On nonexistence for stationary solutions to the Navier-Stokes equations with a linear strain

Pen-Yuan Hsu
Graduate School of Mathematical Sciences
University of Tokyo

Joint work with Y. Maekawa (Kobe University)

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Equations

We consider stationary solutions to the 3D Navier-Stokes equations for viscous incompressible flows with a linear strain

$$\begin{cases} -\triangle U + (U, \nabla)U + (Mx, \nabla)U + MU + \nabla P = 0, & x \in \mathbb{R}^3 \\ \nabla \cdot U = 0, & x \in \mathbb{R}^3. \end{cases} \tag{$\mathrm{NS_M}$}$$

$$M = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \quad \lambda_i \in \mathbb{R}. \tag{1}$$

Here $U(x) = (U_1(x), U_2(x), U_3(x)) \in \mathbb{R}^3$ and $P(x) \in \mathbb{R}$.

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The second equation of $(\mathrm{NS_M})$ is also satisfied when $\mathrm{Tr}(M) = 0$.
When $\mathrm{Tr}(M) \neq 0$: self-similar solution with a linear strain.

To formulate the relations in a more precise way, we start from the 3D incompressible Navier-Stokes equations with unit viscosity and zero external force (NSE):

$$\begin{cases} v_t - \triangle v + (v, \nabla) v + \nabla p &= 0 \\ \nabla \cdot v &= 0, \end{cases} \tag{2}$$

where $v = v(x, t) \in \mathbb{R}^3$, $p = p(x, t) \in \mathbb{R}$ and $x = (x_1, x_2, x_3)^t \in \mathbb{R}^3$.

Relations to the NSE

- $\mathrm{tr}(M) = 0$: stationary solutions to (NSE)

$$\begin{cases} v(x) &= U(x) + Mx, \\ p(x) &= P(x) - \frac{1}{2}|Mx|^2, \end{cases}$$

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$$\begin{cases} v(x,t) &= \frac{1}{\sqrt{2\alpha t}}(U + S_1)(\frac{x}{\sqrt{2\alpha t}}), \\ p(x,t) &= \frac{1}{2\alpha t}(P + S_2)(\frac{x}{\sqrt{2\alpha t}}), \end{cases}$$

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- $\mathrm{tr}(M) > 0$: backward self-similar solutions to (NSE)

$$\begin{cases} v(x,t) &= \frac{1}{\sqrt{2\alpha(T-t)}}(U + S_1)(\frac{x}{\sqrt{2\alpha(T-t)}}), \\ p(x,t) &= \frac{1}{2\alpha(T-t)}(P + S_2)(\frac{x}{\sqrt{2\alpha(T-t)}}), \end{cases} \tag{3}$$

where $T \in \mathbb{R}, \alpha = \frac{|\mathrm{tr}(M)|}{3} > 0$, $S_1(x) = (M - \frac{\mathrm{tr}(M)}{3} I)x$,
$S_2(x) = \frac{1}{2}(\frac{|\mathrm{tr}(M)|^2}{9}|x|^2 - |Mx|^2)$.

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However, it is still not clear whether $(\mathrm{NS_M})$ admits nontrivial solutions or not, except for the following cases:

- (i) $\lambda_i > 0, \quad i = 1, 2, 3$
- (ii) $\lambda_1 < 0,\ \lambda_2 < 0,\ \sum_{i=1}^{3} \lambda_i = 0,$
- (iii) $\lambda_1 = \lambda_2 = \lambda_3 < 0.$

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Before stating our results, we briefly recall the known results on the cases (i)-(iii).

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  - Tsai (1999): $U \in (L^q(\mathbb{R}^3))^3, 3 < q \leq \infty \Rightarrow U$ must be constant.

Related researches (2/3)

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- Burgers (1948): When U is two-dimensional, an explicit solution exists.(Burgers vortex)
- Asymptotic behavior of Burgers vortex are investigated by [Giga-Kambe],[Gallay-Wayne],[Maekawa],...,etc.

Related researches (3/3)

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- Many general class of forward self-similar solutions have been constructed by [Giga-Miyakawa],[Cannone-Planchon],[Kozono-Yamazaki],...,etc.

Main result

For $\lambda_1 = -\lambda < 0, \lambda_2 = 1, \lambda_3 = \mu \geq 1$, let $\Omega(x) = \nabla \times U(x)$ be the vorticity field. Then we assume that

(C0) $|U(x)| + \dfrac{|P(x)|}{1+|x|} \in L^\infty(\mathbb{R}^3)$;

(C1) $\exists (y_2, y_3) \in \mathbb{R}^2$ $s.t.$ $P(x_1, y_2, y_3) = o(|x_1|)$ at $|x_1| \to \infty$;

(C2) $(1+|x|)|\Omega(x)| \in L^{p_0}(\mathbb{R}^3)$ for some $p_0 \in [1, 3)$;

(C3) there is $\theta_0 > \lambda$ such that

either (i) $|\Omega(x)| \leq C(1+|x_2|)^{-\theta_0 - 1}$

or (ii) $|\Omega(x)| \leq C(1+|x_3|)^{-(\theta_0/\mu)-1}$ holds.

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Theorem 1.1

Let $(U, P) \in (C^2(\mathbb{R}^3))^3 \times C^1(\mathbb{R}^3)$ be a solution to $(\mathrm{NS_M})$. Assume that **(C0)-(C3)** *hold. Then* $U \equiv \mathrm{const.}$

Idea of proof

Fundamental equality

$$\Pi(x) = \frac{1}{2}|U(x)|^2 + Mx \cdot U(x) + P(x), \qquad (4)$$

Let $\mathcal{L}$ be the differential operator defined by

$$\mathcal{L}f = \Delta f - Mx \cdot \nabla f. \qquad (5)$$

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Proposition 2.1

$$\begin{aligned} \mathcal{L}\Pi - U \cdot \nabla \Pi &= |\Omega|^2, & (6)\\ -\Delta U_j - (U \times \Omega)_j + \partial_j \Pi &= -Mx \cdot (\nabla U_j - \partial_j U), & (7)\\ \mathcal{L}\Omega + (M - \mathrm{Tr}(M)I)\Omega &= U \cdot \nabla \Omega - \Omega \cdot \nabla U. & (8) \end{aligned}$$

Estimates for Π

At first we establish estimates for Π from using the relation between Π and Ω. From (7) we have

$$-\Delta\Pi = -\nabla\cdot(U\times\Omega) + \sum_j \partial_j\big(Mx\cdot(\nabla U_j - \partial_j U)\big). \qquad (9)$$

Motivated by (16) we set

$$\Pi_0(x) := -(-\Delta)^{-1}\nabla\cdot(U\times\Omega) + \sum_j (-\Delta)^{-1}\partial_j\big(Mx\cdot(\nabla U_j - \partial_j U)\big)$$

$$= C\sum_j \int_{\mathbb{R}^3} \frac{x_j - y_j}{|x-y|^3}\Big((U(y)\times\Omega(y))_j + My\cdot(\nabla U_j(y) - \partial_j U(y))\Big)\,\mathrm{d}y.$$

Proposition 2.2

Assume that **(C0)**,**(C2)** *hold. Then*

$$\lim_{R\to\infty} \sup_{|x|\geq R} \left(|\Pi_0(x)| + |\nabla \Pi_0(x)|\right) = 0. \quad (10)$$

Moreover, if **(C3)** *holds in addition, then there is* $\delta > 0$ *such that*

$$|\Pi_0(0, x_2, 0)| \leq C(1 + |x_2|)^{-\delta} \quad \text{if (i) of } (\mathbf{C3}) \text{ holds,} \quad (11)$$

$$|\Pi_0(0, 0, x_3)| \leq C(1 + |x_3|)^{-\delta} \quad \text{if (ii) of } (\mathbf{C3}) \text{ holds.} \quad (12)$$

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The condition **(C0)** and Proposition 2.2 implies $\Pi = a_0 + \Pi_0$ and hence,

$$\mathcal{L}\Pi_0 - U \cdot \nabla \Pi_0 = |\Omega|^2. \tag{13}$$

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Since $|\Pi_0(x)| \to 0$ as $|x| \to \infty$ by Proposition 2.2, the strong maximum principle implies

Corollary 2.3

Assume that **(C0)**,**(C2)**,**(C3)***hold. Then either* $\Pi_0 \equiv 0$ *or* $\Pi_0(x) < 0$ *for all* $x \in \mathbb{R}^3$.

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For the moment we consider a smooth nontrivial function Π_0 which satisfies

$$\mathcal{L}\Pi_0 - U \cdot \nabla\Pi_0 \geq 0. \tag{14}$$

The strong maximum principle implies that $\Pi_0(x) < 0$ for all $x \in \mathbb{R}^3$.
Our aim is to derive a lower bound on the spatial decay of $-\Pi_0$.

Proposition 2.4

Assume that **(C0)**-**(C3)** *hold and that* $\Pi_0 \not\equiv 0$. *Then for any* $l > 0$ *there is* $C > 0$ *such that*

$$-\Pi_0(0, x_2, 0) \geq C(1 + |x_2|)^{-l} \qquad \text{if (i) of } \textbf{(C3)} \text{ holds,} \tag{15}$$

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In the proof of Proposition the sign of λ_i is essential. Indeed, we carefully estimate the property that the positivity of λ_2 (λ_3) leads to slower spatial decay of $-\Pi_0$ in $x_2(x_3)$ direction.
We note that, since $\lambda_1 < 0$, we can not use the argument in [Tsai].

Conclusion

Proof of Theorem 1.1

If $\Pi_0 \not\equiv 0$ then the lower bound for Π_0 in Proposition 2.4 contradicts with the decay estimate of Π_0 in Proposition 2.2. Hence $\Pi_0 \equiv 0$, i.e., $\Pi \equiv \mathrm{const}$. Thus we have $\Omega \equiv 0$ from (6), which implies $U = \mathrm{const}$.

Estimate for velocity

Set

$$V(x) = (-\Delta)^{-1}\nabla \times \Omega = C \int_{\mathbb{R}^3} \frac{(x-y)}{|x-y|^3} \times \Omega(y)\,\mathrm{d}y. \qquad (17)$$

Then by **(C0)** we have

$$U = u_c + V \qquad u_c : \text{ a constant vector.} \qquad (18)$$

Proposition 2.5

Assume that **(C0)**,**(C2)** *hold. Then*

$$|V(x)| \leq C(1 + |x|)^{-1}. \qquad (19)$$

Estimates for vorticity

Proposition 2.6

Assume that **(C0)**,**(C2)**,**(C3)** *hold. Then,for* $k = 0, 1, 2$,

$$(1 + |x|)|\nabla^k \Omega(x)| \in L^p(\mathbb{R}^3) \quad \text{for all} \quad p \in [p_0, \infty], \tag{20}$$

$$|\nabla^k \Omega(x)| \leq C(1 + |x_2|)^{-\theta_0 - 1} \qquad \text{if} \quad \text{(i) of } (\textbf{C3}) \text{ holds}, \tag{21}$$

$$|\nabla^k \Omega(x)| \leq C(1 + |x_3|)^{-\frac{\theta_0}{\mu} - 1} \qquad \text{if} \quad \text{(ii) of } (\textbf{C3}) \text{ holds}. \tag{22}$$