\frac{n}{2}\frac{1}{q} = a$.

q.e.d.

Actually, the above result, which proves the embedding in $L^r, r = \infty$, is from one side a correction to the one obtained in [Maremonti (1998)], and other side it is also a special case of the embedding proved in [Maremonti (1998)]. Indeed the embedding holds in L^r for suitable $r \in (1, \infty)$. As far as concerns the technique, it is an extension to the "second order of derivatives" of interpolation inequalities developed for the first order derivatives, which goes back to [Nash 1958] (on the topic see [M.Giga, Giga & Saal] ch. 6).

7. The maximum modulus theorem: statement of the problem

Before getting to the heart of the matter, I would like to recall some aspects of the questions:

a) The Maximum Modulus Theorem is one of the classical result in the theory of partial differential equations. However for the equations of the hydrodynamics the problem has been an open problem for long time. If we put aside the Cauchy problem, which is analogous to the heat equation, the first contribute just goes back about 10 years ago and it concerns the initial boundary value problem in a convex bounded domain and in a half-space (cf. [Desch, Hieber & Prüss (2001), Solonnikov $(2002)_1$, Solonnikov $(2002)_2$, Solonnikov $(2002)_3$, Solonnikov $(2003)_1$], [Solonnikov $(2003)_2$, Maremonti $(2008)_1$, Maremonti (2012), Maremonti $(2008)_2$]).

b) Now, what is new in the literature, are the contributes: [Abe & Giga (2011), Abe & Giga (2012)] (forthcoming papers, Luminy (2011) conference by Y. Giga). These authors are able to develop a proof of the results concerning the maximum modulus theorem, which makes use of a functional analysis approach in opposition to the other contributes where a potential theory is employed or it is latent. Hence the functional analysis approach, remarkable aspect, gives a new light to the theory. However the question is solved in its completeness only in the case of an initial boundary value in a bounded domain, partially in the case of an exterior domain. Our aim is just to fill the gap of the last case.

We start partially recalling (in the sense of what we need) the result by Abe and Giga:

Theorem 7.1 *The Stokes operator forms a continuous analytic semigroup in $\mathscr{C}_{|0}(D)$*[8] *(D C^3-smooth bounded domain) with*

$$|u(t,x)| \le ce^{-\gamma t}\|u_\circ\|_\infty, \text{ for all } (t,x) \in [0,T)\times\overline{D}, \tag{7.1}$$

with c independent of $u_\circ$. Moreover, the Stokes operator forms a non continuous analytic semigroup in $L^\infty_\sigma(D)$ and (7.1) holds. In both the cases the following estimate holds:

$$|D^2u(t)|_\infty + |u_t(t)|_\infty \le ct^{-1}|u_\circ|_\infty,$$

for all $t \in (0,T)\times\overline{\Omega}$, where c is independent of $u_\circ$.

Remark 7.1 As pointed out by Abe and Giga, since $C_0(\Omega)$ is not dense in $L^\infty(\Omega)$, then the Stokes operator cannot be a continuous analytic semigroup in $L^\infty_\sigma(\Omega)$, hence the above result in L^∞ can be considered sharp.

Our aim is to extend Theorem 7.1 to the case of an initial boundary value problem in exterior domains. Actually we are able to prove [9]

[8] By the symbol $\mathscr{C}_{|0}(D)$ we mean the set of vector functions

$$\{u : u \in C(\overline{D}), u = 0 \text{ on } \partial D \text{ and } \nabla\cdot u = 0 \text{ in weak sense}\}.$$

[9] We introduce some notation that will be employed:

$$\mathscr{C}(\Omega) = \{u : u \in C(\Omega)\cap L^\infty(\Omega),\ u = 0 \text{ on } \partial\Omega,\ \nabla\cdot u = 0 \text{ in weak sense}\}$$

Theorem 7.2 (Maximum Modulus Theorem) *For each* $u_\circ \in L^\infty(\Omega)$ *satisfying* (1.2), *there exists a solution* (u, π_u) *of problem* (1.1) *such that*

$$|u(t,x)| \le c\|u_\circ\|_\infty, \text{ for all } (t,x) \in (0,T) \times \Omega\,. \tag{7.2}$$

Moreover,

$$\begin{aligned} &\text{for each } \lambda \in (0,1) \text{ and } t > 0, u \in C^{2,\lambda}(\overline{\Omega}) \cap \mathscr{C}(\overline{\Omega}) \text{ and } u_t, \nabla\nabla u \in C^{0,\frac{\lambda}{2}}((0,T)\times\overline{\Omega});\\ &\text{for each } \eta \in (0, \tfrac{1}{2}), |\pi_u(t,x)| \le c(t^{-\frac{1}{2}-\eta}+1)(|x|+1)^{2-n}\|u_\circ\|_\infty,\ (t,x) \in (0,T)\times\Omega; \qquad (7.3)\\ &\text{for each } R > 0 \text{ and } p \in [1,\infty),\ \lim_{t\to 0}\|u(t) - u_\circ\|_{L^p(\Omega_R)} = 0\,. \end{aligned}$$

In estimates (7.2)-(7.3) *the constant* $c \ge 1$ *is independent of* $u_\circ$.

If $u_\circ \in \mathscr{C}(\Omega)$, *then, for each* $t \ge 0$, $u(t,x) \in \mathscr{C}(\Omega)$ *and we also get, for each* $x \in \Omega$, $\lim_{t\to 0} u(t,x) = u_\circ(x)$.

If $u_\circ \in \mathscr{C}(\overline{\Omega})$, *then, for each* $t \ge 0$, $u(t,x) \in \mathscr{C}(\overline{\Omega})$ *and we also get* $\lim_{t\to 0}|u(t) - u_\circ|_0$. *Finally, up to a function* $k(t)$ *for the pressure field, a solution* (u,π_u) *verifying* (7.2)-(7.3) *is unique.*

Our result is in the wake (functional analysis approach) of the paper by Abe-Giga, in the sense that we prove the result by means of duality arguments and employing the semigroup properties of the resolving operator defined on $L^1(\Omega)$.

We do not give asymptotic semigroup properties for the solutions. As far as the behavior of $\nabla u(t,x)$ is concerned, we refer to [Maremonti (2012)].

We do not give estimates on the constant c in (7.2). However in [Krazt (1997)] and in [Maremonti & Russo (1994)] has been proved that in the case of the *Maximum Modulus Theorem* for the Stokes boundary steady problem the constant c cannot be equal to 1. This is in contrast with the case of the elliptic equations, but in accord with the case of elliptic systems in divergence form, see [Fichera (1961)] and [Canfora (1966)]. Now, in the cases of the parabolic equation and of the parabolic system, with elliptic part in divergence form, the solutions verify the estimate of the kind (7.2) with the constant $c = 1$. Hence not only we do not give an estimate of c, but it becomes difficult to conjecture a value for c (for the Cauchy problem trivially is $c = 1$).

The chief items to achieve the proof of the result

a) For each $u_\circ \in L^\infty(\Omega)$ with null divergence there exists a sequence $\{u^m\} \subset \mathscr{C}(\Omega) \cap C^1(\Omega)$ such that

$$u_m(x) \to u(x) \text{ a.e in } \Omega;$$

$$|u^m(x)| \le \|u_\circ\|_\infty, \text{ for all } x \in \Omega.$$

---

and

$$\mathscr{C}(\overline{\Omega}) = \{u : u \in \mathscr{C}(\Omega) \cap C(\overline{\Omega})\}.$$

It is possible to get

$\mathscr{C}_{|0}(\Omega) \equiv \mathscr{C}_0(\Omega)$ completion in $C(\overline{\Omega})$ if Ω is bounded,
$\mathscr{C}_{|0}(\Omega) \equiv \mathscr{C}_0(\Omega)$ completion in $C(\overline{\Omega})$ with $u(x) \to 0$ for $|x| \to \infty$ if Ω is exterior.

These completion spaces were proved in [Maremonti $(2008)_1$, Maremonti (2010)] assuming Ω of $C^{1,h}$-smooth. Recently in they are proved in [Abe & Giga (2011), Abe & Giga (2012)] with $\partial\Omega$ Lipschitz domain.

b) The first result concerns the case of $u_\circ \in \mathscr{C}(\overline{\Omega}) \cap C^1(\overline{\Omega})$. In this case we look be solution $u(t,x) = W(t,x) + w(t,x)$, with $W(t,x)$ solution to the Cauchy problem with an initial data $W(0,x) = u_\circ(x)$ and $w(t,x)$ solution to the problem

$$\begin{aligned} & w_t - \Delta w = -\nabla \pi_w \,, \text{ in } (0,T)\times\Omega, \\ & \nabla\cdot w = 0, \text{ in } (0,T)\times\Omega, \\ & w(t,x) = -W(t,x), \text{ on } (0,T)\times\partial\Omega, \; w(0,x)=0, \text{ on } \{0\}\times\Omega. \end{aligned} \tag{7.4}$$

The peculiarity of this decomposition is the fact that u is the sum of W "nondecaying" and w "usual solution" of the L^p-theory.

c) Thanks to property a), we extend the result of item b) to the case of $u_\circ \in L^\infty(\Omega)$. Indeed, by means of a)-b) we prove the existence of a sequence of solutions $\{u^m, \pi_{u^m}\}$. This sequence converges with respect to a suitable family of seminorms ($d = diam(\mathbb{R}^n - \Omega)$),

for each $p \in (\frac{n}{n-2}, \infty)$, $\rho > d$ and $q \in (\frac{n}{2}, \infty]$,

$$\begin{aligned} \tfrac{t}{t+1}\|u_t(t)\|_{L^p(\Omega_\rho)} + \sum_{|\alpha|=0}^{2} \big(\tfrac{t}{t+1}\big)^{\frac{|\alpha|}{2}} \|D^\alpha u(t)\|_{L^p(\Omega_\rho)} + \big(\tfrac{t}{t+1}\big)^{\frac12+\eta}\|\pi_u(t)\|_{L^p(\Omega_\rho)} \\ \le c \sup_{B(O,\rho)} M_q(t,y,L,u_\circ)\,; \end{aligned} \tag{7.5}$$

for all $\rho > d,\ \eta \in (0, \frac12)$ and $q \in (\frac{n}{2}, \infty]$

$$\big(\tfrac{t}{t+1}\big)^{\frac12+\eta}|\pi_u(t,x)| \le c \sup_{B(O,\rho)} M_q(t,y,L,u_\circ)(1+|x|)^{2-n},\ (t,x)\in(0,T)\times\overline{\Omega}\,,$$

where

$$M_q(t,x,L,u_\circ) = \|u_\circ(x)\|_{L^q(B_L)} t^{-\frac{n}{2}\frac{1}{q}} + \|u_\circ\|_\infty \frac{t^{\frac{\mu}{2}}}{(L+t^{\frac12})^\mu}, \tag{7.6}$$

$(t,x,L,u_\circ) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^+ \times L^\infty(\mathbb{R}^n)$, $B_L := B(O,L) \subset \mathbb{R}^n$.

8. Achievement of item a)

We start recalling the following lemma due to Abe and Giga [Abe & Giga (2011)]:

Lemma 8.1 *Let $D \subset \mathbb{R}^n$ be a bounded Lipschitz domain. There exists a constant c such that for each $u \in L^\infty(D)$ satisfying* (1.2) *there exists a sequence $\{u^m\} \subset \mathscr{C}_0(D)$ such that*

$$\|u^m\|_\infty \le c\|u\|_\infty\,, \textit{ for all } m\in\mathbb{N}, \textit{ and } u^m \to u \textit{ a.e. in } D. \tag{8.1}$$

Moreover, if $u \in \mathscr{C}_{|0}(\overline{D})$, then the convergence of the sequence is uniform on $\overline{\Omega}$.

For the proof see [Abe & Giga (2011)] Lemma 6.3. Now, we prove

Lemma 8.2 *Let Ω be a Lipschitz exterior domain. For each $u \in L^\infty(\Omega)$ satisfying* (1.2), *there exists a sequence $\{u^m\} \subset C^1(\overline{\Omega}) \cap \mathscr{C}(\overline{\Omega})$ such that*

$$\|u^m\|_\infty \le c\|u\|_\infty\,, \textit{ for all } m\in\mathbb{N}, \textit{ and } u^m \to u \textit{ a.e. in } \Omega, \tag{8.2}$$

with c independent of u. Then, in particular we get

$$\text{for each } p\in[1,\infty) \text{ and } \rho>0,\ \lim_m \|u^m-u\|_{L^p(\Omega_\rho)}=0\,. \tag{8.3}$$

Moreover, if $u\in\mathscr{C}(\Omega)$, *then, the convergence is uniform on compact subsets: for each* $\rho>0$, $\lim\limits_m |u^m-u|_{C(\overline{\Omega}_\rho)}=0$. *If* $u\in\mathscr{C}(\overline{\Omega})$, *then, the convergence is uniform on* $\overline{\Omega}$, *that is* $\lim\limits_m |u^m-u_\circ|_0=0$.

Proof. We introduce a smooth cutoff function h_R such that $h_R=1$ in $\Omega_{\frac{R}{3}}$ and ∇h_R has support in $\overline{\Omega}_{2R+\varepsilon,R-\varepsilon}$. Then, we set $u=uh_R+(1-h_R)u$. By virtue of Bogovski's result (see next Lemma 10.1), we denote by u_R a solution to problem

$$\nabla\cdot u_R=u\cdot\nabla h_R \text{ in } \Omega_{2R+\varepsilon,R-\varepsilon},\ u_R=0 \text{ on } \partial\Omega_{2R+\varepsilon,R-\varepsilon}.$$

The compatibility condition is satisfied since the field u is divergence free. Since the right hand side belongs to $L^p(\Omega)$, for all $p\in(1,\infty)$, we claim that $u_R\in C_0(\Omega)$ and $|u_R|_0\le c(R)\|u\|_\infty$. Hence, the field $w_R:=uh_R-u_R$ belongs to $L^\infty(\Omega_R)$ and is divergence free. Analogously, $W_R=u_R+u(1-h_R)\in L^\infty(\Omega)$ and satisfies (1.2), with $W_R=0$ for $|x|\le R$ and $\|W_R\|_\infty\le c\|u\|_\infty$. By virtue of Lemma 8.1, we get the existence of $\{w^m\}\subset\mathscr{C}_0(\Omega_R)\subset\mathscr{C}_0(\Omega)$ such that $|w^m|_0\le c(R)\|u\|_\infty$ and $w^m\to w_R$ a.e. in Ω. Then, we mollify W_R. Hence, setting $W^m:=J_m[W_R]$, we get $\{W^m\}\subset C^1(\overline{\Omega})\cap\mathscr{C}(\overline{\Omega})$, $|W^m|_0\le\|W_R\|_\infty\le c(R)\|u\|_\infty$ and, for each $\rho>d$, $W^m\to W_R$ strongly in $L^q(\Omega_\rho)$ for all $q\in[1,\infty)$. Hence there exists a subsequence, again labelled by m, of $\{W^m\}$ converging a.e. in Ω_ρ. Now, let us consider a sequence of subdomains Ω_{ρ_k} invading Ω such that $\Omega=\bigcup\limits_{k\in\mathbb{N}}\Omega_{\rho_k}$ and $\overline{\Omega}_{\rho_k}\subset\Omega_{\rho_{k+1}}$. By the diagonal trick we find a subsequence, labelled by m again, converging a.e. in Ω to the function W_R. Therefore, setting $u^m:=w^m+W^m$, $\{u^m\}\subset C^1(\overline{\Omega})\cap\mathscr{C}(\overline{\Omega})$ and satisfies (8.2). The claim (8.3) follows from the Lebesgue dominate convergence theorem. If $u\in\mathscr{C}(\Omega)$, then, by virtue of Lemma 8.1 the convergence of $\{w^m\}$ is uniform. For each $\rho>R$, the convergence of $\{W^m\}$ is uniform on Ω_ρ. Hence, for each $\rho>R$, $u^m=w^m+W^m$ converges uniformly on Ω_ρ. Finally, if $u\in\mathscr{C}(\overline{\Omega})$, then $\{W^m\}$ also converges uniformly on $\overline{\Omega}$, hence the same holds for $\{u^m\}$. *q.e.d.*

9. Achievement of item b)

We recall that, for $\mu\ge0$ and $q\in[1,\infty]$, we set

$$M_q(t,x,L,u_\circ)=\|u_\circ(x)\|_{L^q(B_L)}t^{-\frac{n}{2}\frac{1}{q}}+\|u_\circ\|_\infty\frac{t^{\frac{\mu}{2}}}{(L+t^{\frac12})^\mu}\,, \tag{9.1}$$

$(t,x,L,u_\circ)\in\mathbb{R}^+\times\mathbb{R}^n\times\mathbb{R}^+\times L^\infty(\mathbb{R}^n)$.

Lemma 9.1 *Let* $u_\circ$ *be in* $L^\infty(\mathbb{R}^n)$. *Let us consider the heat transformation* $H[u_\circ](t,x)$. *Then, we get,* $k,h\in\mathbb{N}\cup\{0\}$,

$$|D_t^k\nabla^h H[u_\circ](t,x)|\le ct^{-k-\frac{h}{2}}M_q(t,x,L,u_\circ),$$
$$\text{for all } (t,x,L,u_\circ)\in\mathbb{R}^+\times\mathbb{R}^n\times\mathbb{R}^+\times L^\infty(\mathbb{R}^n), \tag{9.2}$$

$$\text{if } u_\circ\in W^{1,\infty}(\mathbb{R}^n),\ |D_t^k H[u_\circ](t,x)|\le ct^{-k+\frac12}|\nabla u_\circ|_\infty$$

Proof. Since, for $k, h \in N \cup \{0\}$ and $\mu > 0$, we can get

$$|D_t^k \nabla^h H(z,t)| \leq ct^{\frac{\mu}{2}}(|z| + t^{\frac{1}{2}})^{-n-h-2k-\mu},$$

then, we deduce

$$\begin{aligned}|D_t^k \nabla^h H[u_\circ](t,x)| &\leq \int\limits_{|z|<L} |D_t^k \nabla^h H(t,z)||u_\circ(x-z)|dz + \int\limits_{|z|>L} |D_t^k \nabla^h H(t,z)||u_\circ(x-z)|dz \\ &\leq c\,\|u_\circ(x)\|_{L^q(B_L)} t^{-\frac{n}{2}\frac{1}{q}-k-\frac{h}{2}} + c\,\|u_\circ\|_\infty \int\limits_{|z|>L} \frac{t^\mu}{(|z| + t^{\frac{1}{2}})^{\mu+n+h+2k}} dz \\ &\leq c\, t^{-k-\frac{h}{2}} M_q(t,x,L,u_\circ),\end{aligned}$$

which proves $(9.2)_1$. Estimate $(9.2)_2$ is well known. *q.e.d.*

Lemma 9.2 *Let $A(t,x)$ be a one parameter ($t \geq 0$) family of functions with $A(t,x) \in C(\partial\Omega)$ and $\int_{\partial\Omega} A(t,x)\cdot n d\sigma = 0$, for all $t \geq 0$. Then there exists an extension $F(t,x)$ inside Ω such that $F(t,x) \in \mathscr{C}(\overline{\Omega}) \cap C^2(\Omega)$, which is divergence free and $suppF(t) \subseteq B(o,3R) \cap \Omega$, uniformly with respect to t, with*

$$|F(t)|_0 \leq c|A(t)|_{C(\partial\Omega)}, \text{ for all } t \geq 0;$$

$$\Delta F = \nabla P + G, \text{ where } G \text{ has support in } \Omega_{2R+\varepsilon, R-\varepsilon}, \text{ for all } t \geq 0,$$

$$|G(t)|_{C^{0,\lambda}}(\Omega) \leq c(R)|A(t)|_{C(\partial\Omega)}, \text{ for all } t \geq 0.$$

Moreover, if, for all $t > 0$, $D_t^k A(t,x) \in C(\partial\Omega)$, then we get

$$|D_t^k F(t)|_0 + |D_t^k G(t)|_0 \leq c|D_t^k A(t)|_{C(\partial\Omega)}, \text{ for all } t > 0\,.$$

Proof. Let consider the steady Stokes problem:

$$\Delta v - \nabla\pi = 0,\ \nabla \cdot v = 0 \text{ in } \Omega_R,\ v = A \text{ on } \partial\Omega \text{ and } v = 0 \text{ on } |x| = 3R.$$

Since $\int_{\partial\Omega} A \cdot n d\sigma = 0$, the compatibility condition is satisfied. By the maximum modulus theorem (cf. [Maremonti (1998)]) we get the existence of $(v,\pi) \in C(\overline{\Omega}_R) \cap C^2(\Omega_R) \times C^1(\Omega_R)$ with

$$|v|_0 \leq c|A|_0,\ \sum_{|\alpha|=1}^{\alpha} |D^\alpha v|_{C^0(K)} \leq c(K)|A|_0, \text{ for all } K \subset \Omega_R.$$

Now, we consider vh_R, where h_R is a smooth cut-off function with $h_R = 1$ for $|x| \leq R$ and $h_R = 0$ for $|x| \geq 2R$. Let V a solution to the Bogovski's problem $\nabla \cdot V = -v\nabla h_R$ in $B(O, 2R+\varepsilon) - B(O, R-\varepsilon)$ and $V = 0$ on $|x| = 2R+\varepsilon$ and $|x| = R-\varepsilon$. The compatibility condition is satisfied. We get

$$|V|_0 \leq c(R)\|v\|_{W^{1,r}(B(O,2R+\varepsilon)-B(O,R-\varepsilon))} \leq c(R)|A|_0.$$

Setting $F = v + V$, we have proved the former claim of the theorem. The latter claim is immediate by the construction. *q.e.d.*

Corollary 1.1 *Let* $A(x) \in C(\partial\Omega) \cap W^{\ell-\frac{1}{q},q}(\partial\Omega)\,, \ell = 1,\dots 3,$ *with* $\int\limits_{\partial\Omega} A(x)\cdot n d\sigma = 0.$ *Then, we get that the extension* F *of Lemma 9.2, which is such that* $\Delta F = \nabla P + G$, *satisfies the following inequalities:*

$$\begin{aligned} &\|P\|_{\ell-1,q} + \|F\|_{\ell,q} \le c|A|_{\ell-\frac{1}{q},q}\,,\\ &\|G\|_{\ell_1-1,q} \le c|A|_{\ell_1-\frac{1}{q},q}\,, \ell_1 = 1,2. \end{aligned} \tag{9.3}$$

Proof. The proof is immediate by the construction of the field F and well known properties for solutions to the Stokes problem. However we refer to [Maremonti (2012)] for details.

q.e.d.

Theorem 9.1 *For each* $u_\circ \in \mathscr{C}(\overline{\Omega}) \cap C^1(\overline{\Omega})$ *there exists a solution* (u,π_u) *to problem* (1.1) *with* $u = W + w$ *and* $\pi_u = \pi_w$, *where* $W = H[u_\circ](t,x)$ *and* $w \in \bigcap\limits_{p>\frac{n}{n-2}} W^{2,p}(\Omega)$, *with* $w_t, \nabla\pi_w \in \bigcap\limits_{p>\frac{n}{n-2}} L^p(\Omega)$. *Moreover, we get:*

$$\begin{aligned} &\textit{for each } p \in (\tfrac{n}{n-2},\infty),\ \rho > d \textit{ and } q \in (\tfrac{n}{2},\infty],\\ &\tfrac{t}{t+1}\|u_t(t)\|_{L^p(\Omega_\rho)} + \sum_{|\alpha|=0}^{2}\big(\tfrac{t}{t+1}\big)^{\frac{|\alpha|}{2}}\|D^\alpha u(t)\|_{L^p(\Omega_\rho)} + \big(\tfrac{t}{t+1}\big)^{\frac12+\eta}\|\pi_u(t)\|_{L^p(\Omega_\rho)}\\ &\qquad\qquad \le c \sup_{B(O,\rho)} M_q(t,y,L,u_\circ)\,;\\ &\lim_{t\to 0}|u(t)-u_\circ|_0 = 0\,;\\ &\textit{for all } \rho > d,\ \eta \in (0,\tfrac12) \textit{ and } q \in (\tfrac{n}{2},\infty],\\ &\big(\tfrac{t}{t+1}\big)^{\frac12+\eta}|\pi_u(t,x)| \le c \sup_{B(O,\rho)} M_q(t,y,L,u_\circ)(1+|x|)^{2-n},\ (t,x)\in(0,T)\times\overline{\Omega}\,. \end{aligned} \tag{9.4}$$

In estimate (9.4) *the constant* c *is independent of* t, L *and* $u_\circ$. *For all* $t > 0$, $(u,\pi_u) \in C^{2,\lambda}(\overline{\Omega}) \times C^{1,\lambda}(\overline{\Omega})$ *and* $u_t, \nabla\nabla u \in C^{0,\frac{\lambda}{2}}((0,T)\times\overline{\Omega})$ *and* (u,π_u), *in its class of existence, is unique up to a function* $k(t)$ *for the pressure field.*

Sketch of the proof. *Existence.* We set $W(t,x) := H[u_\circ](t,x)$, where H is the heat kernel. The field W is divergence free. We associate to W a pressure field identically equal to zero. We look for a solution to problem (1.1) in the form $u = W + w$ and $\pi_u = \pi_w$, where

$$\begin{aligned} &w_t - \Delta w = -\nabla\pi_w\,,\ \text{in } (0,T)\times\Omega,\\ &\nabla\cdot w = 0,\ \text{in } (0,T)\times\Omega,\\ &w(t,x) = -W(t,x),\ \text{on } (0,T)\times\partial\Omega,\ w(0,x) = 0,\ \text{on } \{0\}\times\Omega. \end{aligned} \tag{9.5}$$

We look for a solution of (9.5) in the form $w = F(t,x) + \omega(t,x)$. The field $F(t,x)$ is the extension in $(0,T)\times\Omega$ of the boundary data $-W(t,x)$, whose existence is ensured by the last lemma, and ω is the solution to

$$\begin{aligned} &\omega_t - \Delta\omega = -\nabla(\pi_w - P) - F_t + G\,,\ \text{in } (0,T)\times\Omega,\\ &\nabla\cdot\omega = 0,\ \text{in } (0,T)\times\Omega,\\ &\omega(t,x) = 0,\ \text{on } (0,T)\times\partial\Omega,\ \omega(0,x) = 0,\ \text{on } \{0\}\times\Omega. \end{aligned} \tag{9.6}$$

By Lemma 9.2 concerning F and the one concerning the solutions to the heat equation, we deduce, $k \geq 0$,

$$\begin{aligned}
&|D_t^k F(t)|_0 \leq c|D_t^k W(t)|_{C(\partial\Omega)} \leq ct^{-k} \sup_{\partial\Omega} M_q(t,\xi,L,u_\circ),\ t>0;\\
&|D_t^k G(t)|_0 \leq c|D_t^k W(t)|_{C(\partial\Omega)} \leq ct^{-k} \sup_{\partial\Omega} M_q(t,\xi,L,u_\circ),\ t>0;\\
&|F_t^k(t)|_0 \leq c|W_t(t)|_{C(\partial\Omega)} \leq ct^{\frac{1}{2}-k}\|\nabla u_\circ\|_\infty,\ t>0\,;\\
&|D_t^k G(t)|_0 \leq c|D_t^k W(t)|_{C(\partial\Omega)} \leq ct^{\frac{1}{2}-k}\|\nabla u_\circ\|_\infty,\ t>0\,.
\end{aligned} \tag{9.7}$$

Hence $-F_t + G \in L^s(0,T;L^p(\Omega))$, for all $p \in (1,\infty)$ and $s \in [1,2)$. By virtue of the existence theorem in anisotropic space-time Sobolev spaces (see for ex. [**?**]), we get the existence of $\omega \in \bigcap_{p>1}\big[C([0,T);J^p(\Omega)) \cap L^s(0,T;W^{2,p}(\Omega)\cap J^{1,p}(\Omega))\big]$ with $\omega_t, \nabla\pi_\omega \in \bigcap_{p>1} L^s(0,T;L^p(\Omega))$. As a consequence we have proved the existence of $w = F+\omega$ and $\pi_w = P + \pi_\omega$.

A special estimate for $\|\omega(t)\|_p$ and $\|\omega_t(t)\|$ in a neighborhood of $t=0$. We multiply equation $(9.6)_1$ by $\vartheta(t-\tau,x)$, $\tau \in (0,T)$, where $\vartheta(s,x)$, $s>0$, is a solution to Stokes problem (6.1) with an initial data $\vartheta(0,x) = \vartheta_0(x)$ belonging to $\mathscr{C}_0(\Omega)$. The existence of ϑ is ensured by Theorem 6.1. Bearing estimates $(9.7)_3$ in mind, an integration by parts on $(0,t)\times\Omega$ furnishes:

$$|(\omega(t),\vartheta_0)| = |\int_0^t (F_\tau(\tau),\vartheta(t-\tau))d\tau - \int_0^t (G(\tau),\vartheta(t-\tau))d\tau|. \tag{9.8}$$

Applying the Hölder inequality, since F_t and G have compact support and enjoy estimates $(9.7)_2$ and $(9.7)_3$, by virtue of semigroup properties for the solution $\vartheta(s,x)$, we get

$$|(\omega(t),\vartheta_0)| \leq \int_0^t \|F_\tau(\tau)\|_p\|\vartheta(t-\tau)\|_{p'}d\tau + \int_0^t \|G(\tau)\|_p\|\vartheta(t-\tau)\|_{p'}d\tau \leq c(t^{\frac{1}{2}}+t)\|\nabla u_\circ\|_\infty\|\vartheta_0\|_{p'}\,.$$

The last estimate implies

$$\|\omega(t)\|_p \leq c(t^{\frac{1}{2}}+t)\|\nabla u_\circ\|_\infty,\ t>0. \tag{9.9}$$

We can assume that ω_t is differentiable with respect to t. From the equation of ω_{tt}, multiplying by θ solution backward in time of (6.1), by an integration by parts $(s,t)\Omega$, we get

$$\begin{aligned}
|(t\omega_t(t),\vartheta_0)| = |&\int_s^t (\tfrac{\partial}{\partial\tau}(\tau F_\tau(\tau)),\vartheta(t-\tau))d\tau - \int_s^t (\tfrac{\partial}{\partial\tau}(\tau G(\tau)),\vartheta(t-\tau))d\tau\\
&+\int_s^t (\omega_t(\tau),\vartheta(t-\tau))d\tau - (\omega(t),\vartheta_0) + s(u_s(s),\vartheta(t-s))|\,.
\end{aligned} \tag{9.10}$$

Applying the Hölder inequality, since F_t, F_{tt} and G, G_t have compact support and enjoy estimates (9.7), we get

$$|(t\omega_t(t),\vartheta_0)|\leq \int\limits_0^t \|F_\tau(\tau)\|_p\|\vartheta(t-\tau)\|_{p'}d\tau+\int\limits_0^t \tau\|F_{\tau\tau}(\tau)\|_p\|\vartheta(t-\tau)\|_{p'}d\tau$$
$$+\int\limits_0^t \|G(\tau)\|_p\|\vartheta(t-\tau)\|_{p'}d\tau+\int\limits_0^t \tau\|G_\tau(\tau)\|_p\|\vartheta(t-\tau)\|_{p'}d\tau+\|\omega(t)\|_p\|\vartheta_0\|_{p'}$$
$$+s\|\omega_s\|_p\|\vartheta(t-s)\|_{p'}\leq c(t^{\frac12}+t)\|\nabla u_\circ\|_\infty\|\vartheta_0\|_{p'}+s\|\omega_s\|_p\|\vartheta_0\|_{p'}\,,$$

which implies

$$t\|\omega_t(t)\|_p\leq c(t^{\frac12}+t)\|\nabla u_\circ\|_\infty+s\|\omega(s)\|_p,\; t>s>0\,.$$

Integrating the last inequality on $(0,t)$, we also deduce

$$t^2\|\omega_t(t)\|_p\leq ct(t^{\frac12}+t)\|\nabla u_\circ\|_\infty+t\int\limits_0^t\|\omega(s)\|_p ds,\; t>s>0\,.$$

Hence

$$\lim_{t\to 0} t\|\omega_t(t)\|_p=0. \tag{9.11}$$

Estimate (9.4). We look for $u=W+w=W+F+\omega$ again. Of course Lemma 9.1 ensures that

$$\tfrac{t}{t+1}\|W_t(t)\|_{L^p(\Omega_\rho)}+\sum_{|\alpha|=0}^2\big(\tfrac{t}{t+1}\big)^{\frac{|\alpha|}{2}}\|D^\alpha W(t)\|_{L^p(\Omega_\rho)}\leq c\sup_{B(O,\rho)}M_q(t,y,L,u_\circ), \tag{9.12}$$

which is uniform with respect to t, L and $u_\circ$. Now, we have to deduce the same kind of estimates for the field $w(t,x)$. By construction, F also enjoys the same property. So we restrict our considerations to the field ω. We assume that the proof of the following estimate is achieved[10]

$$\|\omega(t)\|_p\leq c\sup_{\partial\Omega}M_q(t,\xi,L,u_\circ), \tag{9.13}$$

uniformly in $t>0$, $L>0$ and $u_\circ\in\mathscr{C}(\overline{\Omega})\cap C^1(\overline{\Omega})$. We prove the analogous of (9.13) for ω_t. Thanks to (9.11), we consider (9.10) for $s=0$:

$$|(t\omega_t(t),\vartheta_0)|=|\int\limits_0^t(\tfrac{\partial}{\partial\tau}(\tau F_\tau(\tau)),\vartheta(t-\tau))d\tau-\int\limits_0^t(\tfrac{\partial}{\partial\tau}(\tau G(\tau)),\vartheta(t-\tau))d\tau$$
$$+\int\limits_0^t(\omega_t(\tau),\vartheta(t-\tau))d\tau-(\omega(t),\vartheta_0)|\,.$$

[10] The proof of the estimate can be found in [Maremonti (2012)]. However, next we consider the one concerning ω_t. This last is more involved with respect to the estimate of ω, but is analogous in the arguments.

Via further integrations by parts in (9.10) we get

$$|(t\omega_t(t),\vartheta_0)|\leq\int\limits_0^{\frac t2}|\tau(F_\tau(\tau),\vartheta_\tau(t-\tau))|d\tau+\int\limits_{\frac t2}^{t}|(\tfrac{\partial}{\partial\tau}(\tau F_\tau(\tau)),\vartheta(t-\tau))|d\tau$$
$$+\int\limits_0^{\frac t2}|(\tfrac{\partial}{\partial\tau}(\tau G(\tau)),\vartheta(t-\tau))|d\tau+\int\limits_{\frac t2}^{t-1}|(\tfrac{\partial}{\partial\tau}(\tau G(\tau)),\vartheta(t-\tau))|d\tau$$
$$+\int\limits_{t-1}^{t}|(\tfrac{\partial}{\partial\tau}(\tau G(\tau)),\vartheta(t-\tau))|d\tau+|\tfrac t2(F_t((\tfrac t2),\vartheta(\tfrac t2))|+|(\omega(t(1-\sigma)),\vartheta(\sigma t))|$$
$$+\int\limits_0^{t(1-\sigma)}|(\omega(\tau),\vartheta_\tau(t-\tau))|d\tau+\int\limits_{t(1-\sigma)}^{t}|(\omega_\tau(\tau),\vartheta(t-\tau))|d\tau+|(\omega(t),\vartheta_0)|=\sum_{i=1}^{10}I_i,$$

the last for all $\sigma\in(0,1)$. Applying the Hölder inequality, we get

$$t|(\omega_t(t),\vartheta_0)|\leq\int\limits_0^{\frac t2}\|\tau F_\tau(\tau)\|_p\|\vartheta_\tau(t-\tau)\|_{p'}d\tau+\int\limits_{\frac t2}^{t}\|\tfrac{\partial}{\partial\tau}(\tau F_\tau(\tau))\|_p\|\vartheta(t-\tau))\|_{p'}d\tau$$
$$+\int\limits_0^{\frac t2}\|\tfrac{\partial}{\partial\tau}(\tau G(\tau))\|_r\|\vartheta(t-\tau)\|_{r'}d\tau+\int\limits_{\frac t2}^{t-1}\|\tfrac{\partial}{\partial\tau}(\tau G(\tau))\|_1\|\vartheta(t-\tau)\|_\infty d\tau$$
$$+\int\limits_{t-1}^{t}\|\tfrac{\partial}{\partial\tau}(\tau G(\tau))\|_p\|\vartheta(t-\tau))\|_{p'}d\tau+\tfrac t2\|F_t(\tfrac t2)\|_p\|\vartheta(\tfrac t2)\|_{p'}+\|\omega(t(1-\sigma))\|_p\|\vartheta(\sigma t)\|_{p'}$$
$$+\int\limits_0^{t(1-\sigma)}\|\omega(\tau)\|_p\|\vartheta_\tau(t-\tau)\|_{p'}d\tau+\int\limits_{t(1-\sigma)}^{t}\|\omega_\tau(\tau)\|_p\|\vartheta(t-\tau)\|_{p'}d\tau+\|\omega(t)\|_p\|\vartheta_0\|_{p'}.$$

Since F and G have compact support and enjoy estimates $(9.7)_{1,2}$, by virtue of estimate (9.13), then, for $r=\frac{np}{2p+n}$, $p\in(\frac{n}{n-2},\infty)$, we get

$$t\frac{|(\omega_t(t),\vartheta_0)|}{c\|\vartheta_0\|_{p'}}\leq\int\limits_0^{\frac t2}\sup_{\partial\Omega}M_q(\tau,y,L,u_\circ)(t-\tau)^{-1}d\tau+\int\limits_{\frac t2}^{t}\sup_{\partial\Omega}M_q(\tau,y,L,u_\circ)\tau^{-1}d\tau$$
$$+\int\limits_{\frac t2}^{t-1}\sup_{\partial\Omega}M_q(\tau,y,L,u_\circ)(t-\tau)^{-\frac n2\frac1{p'}}d\tau+\int\limits_{t-1}^{t}\sup_{\partial\Omega}M_q(\tau,y,L,u_\circ)d\tau$$
$$+c(\sigma)\sup_{\partial\Omega}M_q(t,y,L,u_\circ)+\int\limits_0^{t(1-\sigma)}M_q(\tau,y,L,u_\circ)(t-\tau)^{-1}d\tau+\int\limits_{t(1-\sigma)}^{t}\|\omega_\tau(\tau)\|_pd\tau\,,$$

which implies via (9.1) and (9.13), uniformly in t, L and $u_\circ$,

$$t\frac{|(\omega_t(t),\vartheta_0)|}{c\|\vartheta_0\|_{p'}} \le c(\sigma)\sup_{\partial\Omega} M_q(t,y,L,u_\circ) + ct^{\frac{n}{2}\frac{1}{q}}\sup_{\partial\Omega} M_q(t,y,L,u_\circ)\Bigg[\int_0^{\frac{t}{2}}\tau^{-\frac{n}{2}\frac{1}{q}}(t-\tau)^{-1}d\tau$$

$$+\int_{\frac{t}{2}}^{t}\tau^{-1-\frac{n}{2}\frac{1}{q}}d\tau + \int_{\frac{t}{2}}^{t-1}\tau^{-\frac{n}{2}\frac{1}{q}}(t-\tau)^{-\frac{n}{2}\frac{1}{p'}}d\tau + \int_{t-1}^{t}\tau^{-\frac{n}{2}\frac{1}{q}}d\tau + \int_0^{t(1-\sigma)}\tau^{-\frac{n}{2}\frac{1}{q}}(t-\tau)^{-1}d\tau\Bigg]$$

$$+\int_{t(1-\sigma)}^{t}\|\omega_t(\tau)\|_p d\tau \le c(\sigma)\sup_{\partial\Omega} M_q(t,y,L,u_\circ) + \int_{t(1-\sigma)}^{t}\|\omega_t(\tau)\|_p d\tau.$$

Then, we get

$$t\|\omega_t\|_p \le c(\sigma)\sup_{\partial\Omega} M_q(t,\xi,L,u_\circ) + c\int_{t(1-\sigma)}^{t}\|\omega_t(\tau)\|_p d\tau,\ t>0. \tag{9.14}$$

In step 1) the existence of (ω,π_ω) is ensured with $\omega_t\in L^s(0,T;L^p(\Omega))$, for some $s\in(1,2)$, for $q=\infty$ estimate (9.14) implies that

$$t\|\omega_t(t)\|_p<\infty,\ \text{for all } t>0. \tag{9.15}$$

Multiplying inequality (9.14) by $t^{\frac{n}{2}\frac{1}{q}}$, setting $\zeta(t)=t^{1+\frac{n}{2}\frac{1}{q}}\|\omega(t)\|_p$, by means of a computation, we easily obtain

$$\zeta(t)\le c(\sigma)t^{\frac{n}{2}\frac{1}{q}}\sup_{\partial\Omega} M_q(t,\xi,L,u_\circ) + \frac{c}{(1-\sigma)^{\frac{n}{2}\frac{1}{q}}}\int_{t(1-\sigma)}^{t}\frac{1}{\tau}\zeta(\tau)d\tau,\ t>0.$$

By definition of (9.1), $t^{\frac{n}{2}\frac{1}{q}}\sup\limits_{\partial\Omega} M_q(t,\xi,L,u_\circ)$ is an increasing monotone function of $t>0$. Moreover, we choose $\sigma\in(0,1)$ such that $-c\frac{\log(1-\sigma)}{(1-\sigma)^{\frac{n}{2}\frac{1}{q}}}<1$. Hence, from the last estimate we get

$$\zeta(t)\le c(\sigma)t^{\frac{n}{2}\frac{1}{q}}\sup_{\partial\Omega} M_q(t,\xi,L,u_\circ) - c\frac{\log(1-\sigma)}{(1-\sigma)^{\frac{n}{2}\frac{1}{q}}}\sup_{(t(1-\sigma),t)}\zeta(\tau),\ t>0\,, \tag{9.16}$$

uniformly in L and $u_\circ$. Let $s>0$, the above considerations concerning $t^{\frac{n}{2}\frac{1}{q}}\sup\limits_{\partial\Omega} M_q(t,\xi,L,u_\circ)$, the value of σ, estimate (9.15) and estimate (9.16) furnish

$$\zeta(t)\le c(\sigma)s^{\frac{n}{2}\frac{1}{q}}\sup_{\partial\Omega} M_q(s,\xi,L,u_\circ) - c\frac{\log(1-\sigma)}{(1-\sigma)^{\frac{n}{2}\frac{1}{q}}}\sup_{(0,s]}\zeta(\tau),\ \forall t\in(0,s]\,.$$

Hence, we get

$$s^{1+\frac{n}{2}\frac{1}{q}}\|\omega_s(s)\|_p=\zeta(s)\le\sup_{(0,s]}\zeta(\tau)\le c(\sigma)s^{\frac{n}{2}\frac{1}{q}}\sup_{\partial\Omega} M_q(s,\xi,L,u_\circ)\,,$$

which implies

$$s\|\omega_s(s)\|_p \leq c(\sigma) \sup_{\partial\Omega} M_q(s,\xi,L,u_\circ)\,. \tag{9.17}$$

Since $w=\omega+F$ and $u=W+w$, via $(9.7)_3$ and (9.12), we also deduce $(9.4)_1$ for u_t. Now we look for the estimates of second spatial derivatives of ω and of the gradient of π_ω. To this end, for $t>0$, we regard the Stokes problem (9.6) as a Stokes steady problem with *body force* $\mathfrak{f}=\omega_t+F_t-G$ and homogeneous boundary data. Bearing this in mind, then, by well known estimates on the second derivatives and on the gradient of the pressure, we get the following

$$\begin{aligned}\|\nabla\pi_\omega(t)\|_p+\|D^2\omega(t)\|_p &\leq c(\|\omega_t(t)+F_t-G\|_p+\|\omega(t)\|_p)\\ &\leq c\sup_{\partial\Omega} M_q(t,\xi,L,u_\circ)(t^{-1}+1),\end{aligned} \tag{9.18}$$

with c independent of t, L and $u_\circ$. We can conclude that for $w=F+\omega$ estimate (9.4) is proved.

We do not prove properties $(9.4)_{2,3}$ (cf. [Maremonti (2012)]). In this connection it is enough to remark that

- $u=W+F+\omega\in\mathscr{C}(\overline{\Omega})$, with

$$\lim_{t\to0}|W(t)-u_\circ|_0=0,\ \lim_{t\to0}|F(t)|_0=0, \text{ and, since } \omega(0,x)=0,\ \lim_{t\to0}|\omega(t)|_0=0;$$

- π_u is harmonic and estimate (9.4) holds for $t>0$ and in a neighborhood of $\partial\Omega$.

q.e.d.

10. Achievement of item c)

We start giving a "special cases of MMT".

Theorem 10.1 *Assume that* $u_\circ\in\mathscr{C}(\overline{\Omega})\cap C^1(\overline{\Omega})$ *and, for some* $\delta>0$, *assume also* $dist(\operatorname{supp}u_\circ,\partial\Omega)>3R+\delta$. *Then, there exists a* $c(\delta)$ *such that the solution* (u,π) *corresponding to* $u_\circ$ *by virtue of Theorem 9.1 satisfies the maximum modulus estimate in the following form* [11]

$$|u(t,x)|_0\leq c\big[|u_\circ|_0^{\frac{n}{2}\frac{1}{q}}\big(\sup_{\partial\Omega}M_\infty(t,\xi,L,u_\circ)\big)^{1-\frac{n}{2}\frac{1}{q}}+M_\infty(t,x,L,u_\circ)\big], \tag{10.1}$$

for all $q>\frac{n}{2}$, *where* $c=c(\delta)$ *is independent of* $(t,x)\in[0,T)\times\overline{\Omega},L$ *and* $u_\circ$.

Proof. Since $|H_t(t,z)|\leq c\left(|z|^2+t\right)^{-\frac{n}{2}-1}$, for all $(t,z)\in(0,T)\times\mathbb{R}^n$, then, for the time derivative of $W(t,x)$ we get the following inequality:

$$|W_t(t,x)|\leq c\int\limits_{|y|>3R+\delta}(|x-y|^2+t)^{-\frac{n}{2}-1}|u_\circ(y)|dy.$$

[11] We remark that by a simple estimate of the function M_∞ one proves that (10.1) implies the classical estimate of the maximum modulus theorem.

Since for $x \in B(O,3R)$ and $|y|>3R+\delta$ we have $|x-y|\geq |y|-3R>\delta$, by means of an elementary computation we get

$$|W_t(t,x)|\leq c\delta^{-2}|u_\circ|_0, \text{ for all } (t,x)\in[0,T)\times B(O,3R),$$

which implies the properties:

$$\begin{aligned}&|W(t,x)|\leq c\delta^{-2}|u_\circ|_0 t\,, \text{ on } [0,T)\times\partial\Omega\,,\\ &|W_t(x,t)|\leq c\delta^{-2}|u_\circ|_0\,, \text{ on } [0,T)\times\partial\Omega\,.\end{aligned}\tag{10.2}$$

Thanks to the estimates (10.2), we can modify the ones (9.7). In particular, via an interpolation, we deduce the following ones, for $\sigma\in[0,1]$:

$$\begin{aligned}&|F(t)|_0\leq c|W(t)|_{C(\partial\Omega)}\leq c(\delta)t^{\sigma}|u_\circ|_0^{\sigma}\big(\sup_{\partial\Omega}M_\infty(t,\xi,L,u_\circ)\big)^{1-\sigma},\\ &|F_t(t)|_0\leq c|W_t(t)|_{C(\partial\Omega)}\leq c(\delta)t^{-1+\sigma}|u_\circ|_0^{\sigma}\big(\sup_{\partial\Omega}M_\infty(t,\xi,L,u_\circ)\big)^{1-\sigma},\end{aligned}\tag{10.3}$$

which are uniform with respect to t, L, and $u_\circ$. We multiply the Stokes equation ω by ψ, where ψ is the solution of the problem (6.1) corresponding to $w_0\in C_0^1(\Omega)$, whose existence is ensured by Theorem 6.2. Integrating by parts on $(0,t)\times\Omega$ and applying the Hölder inequality, we get

$$\begin{aligned}|(\omega(t),w_0)|\leq&\int_0^{\frac{t}{2}}\|F(\tau)\|_q\|\psi_\tau(t-\tau)\|_{q'}d\tau+\int_{\frac{t}{2}}^{t}\|F_\tau(\tau)\|_q\|\psi(t-\tau)\|_{q'}d\tau+\|F(\tfrac{t}{2})\|_q\|\psi(\tfrac{t}{2})\|_{q'}\\ &+\int_0^{t-1}\|G(\tau)\|_1\|\psi(t-\tau)\|_\infty d\tau+\int_{t-1}^{t}\|G(\tau)\|_q\|\psi(t-\tau)\|_{q'}d\tau\,,\end{aligned}$$

where, for $t\in(0,1)$, if $\tau<0$, it is assumed $G(\tau)=0$. Now we estimate the right hand-side. We consider some $q>\frac{n}{2}$. We start by recalling that F and G have compact support. For the integral related to F and F_t, we employ (10.3) with $\sigma=\frac{n}{2}\frac{1}{q}$ and we employ estimates $(6.3)_{1,3}$ for ψ. For the integrals related to G we employ $(9.7)_2$ and $(6.3)_1$ for ψ. Recalling that M_∞ is an increasing function of t, we get

$$|(\omega(t),w_0)|\leq c|u_\circ|_0^{\frac{n}{2}\frac{1}{q}}\big(\sup_{\partial\Omega}M_\infty(t,\xi,L,u_\circ)\big)^{1-\frac{n}{2}\frac{1}{q}}\|w_0\|_1+c\sup_{\partial\Omega}M_\infty(t,\xi,L,u_\circ)\|w_0\|_1,$$

which implies $|\omega(t)|_0\leq c(\delta)|u_\circ|_0^{\frac{n}{2}\frac{1}{q}}\big(\sup_{\partial\Omega}M_\infty(t,\xi,L,u_\circ)\big)^{1-\frac{n}{2}\frac{1}{q}}+c\sup_{\partial\Omega}M_\infty(t,\xi,L,u_\circ)$ uniformly in $t>0, L>0$ and $u_\circ$ with $dist(\operatorname{supp}u_\circ,\partial\Omega)>3R+\delta$. Since $u=W+w=W+F+\omega$ and from (9.2) we have $|W(t,x)|\leq cM_\infty(t,x,L,u_\circ)$, then, we derive (10.1) too.

q.e.d.

We recall some properties of the solutions to the Bogovski problem:

$$\nabla\cdot v=g,\ \text{ in } E,\ v=0 \text{ on } \partial E \text{ Lipschlitz domain}, \int_\Omega g dx=0\,.\tag{10.4}$$

Lemma 10.1 *If $g \in C_0^\infty(E)$, then there exists at least a solution $v \in C_0^\infty(E)$ to problem* (10.4) *such that, for $m \in \mathbb{N}$ and $r \in (1,\infty)$,*

$$\|v\|_{m,r} \le c\|g\|_{m-1,r}\,.$$

For the proof of the Lemma we refer to [M.Giga, Giga & Saal].
In section 5 we need of the following properties on a Bogovski solution v. It is known that the domain E can be represented as a union of domains C_k, $k = 1,\dots,N$, star-shaped with respect to the balls B_k of a fixed radius; moreover, there exists a smooth partition of unity, say $\sum\limits_{k=1}^{N}\psi_k(x) = 1$, with supp$\psi_k \subset S_k$. Then, a vector field satisfying (**??**) can be written in the form

$$v(x) = \mathbb{B}[g] = \sum_{k=1}^{N} v_k(x)\,,$$

where

$$v_k(x) = \mathbb{B}^k[\psi_k g] = \int\limits_{S_k} \mathbb{B}^k(x-y,y)\psi_k(y)g(y)dy\,,$$

$$\mathbb{B}^k(z,y) = \frac{z}{|z|^n}\int\limits_{|z|}^{\infty} q^k(y+\xi\frac{z}{|z|})\xi^{n-1}d\xi\,,$$

$$q^k(x) \in C_0^\infty(B_k) \text{ and } \int\limits_{B_k} q_k(y)dy = 1\,.$$

We also recall that, for each $k = 1,\dots,N$, $\mathbb{B}^k$ is an operator with weakly singular kernel and $\frac{\partial}{\partial x_j}\mathbb{B}^k$ is an operator with singular kernel of Calderon-Zigmund kind. Finally, we observe that

$$\frac{\partial}{\partial x_j}\mathbb{B}^k[\psi_k g] = \mathbb{B}^k[\frac{\partial}{\partial y_j}(\psi_k g)] + \mathbb{B}_j^k[\psi_k g], \tag{10.5}$$

where $\mathbb{B}_j^k[\,\cdot\,]$ is the integral operator with the kernel

$$\mathbb{B}_j^k(x-y,y) = \frac{x-y}{|x-y|^n}\int\limits_{|x-y|}^{\infty} \frac{\partial}{\partial y_j}q^k(y+\xi\frac{x-y}{|x-y|})\xi^{n-1}d\xi\,.$$

The Maximum Modulus Theorem

We distinguish three different cases:

a) $u_\circ \in L^\infty(\Omega)$ and $(u_\circ,\nabla\varphi) = 0$ for all $\varphi \in \widehat{W}^{1,1}(\Omega)$;

b) $u_\circ \in \mathscr{C}(\Omega)$, that is $u_\circ \in C(\Omega)\cap L^\infty(\Omega)$, $u_\circ = 0$ on $\partial\Omega$ and $(u_\circ,\nabla\varphi) = 0, \varphi \in \widehat{W}^{1,1}(\Omega)$;

c) $u_\circ \in \mathscr{C}(\overline{\Omega})$, that is $u_\circ \in \mathscr{C}(\Omega)$ and $u_\circ(x)$ uniformly continuous.

We only discuss the case a).
We reacall that for all $u_\circ$ satisfying the assumption a), there exists a sequence $\{u_\circ^m\} \subset \mathscr{C}(\overline{\Omega})\cap C^1(\overline{\Omega})$ such that $u_\circ^m(x) \to u_\circ$ a.e. in $x \in \Omega$ and $|u_\circ^m|_0 \le \|u_\circ\|_\infty$, for all $m \in \mathbb{N}$, which in turn implies that, for all $\rho > 0, \lim\limits_m \|u_\circ^m - u_\circ\|_{L^p(\Omega_\rho)} = 0$. We denote by

$\{(u^m,\pi_{u^m})\}$ the sequence of solutions whose existence is ensured by the above results of item b) (see section 3).

Assume that $u^m(t,x)$ satisfies the following estimate:

$$|u^m(t,x)|\le c|u_\circ^m|_0\le c\|u_\circ\|_\infty,\ \textit{for all } (t,x)\in(0,T)\times\Omega \textit{ and } m\in\mathbb{N}\,.$$

Then, for $p\in(\frac{n}{n-2},\infty)$, $q\in(\frac{q}{2},\infty]$ and $\rho>d$, the following seminorm family

$$\tfrac{t}{t+1}\|u_t(t)\|_{L^p(\Omega_\rho)}+\sum_{|\alpha|=0}^{2}\big(\tfrac{t}{t+1}\big)^{\frac{|\alpha|}{2}}\|D^\alpha u(t)\|_{L^p(\Omega_\rho)}+\big(\tfrac{t}{t+1}\big)^{\frac12+\eta}\|\pi_u(t)\|_{L^p(\Omega_\rho)}$$
$$\le c\sup_{B(O,\rho)}M_q(t,y,L,u_\circ)$$

$\eta\in(0,\frac12)$, ensures a suitable convergences of the sequence $\{(u^m,\pi_{u^m})\}$ to a limit (u,π_u), which is a solution to the Stokes problem and such that

$$|u(t,x)|\le c\|u_\circ\|_\infty,\ \textit{for all } (t,x)\in(0,T)\times\Omega\,.$$

Indeed from the linearity of the problem we get

$$\tfrac{t}{t+1}\|u_t^m(t)-u_t^r(t)\|_{L^p(\Omega_\rho)}+\sum_{|\alpha|=0}^{2}\big(\tfrac{t}{t+1}\big)^{\frac{|\alpha|}{2}}\|D^\alpha u^m(t)D^\alpha u^r(t)\|_{L^p(\Omega_\rho)}$$
$$+\big(\tfrac{t}{t+1}\big)^{\frac12+\eta}\|\pi_{u^m}(t)-\pi_{u^r}(t)\|_{L^p(\Omega_\rho)}\le c\sup_{B(O,\rho)}M_q(t,y,L,u_\circ^m-u_\circ^r).$$

For all compact set $[\eta,T_0]\times\overline{\Omega}_\rho,\eta>0$, we get the following estimates:

- given $\varepsilon>0$, there exists a $L_0>0$ such that $\|u_\circ\|_\infty\frac{T_\circ^{\frac{\mu}{2}}}{(L_0+T_0^{1/2})^\mu}<\varepsilon$;
- given $\varepsilon>0$, there exists $\nu(\eta,L_0,\rho)$ such that

$$\|u_\circ^m-u_\circ^r\|_{L^p(\Omega_\rho)}<\varepsilon\eta^{\frac{n}{2}\frac{1}{q}},\ \text{for all } m,r\ge\nu.$$

Hence from the definition of M_q, we get

$$\sup_{B(O,\rho)}M_q(t,y,L,u_\circ^m-u_\circ^r)<2\varepsilon,\ \text{for all } L>L_0,m,r\ge\nu,$$

which implies, among others, the Cauchy condition in $C((\eta,T_0)\times\overline{\Omega}_\rho)$ for the sequence $\{(u^m,\pi_{u^m})\}$. Denoting the limit by (u,π_u), we easily deduce

$$|u(t,x)|\le|u(t,x)-u^m(t,x)|+|u^m(t,x)|\le c\|u_\circ\|_\infty,\ \text{for all } (t,x)\in(0,T)\times\Omega\,.$$

q.e.d.

Therefore, we just restrict our considerations to the estimate

$$|u^m(t,x)|\le c|u_\circ^m|_0,\ \text{for all } (t,x)\in(0,T)\times\Omega,\ m\in\mathbb{N}. \tag{10.6}$$

To this end, we start with

Lemma 10.2 *Let* (u, π_u) *be a solution to problem* (6.1), *where we assume* $u(0,x) = 0$ *and* $f \in L^2(\Omega)$ *with* $\|f(t)\|_2 \le c(t^{-\frac{1}{2}-\eta} + 1)$, $\eta \in (0, \frac{1}{2})$. *Moreover, assume that*

- $u(t,x) \in C((0,T) \times \overline{\Omega})$, *with* $|u(t)|_0 \le A$, *for all* $t > 0$ *and, for all* $\rho > d$, $\lim\limits_{t\to 0} \|u(t)\|_{L^2(\Omega_\rho)} = 0$;
- *for all* $\eta > 0$, $u_t, D^2u, \nabla\pi \in C((\eta, T) \times \overline{\Omega})$;
- $|\pi_u(t,x)| \le \Pi(1 + |x|)^{-n+2}(t^{-\frac{1}{2}-\eta} + 1)$.

Then, for all $t > 0$ *and* $r \ge 2$,

$$\|u(t)\|_r \le \Big[\frac{1}{2} e^{t^{\frac{1}{2}-\eta}} \int_0^t e^{-s^{\frac{1}{2}+\eta}} \|f(s)\|_2^2 s^{\frac{1}{2}+\eta} ds\Big]^{\frac{a}{2}} A^{1-a}, \text{ with } a = 1 - \frac{2}{r}.$$

Proof. We introduce a weighted energy inequality for u, of the same kind of the one exhibited in [Galdi&Rionero]. We multiply the Stokes equation of u by $g^2 u$, where $g = e^{-\frac{\alpha}{2}|x|}$, $\alpha \in (0, \alpha_0)$. Hence, an integration by parts on $(\tau, t) \times \Omega$ and an application of the Schwartz inequality easily furnish

$$\|u(t)g\|_2^2 + 2\int_\tau^t \|\nabla u(s)g\|_2^2 ds \le \|u(\tau)g\|_2^2 + \frac{1}{2}(\alpha^2 + (n-1)\alpha) \int_\tau^t \|u(s)g\|_2^2 ds + \int_\tau^t \frac{\|u(s)g\|_2^2}{s^{\frac{1}{2}+\eta}} ds$$
$$+ \frac{\alpha^2}{2} \int_\tau^t \|\pi(s)g\|_2^2 s^{\frac{1}{2}+\eta} ds + \frac{1}{2} \int_\tau^t \|f(s)\|_2^2 s^{\frac{1}{2}+\eta} ds.$$

By the first assumption on u we get

$$\lim_{\tau\to 0} \|u(\tau)g\|_2^2 \le \lim_{\tau\to 0} \|u(\tau)\|_{L^2(\Omega_\rho)}^2 + A^2 c(\alpha) e^{-\alpha\rho} = A^2 c(\alpha) e^{-\alpha\rho},$$

which implies

$$\lim_{\tau\to 0} \|u(\tau)g\|_2^2 = 0.$$

Hence from the energy weighted inequality, taking the limit for $\tau \to 0$, we obtain

$$\|u(t)g\|_2^2 \le \frac{1}{2}(\alpha^2 + (n-1)\alpha) \int_0^t \|u(s)g\|_2^2 ds + \int_0^t \frac{\|u(s)g\|_2^2}{s^{\frac{1}{2}+\eta}} ds$$
$$+ \frac{\alpha^2}{2} \int_0^t \|\pi(s)g\|_2^2 s^{\frac{1}{2}+\eta} ds + \frac{1}{2} \int_0^t \|f(s)\|_2^2 s^{\frac{1}{2}+\eta} ds.$$

Applying the Gronwall lemma, we deduce

$$\|u(t)g\|_2^2 \le \frac{1}{2} e^{\beta(\alpha)t + t^{\frac{1}{2}-\eta}} \int_0^t e^{-\beta(\alpha)s - s^{\frac{1}{2}+\eta}} \big[\alpha^2 \|\pi(s)g\|_2^2 + \|f(s)\|_2^2\big] s^{\frac{1}{2}+\eta} ds.$$

Since $\|\pi(s)g\|_2^2 \le c\Pi^2 \frac{1}{\alpha}(s^{-\frac12-\eta}+1)^2$, applying the Lebesgue dominated convergence theorem, in the limit of $\alpha \to 0$ we get

$$\|u(t)\|_2^2 \le \frac{1}{2} e^{t^{\frac12-\eta}} \int_0^t e^{-s^{\frac12+\eta}} \|f(s)\|_2^2 s^{\frac12+\eta} ds, \text{ for all } t > 0.$$

Since $|u(t)|_0 < A$, by interpolation we get the result for all $r \ge 2$. *q.e.d.*

From the above lemma easily follows

Theorem 10.2 *In the same hypotheses of Lemma 10.2, if $f = 0$, then, up to a function $k(t)$ for the pressure field, (u, π_u) is the null solution.*

For $\overline{R} > R$, let us consider a smooth nonnegative cut-off function $\chi_{\overline{R}}$ such that $\chi_{\overline{R}}(x) = 1$ in $\Omega_{\overline{R}/3}$ and $\chi_{\overline{R}}(x) = 0$ in $\Omega_{\overline{R}} - \Omega_{2\overline{R}/3}$. We set

$$u = u\chi_{\overline{R}} + u(1-\chi_{\overline{R}}) = u^1 + u^2 .$$

From the equation for (u, π_u), we get

$$\begin{aligned}
&u^1_t - \Delta u^1 = -\nabla\pi_{u^1} + \pi_u \nabla\chi_{\overline{R}} - 2\nabla\chi_{\overline{R}} \cdot \nabla u - u\Delta\chi_{\overline{R}}, \text{ in } (0,T)\times\Omega_{\overline{R}},\\
&\nabla\cdot u^1 = u\cdot\nabla\chi_{\overline{R}}, \text{ in } (0,T)\times\Omega_{\overline{R}}, \ u^1 = 0 \text{ on } (0,T)\times\partial\Omega_{\overline{R}},\\
&u^1(0,x) = u_\circ\chi_{\overline{R}} \text{ on } \{0\}\times\Omega_{\overline{R}} .
\end{aligned}$$

Analogously, we get

$$\begin{aligned}
&u^2_t - \Delta u^2 = -\nabla\pi_{u^2} - \pi_u \nabla\chi_{\overline{R}} + 2\nabla\chi_{\overline{R}} \cdot \nabla u + u\Delta\chi_{\overline{R}}, \text{ in } (0,T)\times\Omega,\\
&\nabla\cdot u^2 = -u\cdot\nabla\chi_{\overline{R}}, \text{ in } (0,T)\times\Omega, \ u^2 = 0 \text{ on } (0,T)\times\partial\Omega,\\
&u^2(0,x) = u_\circ(1-\chi_{\overline{R}}) \text{ on } \{0\}\times\Omega .
\end{aligned}$$

We perform the decomposition $u^1 = u^{11} + u^{12} + u^{13}$, where

$$u^{11} := \mathbb{B}[u\cdot\nabla\chi_{\overline{R}}](t,x), \ (t,x) \in [0,T)\times\Omega,$$

has compact support in $\Omega \cap B(O, 2\overline{R}+\varepsilon) \setminus B(O, \overline{R}-\varepsilon)$, for all $t \in [0,T)$, and it is a solution to the Bogovski equation

$$\nabla\cdot u^{11} = u\cdot\nabla\chi_{\overline{R}}\,, \text{ in } (0,T)\times\Omega;$$

further

$$\begin{aligned}
&u^{12}_t - \Delta u^{12} = -\nabla\pi_{u^{12}}, \text{ in } (0,T)\times\Omega_{\overline{R}},\\
&\nabla\cdot u^{12} = 0, \text{ in } (0,T)\times\Omega_{\overline{R}}, \ u^{12} = 0 \text{ on } (0,T)\times\partial\Omega_{\overline{R}},\\
&u^{12}(0,x) = u_\circ\chi_{\overline{R}} - u^{11}(0,x), \text{ on } \{0\}\times\Omega_{\overline{R}},
\end{aligned}$$

finally

$$\begin{aligned}
&u^{13}_t - \Delta u^{13} = -\nabla\pi_{u^{13}} + \mathbb{G} + \mathbb{F}, \text{ in } (0,T)\times\Omega_{\overline{R}},\\
&\nabla\cdot u^{13} = 0, \text{ in } (0,T)\times\Omega_{\overline{R}}, \ u^{13} = 0 \text{ on } (0,T)\times\partial\Omega_{\overline{R}},\\
&u^{13}(0,x) = 0 \text{ on } \{0\}\times\Omega_{\overline{R}},
\end{aligned}$$

where

$$\mathbb{G}=\pi_u\nabla\chi_{\overline{R}}-2\nabla\chi_{\overline{R}}\cdot\nabla u-u\Delta\chi_{\overline{R}};$$

$$\mathbb{F}=\mathbb{B}[-\Delta\chi_{\overline{R}}\pi_u]+\tfrac{\partial}{\partial x_j}\mathbb{B}[\pi_u\tfrac{\partial}{\partial y_j}\chi_{\overline{R}}]-\sum_{k=1}^N\mathbb{B}^k[\pi_u\nabla\chi_{\overline{R}}\cdot\nabla\psi_k]-\sum_{k=1}^N\sum_{j=1}^n\mathbb{B}^k_j[\pi_u\psi_k\tfrac{\partial}{\partial y_j}\chi]$$
$$+\sum_{k=1}^N\sum_{j=1}^n\frac{\partial}{\partial x_j}\mathbb{B}^k_j[\psi_k\nabla\chi_{\overline{R}}\cdot u]+\sum_{k=1}^N\sum_{j=1}^n\mathbb{B}^k_j[\frac{\partial}{\partial y_j}(\psi_k\nabla\chi_{\overline{R}}\cdot u)]$$
$$+\sum_{k=1}^N\mathbb{B}^k[2\nabla(\psi_k\chi_{\overline{R}})\cdot\nabla u]+\sum_{k=1}^N\mathbb{B}^k[\Delta(\psi_k\nabla\chi_{\overline{R}})\cdot u)].$$

We have deduced the above expression of $\mathbb{F}$ as a consequence of the following steps:
i) we apply the heat operator $\frac{\partial}{\partial t}-\Delta$ to the function $u^{11}(t,x)$;
ii) we remember that $u_t-\Delta u=-\nabla\pi_u$;
iii) we apply the properties of the Bogovski operator $\mathbb{B}$.
Analogously, we perform the decomposition $u^2=u^{21}+u^{22}+u^{23}$, where

$$u^{21}:=-\mathbb{B}[u\cdot\nabla\chi_{\overline{R}}](t,x)\,,(t,x)\in[0,T)\times\Omega,$$

has compact support in $\Omega\cap B(O,2\overline{R}+\varepsilon)\setminus B(O,\overline{R}-\varepsilon)$, for all $t\in[0,T)$, and it is a solution to the Bogovski equation

$$\nabla\cdot u^{21}=-u\cdot\nabla\chi_{\overline{R}}\,,\text{ in }(0,T)\times\Omega;$$

also

$$\begin{aligned}&u^{22}_t-\Delta u^{22}=-\nabla\pi_{u^{22}},\text{ in }(0,T)\times\Omega,\\&\nabla\cdot u^{22}=0,\text{ in }(0,T)\times\Omega,\ u^{22}=0\text{ on }(0,T)\times\partial\Omega,\\&u^{22}(0,x)=u_\circ(1-\chi_{\overline{R}})-u^{21}(0,x),\text{ on }\{0\}\times\Omega\,;\end{aligned}$$

finally,

$$\begin{aligned}&u^{23}_t-\Delta u^{23}=-\nabla\pi_{u^{23}}-\mathbb{G}-\mathbb{F},\text{ in }(0,T)\times\Omega,\\&\nabla\cdot u^{23}=0,\text{ in }(0,T)\times\Omega,\ u^{23}=0\text{ on }(0,T)\times\partial\Omega,\\&u^{23}(0,x)=0\text{ on }\{0\}\times\Omega\,,\end{aligned}$$

where setting $-(\mathbb{G}+\mathbb{F})$ we have considered the fact that $\nabla\chi_{\overline{R}}=-\nabla(1-\chi_{\overline{R}})$ as well as $u^{11}=-u^{21}$.

The above decomposition allows to claim that

$$|u(t,x)|=|u^1+u^2|=|u^{11}+u^{12}+u^{13}+u^{21}+u^{22}+u^{23}|=|u^{12}+u^{13}+u^{22}+u^{23}|\,.$$

From the quoted theorem by Abe and Giga, we get

$$|u^{12}(t,x)|\le c|u_\circ\chi_{\overline{R}}-u^{11}(0,x)|\le c|u_\circ|_0,\text{ for all }(t,x)\in(0,T)\times\Omega_{\overline{R}}\,.$$

Since the $dist(supp\,u^{22}(0,x),\partial\Omega)>R-\varepsilon>d$, via the latter special case of MMT we get

$$|u^{22}(t,x)|\le c|u_\circ(1-\chi_{\overline{R}})-u^{21}(0,x)|\le c|u_\circ|_0,\text{ for all }(t,x)\in(0,T)\times\Omega_{\overline{R}}\,.$$

Hence we have to estimate $|u^{13}|$ and $|u^{33}|$, whose Stokes equations have the right hand sides which differ by a sign. The kind of estimate and its proof is just the same. We furnish the second one. We begin remarking that

a) since $\mathbb{G}+\mathbb{F}$ has compact support:

$$\|\mathbb{G}(t)\|_r+\|\mathbb{F}(t)\|_r\le c(\|\pi_u(t)\|_{L^r(\Omega_{\overline{R}})}+\|u\|_{W^{1,r}(\Omega_{\overline{R}})}),\ \text{uniformly in } t>0;$$

By means of estimate $(9.4)_1$, for $r>\frac{n}{n-2}$, we can claim

$$\|\mathbb{F}(t)\|_r+\|\mathbb{G}(t)\|_r \le c\sup_{B(O,\overline{R})} M_\infty(t,y,L,u_\circ)(t^{-\frac12-\eta}+t^{-\frac12}+1),\ t>0\,, \tag{10.7}$$

with c independent of t, L and $u_\circ$. We multiply the equation of u^{33} by $\psi(t-\tau,x)$, where, by virtue of Theorem 6.2, $\psi(s,x)$ corresponds to $w_0\in C^1_0(\Omega_{\overline{R}})$. Integrating by parts on $(0,T)\times\Omega_{\overline{R}}$, applying the Hölder inequality, via items a)-b) and (10.7), we obtain for $t\ge 2$ and $r>\frac{n}{2}$,

$$|(u^{33}(t),w_0)|\le\int_0^{t-1}(\|\mathbb{F}(\tau)\|_r+\|\mathbb{G}(\tau)\|_r)\,\|\psi(t-\tau)\|_\infty d\tau + \int_{t-1}^{t}(\|\mathbb{F}(\tau)\|_r+\|\mathbb{G}(\tau)\|_r)\,\|\psi(t-\tau)\|_{r'} d\tau\,.$$

In the case of $t\le 2$ with $r>\frac{n}{2}$, we just consider:

$$|(u^{33}(t),w_0)|\le\int_0^{t}(\|\mathbb{F}(\tau)\|_r+\|\mathbb{G}(\tau)\|_r)\,\|\psi(t-\tau)\|_{r'} d\tau.$$

In both the cases, since $\sup_{B(O,\overline{R})} M_\infty(t,y,L,u_\circ)$ is an increasing function of t, by virtue of estimates $(6.3)_1$ and (10.7), we get

$$|(u^{33}(t),w_0)|\le c(\overline{R})\sup_{B(O,\overline{R})} M_\infty(t,y,L,u_\circ)\|w_0\|_1,$$

which implies

$$|u^{33}(t)|_0\le c(\overline{R})\sup_{B(O,\overline{R})} M_\infty(t,y,L,u_\circ), \tag{10.8}$$

which is true for all t, L and $u_\circ$.

q.e.d.

We conclude this section furnishing the uniqueness of a solution (u,π_u) in its class of existence. By virtue of Theorem 10.2, in order to achieve the uniqueness it is enough to prove that, for all $u_\circ$, a solution (u,π_u) enjoys the limit property:

$$\lim_{t\to0}\|u(t)-u_\circ\|_{L^r(\Omega_R)}=0. \tag{10.9}$$

Of course, we can restrict ourselves to the case of $r=2$. To this end, we consider the approximation (u^m,π^m) again. By virtue of the properties of (u^m,π^m), for all $R>d$

and $t>0$, we get

$$\frac{1}{2}\frac{d}{dt}\|u^m(t)h_R\|_2^2+\int_0^t\|h_R\nabla u^m(\tau)d\tau\|_2^2d\tau$$
$$\leq\int_0^t\int_\Omega\Big[\frac{1}{2}|u^m(t,x)|^2|\Delta h_R^2(x)|+|\pi^m(\tau,x)\nabla h_R^2\cdot u^m(t,x)|\Big]d\tau dx. \tag{10.10}$$

By virtue of estimates (10.6) and $(9.4)_3$, from estimate (10.10), for each $R>d$ and $t>0$, we deduce

$$\|u^m(t)h_R\|_2^2\leq\|u_\circ^m h_R\|_2^2+c(R,u_\circ)(t+t^{\frac{1}{2}-\eta}).$$

Since by Lebesgue dominated convergence theorem $\lim_m\|u^m(t)h_R\|_2=\|u(t)h_R\|_2$ for all $t\geq 0$, for each $R>d$ and $t>0$, the last estimate implies

$$\|u(t)h_R\|_2^2\leq\|u_\circ h_R\|_2^2+c(R,u_\circ)(t+t^{\frac{1}{2}-\eta}). \tag{10.11}$$

Multiplying equation $(1.1)_1$ by $\varphi\in C_0^\infty(\Omega)$, we get

$$(u^m(t)-u_\circ^m,\varphi)=\int_0^t\Big[(u^m(\tau),\Delta\varphi)+(\pi^m(\tau),\nabla\cdot\varphi)\Big]d\tau.$$

Hence, in the limit for $m\to\infty$, we deduce

$$(u(t)-u_\circ,\varphi)=\int_0^t\Big[(u(\tau),\Delta\varphi)+(\pi_u(\tau),\nabla\cdot\varphi)\Big]d\tau,$$

which, via $(7.3)_{1,2}$, implies

$$\lim_{t\to 0}(u(t)-u_\circ,\varphi)=0,\ \text{for all}\ \varphi\in C_0^\infty(\Omega).$$

Thus, by density we also get

$$\lim_{t\to 0}(u(t)-u_\circ,\varphi)=0,\ \text{for all}\ \varphi\in L^1(\Omega). \tag{10.12}$$

Now, employing (10.11) we get

$$\|u(t)h_R-u_\circ h_R\|_2^2=\|u(t)h_R\|_2^2+\|u_\circ h_R\|_2^2-2(u(t),u_\circ h_R^2)$$
$$\leq 2\|u_\circ h_R\|_2^2-2(u(t),u_\circ h_R^2)+c(R,u\circ)(t+t^{\frac{1}{2}-\eta}).$$

Taking (10.12) into account, from the last estimate we deduce (10.9) in the limit for $t\to 0$. Applying the Hölder inequality, we complete the proof of $(7.3)_3$ for $p\in[1,2]$. For $p\in[2,\infty)$, we deduce $(7.3)_3$ by interpolation between L^2-L^∞. *q.e.d.*

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