A Stochastic FitzHugh-Nagumo Model

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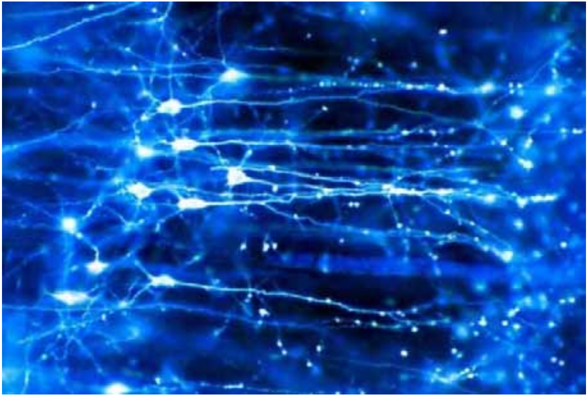

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or as stochastic semilinear evolution equation

$$dX(t) = (AX(t) + F(X(t)))\,dt + \sqrt{Q}\,dW(t),$$
$$X(0) = x \in H = L^2(0,1) \times L^2(0,1)$$

with

$$A = \begin{pmatrix} A_0 - (p(\xi) - \eta) I & -I \\ \gamma I & -\alpha I \end{pmatrix}, \; F(u,v) = \begin{pmatrix} f(u) - \eta u \\ 0 \end{pmatrix} = \begin{pmatrix} -(u-\xi_0)^3 - \xi_0^3 \\ 0 \end{pmatrix},$$

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Outline

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Introduction
- The Stochastic Integral
- Solution Concept: Mild Solution

Existence and Uniqueness of a Mild Solution

Invariant Measure: Existence and Uniqueness

Kolmogorov Operator

Wiener Process
Definition

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Let

- U separable Hilbert space,
- $Q \in \mathcal{L}(U)$ positive and symmetric, $\mathrm{tr}(Q) < \infty$.

A stochastic process $(W(t))_{t\in[0,\infty)}$ of U valued random variables on a filtered probability space $\left(\Omega,\ \mathcal{F},\ (\mathcal{F}_t)_{t\geq 0},\mathbb{P}\right)$ with normal filtration $(\mathcal{F}_t)_{t\geq 0}$ is called U valued Wiener process with covariance operator Q, if

1. $W(0) = 0$ almost surely,
2. $W(t) - W(s) \sim N(0,(t-s)Q)$ for all $0 \leq s \leq t < \infty$,
3. $W(t) - W(s)$ is independent of $\mathcal{F}_s$ for all $0 \leq s \leq t < \infty$.

Wiener Integral:
$\int_0^t \Phi dW(s)$, $\Phi :]0,t] \to \mathcal{L}(U,H)$

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1. Simple functions
   $0 = t_0 < \cdots < t_n = t$, $\Phi_i \in \mathcal{L}(U,H)$, $\Phi(t) = \sum_{i=1}^n \Phi_i \mathbb{1}_{]t_{i-1},t_i]}(t)$,

$$\int_0^t \Phi dW(s) := \sum_{i=1}^n \Phi_i(W(t_i) - W(t_{i-1})).$$

2. Ito Isometry

$$\mathbb{E}\left\|\int_0^t \Phi dW(s)\right\|_H^2 = \sum_{i=1}^n \mathbb{E}\left\|\Phi_i(W(t_i) - W(t_{i-1}))\right\|^2 = \sum_{i=1}^n (t_i - t_{i-1})\|\Phi_i Q^{1/2}\|_{HS}^2$$
$$= \int_0^t \|\Phi(s)Q^{1/2}\|_{HS}^2 ds.$$

3. Linear extension to $L^2(0,t;\mathcal{L}(U,H))$.

The Mild Solution
Definition

For $T \geq 0$ a predictable H-valued process X is called a mild solution for the initial value $x \in H$, if it satisfies

$$\mathbb{P}\left(\int_0^T |X(s)|_H^2 \, ds < \infty\right) = 1$$

and $\mathbb{P}$-a.s. the integral equation

$$X(t) = X(t,x) = \underbrace{S(t)x + \int_0^t S(t-s)F(X(s)) \, ds}_{=:Y(t)} + \underbrace{\int_0^t S(t-s)\sqrt{Q}\, dW(s)}_{=:W_A(t)}, \quad t \in [0,T].$$

Thus

$$Y(t) = S(t)x + \int_0^t S(t-s)F[Y(s) + W_A(s)] \, ds.$$

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Example

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Since

$$Y'(t) = AY(t) + F\left(Y(t) + W_A(t)\right),$$

we can estimate

$$\begin{aligned}\frac{d}{dt}|Y|_H^2 &= 2\langle AY,\ Y\rangle_H + 2\langle F(Y + W_A),\ Y\rangle_H \\ &= 2\langle AY,\ Y\rangle_H + 2\langle F(Y + W_A) - F(W_A),\ Y\rangle_H + 2\langle F(W_A),\ Y\rangle_H \\ &\leq -\omega_1 |Y|_H^2 + \frac{1}{\omega_1}|F(W_A)|_H^2 .\end{aligned}$$

Taking $\mathbb{E}\sup_{t\in[0,T]}$ and Gronwall's lemma lead to:

$$\mathbb{E}\sup_{t\in[0,T]}|Y(t)|_H^2 \leq \left(|x|_H^2 + \frac{1}{\omega_1}\mathbb{E}\int_0^T |F(W_A(s))|_H^2\ ds\right)e^{-\omega_1 T} \leq C\left(1 + |x|_H^2\right).$$

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Approximating System

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Aim:

Existence and uniqueness result for a mild solution similar to S. Bonaccorsi, E. Mastrogiacomo, 2008.

Existence:

- Approximation of X by a Cauchy-sequence $(X_\epsilon)_{\epsilon>0}$ which solves

$$\begin{aligned} dX_\epsilon(t) &= (AX_\epsilon(t) + F_\epsilon(X_\epsilon(t)))\, dt + \sqrt{Q}\, dW(t), \\ X_\epsilon(0) &= x \in H \end{aligned}$$

in a mild sense with $F_\epsilon(X) = \begin{pmatrix} \frac{-ug(u)}{1+\epsilon g(u)} \\ 0 \end{pmatrix}$.

- F_ϵ Lipschitz continuous, $D(F_\epsilon) = H$.

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Essential Results for Existence and Uniqueness

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- Uniformly Boundedness:

$$\mathbb{E}\int_0^T \left|F_\epsilon\left(X_\epsilon(s,x)\right)\right|_H^2 \, ds \;\leq\; C.$$

- Convergence of $(X_\epsilon)_{\epsilon>0}$ in $L^2_W(\Omega;\; C([0,T],\; H))$ for every $x \in H$.
- Continuous dependency on the initial data:

$$\mathbb{E}\left|X(t,x) - X(t,\tilde{x})\right|_H^2 \;\leq\; C\left|x - \tilde{x}\right|_H^2, \quad x,\; \tilde{x} \in H.$$

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Invariant Measure

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- Transition semigroup $(P_t)_{t\geq 0} \subset \mathcal{L}(B_b(H))$:

$$P_t : B_b(H) \to B_b(H),\ \phi(\cdot) \mapsto \mathbb{E}\phi(X(t,\cdot)), \qquad t \geq 0.$$

- Feller property: $P_t(C_b(H)) \subseteq C_b(H)$.
- Existence and uniqueness of an invariant measure $\mu \in M^+(H)$:

$$\int_H P_t\phi(x)\, d\mu(x) = \int_H \phi(x)\, d\mu(x), \quad t \geq 0,\ \phi \in B_b(H).$$

  - Strategy:

$$\text{w–}\lim_{t\to\infty} \mathcal{L}(X(t,x)) = \mu, \quad x \in H.$$

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$$\text{w-}\lim_{t\to\infty} \mathcal{L}(X(t,x)) = \mu, \quad x \in H.$$

Invariant Measure

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- Transition semigroup $(P_t)_{t\geq 0} \subset \mathcal{L}(B_b(H))$:

$$P_t : B_b(H) \to B_b(H), \ \phi(\cdot) \mapsto \mathbb{E}\phi(X(t,\cdot)), \qquad t \geq 0.$$

- Feller property: $P_t(C_b(H)) \subseteq C_b(H)$.
- Existence and uniqueness of an invariant measure $\mu \in M^+(H)$:

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Kolmogorov Operator

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- (Unique) extension of the transition semigroup $(P_t)_{t\geq 0}$ to C_0-semigroup of contractions $\left(\tilde{P}_t\right)_{t\geq 0}$ on $L^1(H,\ \mu)$.

  Infinitesimal generator: $\left(\tilde{L},\ D(\tilde{L})\right)$ dissipative.
- Identification of $\left(\tilde{L},\ D(\tilde{L})\right)$ on test space

$$\mathcal{F}C_b^2(D(A)) := \{\phi \in C_b^2(H) \mid \varphi(x) = \hat{\varphi}_n\left(\langle x,\ e_1\rangle_H, \dots, \langle x,\ e_n\rangle_H\right),$$
$$\hat{\varphi}_n \in C_b^2(\mathbb{R}^n),\ n \in \mathbb{N}\}$$

  as $\left(L,\ \mathcal{F}C_b^2(D(A))\right)$ with

$$L\varphi(x) = \langle x,\ AD\varphi(x)\rangle_H + \langle F(x),\ D\varphi(x)\rangle_H + \frac{1}{2}\mathrm{Tr}_H\left(\sqrt{Q}D^2\varphi(x)\sqrt{Q}\right).$$

- L^1-uniqueness of $\left(L,\ D(L) = \mathcal{F}C_b^2(D(A))\right)$ as provided by A. Es-Sarhir, W. Stannat, 2008.

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Thank you for your attention!