The Stokes flow in exterior domains with non-decaying initial velocity

Ken Abe

Univ. of Tokyo

June 12, 2012

jointly with Prof. Yoshikazu Giga (U. Tokyo)

We consider

$$(S)\begin{cases} v_t - \Delta v + \nabla q = 0 & \text{in } \Omega \times (0, T) \\ \text{div}\, v = 0 & \text{in } \Omega \times (0, T) \\ \text{B. C.} \quad v = 0 & \text{on } \partial\Omega \\ \text{I. C.} \quad v(x,0) = v_0 & \text{on } \{t = 0\} \end{cases}$$

in a domain $\Omega \subset \mathbf{R}^n$ with $n \geq 2$.

$v(x, t)$: unknown velocity field
$q(x, t)$: unknown pressure field
v_0 : a given initial data

$S(t) : v_0 \mapsto v(\cdot, t)(t \geq 0)$ Stokes semigroup

Definition (Analytic semigroup)

X: Banach space, $\{T(t)\}_{t\geq 0} \subset \mathcal{L}(X)$: semigroup,
We say $T(t)$ is analytic if $\exists C > 0$ s.t.

$$\left\|\frac{dT(t)}{dt}\right\|_{\mathcal{L}(X)} \leq \frac{C}{t} \quad \text{for } t \in (0,1].$$

Ex.

- $S(t)$ is analytic in $L^r_\sigma \cdots$ e.g. bounded and exterior domains
- $e^{t\Delta}$ is analytic in $L^\infty \cdots$ Masuda '72, Stewart '74

Analyticity of $S(t)$ in L^∞

$\underline{\Omega = \mathbf{R}^n_+}$

- Desch-Hieber-Prüss '01
  Resolvent approach $\cdots C_{0,\sigma}$, L^∞_σ
- Solonnikov '03, Maremonti-Starita '03
  Explicit formula for $S(t)$

$\Omega =$ admissible (e.g. bdd)
$S(t)$ is analytic in $C_{0,\sigma}$ [A-Giga, preprint]

L^∞-type spaces
$C_{0,\sigma} = L^\infty$-closure of $C^\infty_{c,\sigma}$
$L^\infty_\sigma = \{v \in L^\infty \mid \int_\Omega v\nabla\varphi = 0,\ \nabla\varphi \in L^1\}$: non-decaying

A typical result

Theorem 1

Let Ω be a bounded domain with C^3 boundary. Then there exists $T_0 > 0$ and $C > 0$ s.t. a priori estimate

$$\sup_{0 \le t \le T_0} ||N(v,q)||_\infty(t) \le C||v_0||_\infty$$

holds for all (v, q) with $v_0 \in C^\infty_{c,\sigma}(\Omega)$.

Here

$$N(v,q)(x,t) = |v(x,t)| + t^{\frac{1}{2}}|\nabla v(x,t)| + t|\nabla^2 v(x,t)| + t|\partial_t v(x,t)| + t|\nabla q(x,t)|.$$

Strictly admissible domain

To establish a priori estimate, a key is the estimate

$$\sup_{x\in\Omega} d_\Omega(x)|\nabla q(x,t)| \le C_\Omega ||\nabla v||_{L^\infty(\partial\Omega)}(t). \qquad \text{(P)}$$

(P) follows from the strict estimate

$$\sup_{x\in\Omega} d_\Omega(x)|\nabla P(x)| \le C_\Omega ||W||_{L^\infty(\partial\Omega)}$$

of the problem:

$$\Delta P = 0 \quad \text{in } \Omega, \ \ \partial P/\partial n_\Omega = \mathrm{div}_{\partial\Omega} W \ \ \text{on } \partial\Omega \qquad \text{(NP)}$$

Definition

Let $\Omega \subset \mathbf{R}^n (n \geq 2)$ be a domain with C^1 boundary. We call Ω strictly admissible if $\exists C_\Omega > 0$ s.t. a priori estimate

$$\sup_{x \in \Omega} d_\Omega(x) |\nabla P(x)| \leq C_\Omega ||W||_{L^\infty(\partial\Omega)}$$

holds for all sol. of (NP) with $W \in L^\infty_{\tan}(\partial\Omega)$.

RK

- Strictly admissible $\cdots$ $\mathbf{R}^n_+$, C^3-bounded domain
- Not strictly admissible$\cdots$ Layer domains

More general result

Theorem 2

Let Ω be a strictly admissible, uniformly C^3 domain. Then for any T_0 there exists $C = C(T_0, \Omega)$ a priori estimate

$$\sup_{0 \le t \le T_0} ||N(v, q)||_\infty(t) \le C||v_0||_\infty$$

holds for all (v, q) with $v_0 \in L^\infty_\sigma(\Omega)$.

RK

For a general unbounded domain, the existence is unknown.

Goal: Ω=exterior

Extend $S(t) : v_0 \mapsto v(\cdot, t)(t \geq 0)$

to an analytic semigroup in $L^\infty_\sigma(\Omega)$

Uniqueness in L^∞_σ

$\underline{\Omega = \mathbf{R}^n_+}$ Solution formula for (v, q) with $v_0 \in L^\infty_\sigma$

$$\Rightarrow \sup_{\substack{x \in \Omega \\ t \in (0,T)}} t^{1/2} d_\Omega(x) \left| \nabla q(x,t) \right| \leq C \| v_0 \|_\infty,$$

where $\Omega = \mathbf{R}^n_+$ with $d_{\mathbf{R}^n_+}(x) = x_n$.

<u>Note</u>
$\nabla q \to 0$ as $x_n \to \infty$ is necessary for the uniqueness.
(e.g. Poiseuille flows)

Non-decaying solutions

Definition

Let (v, q) solve (S) in the classical sense with $v_0 \in L^\infty_\sigma(\Omega)$ in the sense that

$$v(\cdot, t) \longrightarrow v_0 \quad * \text{-weakly in } L^\infty \text{ as } t \downarrow 0.$$

We call (v, q) L^∞-solution if $N(v, q)(x, t)$ and

$$t^{1/2} d_\Omega(x) \left| \nabla q(x, t) \right|$$

are bounded in $\Omega \times (0, T)$.

Main results

Theorem 3

Let $\Omega \subset \mathbf{R}^n (n \geq 2)$ be an exterior domain with C^3 boundary.

(i) For $v_0 \in L^\infty_\sigma$ there exists a unique L^∞-solution (v, q) satisfying

$$\sup_{0 \leq t \leq T_0} ||N(v,q)||_\infty(t) \leq C||v_0||_\infty$$

for any T_0 with $C = C(T_0, \Omega)$.

(ii) $S(t) : v_0 \mapsto v(\cdot, t)$ is uniquely extendable to a (non C_0-) analytic semigroup in $L^\infty_\sigma(\Omega)$.

RK

- $S(t)$ is C_0-semigroup in $BUC_\sigma(\subset L^\infty_\sigma)$
- Maximum modulus, i.e. $\exists C_\Omega > 0$ s.t.

$$||S(t)v_0||_\infty \leq C_\Omega ||v_0||_\infty \quad t > 0$$

  for $v_0 \in L^\infty_\sigma$ is proved by Maremonti '12
- $S(t)$ is bounded analytic in $X = L^r_\sigma$ for $r \in (1, \infty)$ in the sense that both

$$||S(t)||_{\mathcal{L}(\mathrm{X})} \text{ and } t||dS(t)/dt||_{\mathcal{L}(\mathrm{X})}$$

  are bounded in $[0, \infty)$ [Borchers- Sohr '87], [Borchers-Varnhorn '93]

Ideas: Extension to L^∞_σ

Prove the existence for $v_0 \in L^\infty_\sigma$

1. Admissibility
   If an exterior domain is strictly admissible,
   $$\sup_{0\le t\le T} ||N(v,q)||_\infty(t) \le C_T ||v_0||_\infty$$
   is available for $v_0 \in C^\infty_{c,\sigma}$.
2. Approximation
   If $\exists C_\Omega > 0$ s.t. $\forall v_0 \in L^\infty_\sigma, \exists \{v_{0,m}\}_{m\ge1} \subset C^\infty_{c,\sigma}$ s.t.
   $$||v_{0,m}||_\infty \le C_\Omega ||v_0||_\infty$$
   $$v_{0,m} \longrightarrow v_0 \quad \text{a.e. in } \Omega$$

By setting $(v_m,\ q_m)$ as a solution for $v_{0,m}$, we have

$$\sup_{0\le t\le T} ||N(v_m, q_m)||_\infty(t) \le C||v_0||_\infty.$$

From a compactness result, we find

$$(v_m, q_m) \rightarrow \exists(v, q)$$

We then define

$$S(t)v_0 := v(\cdot, t) \quad \text{for } v_0 \in L^\infty_\sigma$$

Note: Approximation topology is not uniform.

However, uniqueness automatically follows from a priori estimate

$$\sup_{0 \leq t \leq T} ||N(v, q)||_\infty(t) \leq C_T ||v_0||_\infty$$

with $v_0 = 0$.

Sketch of the proof: Exterior is "admissible"

Argument by contradiction

Suppose that a priori estimate were false, then $\exists\{P_m\}_{m\geq 1}$ and $\{x_m\}_{m\geq 1}\subset\Omega$ s.t.

$$d_\Omega(x_m)|\nabla P_m(x_m)| \geq m||W_m||_{L^\infty(\partial\Omega)}$$

By normalizing P_m (still denoted by P_m),

$$\frac{1}{2} \leq d_\Omega(x_m)\left|\nabla P_m(x_m)\right| \leq \sup_{x\in\Omega} d_\Omega(x)\left|\nabla P_m(x)\right| = 1$$

and $W_m \to 0$ uniformly on $\partial\Omega$.

Case 1 $\overline{\lim}_{m\to\infty} d_\Omega(x_m) < \infty \Longrightarrow$ reduced to $\mathbf{R}^n_+$ and Ω

Case 2 $\overline{\lim}_{m\to\infty} d_\Omega(x_m) = \infty \Longrightarrow$ reduced to $\mathbf{R}^n$

Blow-up sequence

$$Q_m(x) = P_m(x_m + {d_m}^{1/2}x), \quad d_m = d_\Omega(x_m) \uparrow \infty$$

$\Omega_m = (\Omega - \{x_m\})/d_m$: downscaled domain

Ω: exterior $\Rightarrow \Omega_m^c \to \{a\}(a \neq 0)$

The estimates for P_m are inherited to

$$\sup_{x \in \Omega_m} d_{\Omega_m}(x)|\nabla Q_m(x)| \leq 1$$

$$|\nabla Q_m(0)| \geq 1/2$$

Since Q_m is harmonic, $Q_m \to Q$ locally uniformly in $\mathbf{R}^n \backslash \{a\}$, so

$$\underline{|\nabla Q(0)| \geq 1/2}.$$

$\Delta Q = 0$ in $\mathbf{R}^n \backslash \{a\}$ under the bound

$$\sup_{x \in \mathbf{R}^n} |x - a||\nabla Q(x)| \leq 1$$

When $n \geq 3$, $x = a$ is removable so $\nabla Q \equiv 0.$

When $n = 2$, with the help of mean value, i.e.

$$\fint_{\partial B_a(r)} Q d\mathcal{H}^{n-1}$$

which is independent of $r > 0$, $x = a$ is still removable.