The spin-coating process with heat transfer

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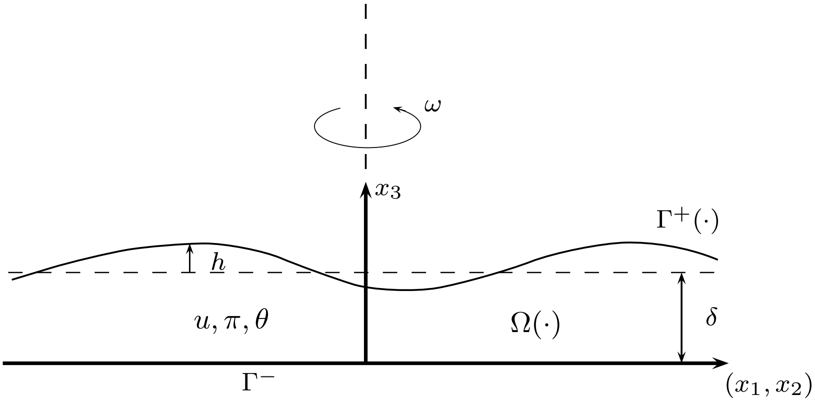

The model

$$
\begin{cases}
\partial_t u + u \cdot \nabla u - \Delta u + \nabla \pi + 2\omega \times u = -\omega \times \omega \times \chi x - \theta e_3 & \text{in } \Omega(t) \\
\partial_t \theta + u \cdot \nabla \theta = 0 & \text{in } \Omega(t) \\
\operatorname{div} u = 0 & \text{in } \Omega(t) \\
-\mathcal{T} \nu = \sigma \kappa \nu & \text{on } \Gamma^+(t) \\
V = u \cdot \nu & \text{on } \Gamma^+(t) \\
(u^1, u^2) = c(h + \delta)^\alpha \partial_3 (u^1, u^2) & \text{on } \Gamma^-(t) \\
u^3 = 0 & \text{on } \Gamma^-(t) \\
u(0) = u_0 & \text{in } \Omega(0) \\
\theta(0) = \theta_0 & \text{in } \Omega(0) \\
\Omega(0) = \{x \in \mathbb{R}^3_+ : x_3 < \delta + h_0(x_1, x_2)\}. &
\end{cases}
$$

Main result

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Theorem

Let $p > 5$ and $J = (0, T)$. There is $\varepsilon > 0$ such that for $\omega \geq 0$ and initial values satisfying certain compatibility conditions and

$$\omega + \|u_0\|_{W^{2-2/p,p}(\Omega(0))} + \|h_0\|_{W^{3-2/p,p}(\mathbb{R}^2)} + \|\theta_0\|_{W^{2-1/p,p}(\Omega(0))} < \varepsilon$$

the spin coating system with heat convection admits a uniquely determined solution

$$u \in \mathbb{E}_u(J) := L^p(J; W^{2,p}(\Omega(\cdot))) \cap W^{1,p}(J; L^p(\Omega(\cdot)))$$

$$\pi \in \mathbb{E}_\pi(J) := \{\pi \in L^p(J; \dot{W}^{1,p}(\Omega(\cdot))) : \gamma\pi \in W^{\frac{p-1}{2p},p}(J; L^p(\Gamma^+(\cdot))) \cap L^p(J; W^{1-\frac{1}{p},p}(\Gamma^+(\cdot)))\}$$

$$h \in \mathbb{E}_h(J) := W^{2-\frac{1}{2p},p}(J; L^p(\mathbb{R}^2)) \cap W^{1,p}(J; W^{2-\frac{1}{p},p}(\mathbb{R}^2)) \cap L^p(J; W^{3-\frac{1}{p},p}(\mathbb{R}^2))$$

$$\theta \in \mathbb{E}_\theta(J) := L^\infty(J; W^{2-1/p,p}(\Omega(\cdot))) \cap W^{1,\infty}(J; W^{1-1/p,p}(\Omega(\cdot))).$$

Transformation to a fixed domain

Hanzawa-Transform

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$$
\begin{cases}
\partial_t u - \Delta u + 2\omega \times u + \nabla \pi = -\theta e_3 + F_1(u, \pi, h) & \text{in } J \times \Omega \\
\partial_t \theta + V(u, h) \cdot \nabla \theta = 0 & \text{in } J \times \Omega \\
\operatorname{div} u = F_d(u, h) & \text{in } J \times \Omega \\
\gamma \mathcal{T}(u, \pi)\nu_D - \sigma \Delta' h \nu_D = G_+(u, \pi, h) & \text{on } J \times \Gamma^+ \\
\partial_t h - \gamma u^3 = H(u, h) & \text{on } J \times \Gamma^+ \\
\gamma(u^1, u^2) - \gamma c \delta^\alpha \partial_3 (u^1, u^2) = G_-(u^1, u^2, h) & \text{on } J \times \Gamma^- \\
\gamma u^3 = 0 & \text{on } J \times \Gamma^- \\
u(0) = u_0 & \text{in } \Omega \\
\theta(0) = \theta_0 & \text{in } \Omega \\
h(0) = h_0 & \text{in } \mathbb{R}^2
\end{cases}
$$

Solving the spin-coating system

Denk/Geißert/Hieber/Saal/Sawada 2011

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There is a Lipschitz continuous operator

$$\Psi\colon U \subset L^p(J; L^p(\Omega)) \to \mathbb{E}_u(J) \times \mathbb{E}_\pi(J) \times \mathbb{E}_h(J), \qquad f_1 \mapsto (u, \pi, h)$$

with

$$\begin{cases} \partial_t u - \Delta u + 2\omega \times u + \nabla\pi = f_1 + F_1(u, \pi, h) & \text{in } J \times \Omega \\ \operatorname{div} u = F_d(u, h) & \text{in } J \times \Omega \\ \gamma \mathcal{T}(u, \pi)\nu_D - \sigma\Delta' h\nu_D = G_+(u, \pi, h) & \text{on } J \times \Gamma^+ \\ \partial_t h - \gamma u^3 = H(u, h) & \text{on } J \times \Gamma^+ \\ \gamma(u^1, u^2) - \gamma c\delta^\alpha \partial_3(u^1, u^2) = G_-(u^1, u^2, h) & \text{on } J \times \Gamma^- \\ \gamma u^3 = 0 & \text{on } J \times \Gamma^- \\ u(0) = u_0 & \text{in } \Omega \\ h(0) = h_0 & \text{in } \mathbb{R}^2. \end{cases}$$

Reduction to a nonlinear transport equation

Let

$$\mathcal{V}(\theta) = V(\Psi_1(-\theta e_3), \Psi_3(-\theta e_3)).$$

Then (u, π, h, θ) solves the spin coating system with heat convection if and only if

$$\begin{cases} \partial_t \theta + \mathcal{V}(\theta) \cdot \nabla \theta = 0 & \text{in } J \times \Omega \\ \theta(0) = \theta_0 & \text{in } \Omega \end{cases}$$

and $(u, \pi, h) = \Psi(-\theta e_3)$.

Lemma

There is an open neighbourhood $U \subset L^p(J; L^p(\Omega))$ of zero such that

- $\mathcal{V} \colon U \subset L^p(J; L^p(\Omega)) \to L^1(J; W^{2-1/p,p}(\Omega)) \cap L^\infty(J; W^{1-1/p,p}(\Omega))$,
- $\|\mathcal{V}(\theta) - \mathcal{V}(\tau)\|_{L^1(J; W^{1-1/p,p}(\Omega))} \leq C \|\theta - \tau\|_{L^p(J; L^p(\Omega))}$ *for* $\theta, \tau \in U$,
- $\mathcal{V}(\theta) \cdot \nu = 0$ *on* $\partial\Omega$.

Solving the linear transport equation

Danchin 2005

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Let

- $\theta_0 \in W^{2-1/p,p}(\Omega)$,
- $v \in L^1(J; W^{2-1/p,p}(\Omega)) \cap L^\infty(J; W^{1-1/p,p}(\Omega))$ with $v \cdot \nu = 0$ on $\partial\Omega$.

Then there is a unique solution $\theta \in \mathbb{E}_\theta(J)$ of

$$\text{(LTE)} \begin{cases} \partial_t\theta + v \cdot \nabla\theta = 0 & \text{in } J \times \Omega \\ \theta(0) = \theta_0 & \text{in } \Omega \end{cases}$$

with

$$\|\theta\|_{L^\infty(J;W^{2-1/p,p}(\Omega))} \leq \|\theta_0\|_{W^{2-1/p,p}(\Omega)} \, e^{C\|v\|_{L^1(J;W^{2-1/p,p}(\Omega))}}.$$

Estimating differences of solutions

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Let θ, τ satisfy

$$\begin{cases} \partial_t \theta + v \cdot \nabla \theta = 0 & \text{in } J \times \Omega \\ \theta(0) = \theta_0 & \text{in } \Omega \end{cases} \quad \text{and} \quad \begin{cases} \partial_t \tau + w \cdot \nabla \tau = 0 & \text{in } J \times \Omega \\ \tau(0) = \theta_0 & \text{in } \Omega \end{cases}$$

for data $\theta_0 \in W^{2-1/p,p}(\Omega)$ and $v, w \in L^1(J; W^{2-1/p,p}(\Omega))$. Then $\rho = \theta - \tau$ satisfies

$$\begin{cases} \partial_t \rho + v \cdot \nabla \rho = (w - v)\nabla \tau & \text{in } J \times \Omega \\ \rho(0) = 0 & \text{in } \Omega \end{cases}$$

and thus

$$\|\theta - \tau\|_{L^\infty(J;W^{1-1/p,p}(\Omega))} \leq C \|v - w\|_{L^1(J;W^{1-1/p,p}(\Omega))} \|\tau\|_{L^\infty(J;W^{2-1/p,p}(\Omega))} e^{C\|v\|_{L^1(J;W^{1,\infty}(\Omega))}}.$$

A modified Contraction Mapping Principle

Kreml/Pokorny 2010

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Lemma

Assume

1. X *reflexive Banach space or has separable pre-dual,* $X \hookrightarrow Y$*,*
2. $H \subset X$ *convex, closed, bounded,*
3. $\Lambda \colon H \to H$ *with*

$$\|\Lambda(u) - \Lambda(v)\|_Y \leq \eta \|u - v\|_Y, \qquad u, v \in H$$

for some $\eta < 1$*.*

Then Λ *has exactly one fixed point in* H*.*

Here:

$$X = L^\infty(J; W^{2-1/p,p}(\Omega)), \qquad Y = L^\infty(J; W^{1-1/p,p}(\Omega)).$$

Solving the nonlinear transport equation

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Proposition

Let $5 < p < \infty$, $J = (0, T)$ *and* $\mathcal{V}$ *as before. There is* $\varepsilon > 0$ *s. t. if*

$$\|\theta_0\|_{W^{2-1/p,p}(\Omega)} \leq \varepsilon$$

then there is exactly one solution

$$\theta \in \mathbb{E}_\theta(J) = L^\infty(J; W^{2-1/p,p}(\Omega)) \cap W^{1,\infty}(J; W^{1-1/p,p}(\Omega))$$

of

$$\text{(NTE)} \begin{cases} \partial_t\theta + \mathcal{V}(\theta) \cdot \nabla\theta = 0 & \text{in } J \times \Omega \\ \theta(0) = \theta_0 & \text{in } \Omega. \end{cases}$$

Solving the nonlinear transport equation

Proof.

- Let $\mathbb{B}_\theta^R$ denote the closed ball of radius R in $X = L^\infty(J; W^{2-1/p,p}(\Omega))$ around zero,
- Define $\Lambda \colon \mathbb{B}_\theta^R \to \mathbb{E}_\theta$:

$$\Lambda(\theta) = \tau \quad :\Longleftrightarrow \quad \begin{cases} \partial_t \tau + \mathcal{V}(\theta) \cdot \nabla \tau = 0 \\ \tau(0) = \theta_0. \end{cases}$$

- $\theta = \Lambda(\theta) \quad \Longleftrightarrow \quad \theta$ solves the nonlinear transport equation (NTE).
- Apply the modified Contraction Mapping Principle.

Solving the nonlinear transport equation

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For $\theta, \tau \in \mathbb{B}_\theta^R$

- $\|\Lambda(\theta)\|_{L^\infty(J;W^{2-1/p,p}(\Omega))} \leq \|\theta_0\|_{W^{2-1/p,p}(\Omega)} e^{C\|\mathcal{V}(\theta)\|_{L^1(J;W^{2-1/p,p}(\Omega))}}$
  $$\leq C\|\theta_0\|_{W^{2-1/p,p}(\Omega)} \overset{!}{\leq} R$$
- $\|\Lambda(\theta) - \Lambda(\tau)\|_{L^\infty(W^{1-1/p,p})} \leq C\|\mathcal{V}(\theta) - \mathcal{V}(\tau)\|_{L^1(W^{1-1/p,p})} \|\tau\|_{L^\infty(W^{2-1/p,p})}$
  $$\leq CR\|\theta - \tau\|_{L^\infty(W^{1-1/p,p})}.$$
- Choose $\|\theta_0\|_{W^{2-1/p,p}(\Omega)}$ small enough.

□

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Thank you for your attention!