On the sectorial $\mathcal{R}$-boundedness of the Stokes operator for the compressible viscous fluid flow in the half-space with slip boundary condition

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Outline

1. Introduction
   - problems
   - known results
   - definition of $\mathcal{R}$-boundedness
2. Main theorem
   - $\mathcal{R}$-boundedness of the solution operator to the resolvent problem
   - the generation of analytic semigroup
   - the maximal L_p-L_q regularity
3. Outline of proof
   - case of whole-space
   - case of half-space

Nonlinear problem

The motion of compressible viscous fluid is formulated by the following initial boundary value problem:

$$
\text{(NP)} \quad \begin{cases} \dfrac{\partial \rho}{\partial t} + \operatorname{div}(\rho u) = 0 & \text{in } \Omega,\ t > 0, \\ \rho\left(\dfrac{\partial u}{\partial t} + u \cdot \nabla u\right) - \operatorname{Div} S(u, P) = F & \text{in } \Omega,\ t > 0, \\ (\rho, u)|_{t=0} = (\rho_0, u_0),\ \lim_{|x|\to\infty} (\rho, u) = (\rho_\infty, 0), & \end{cases}
$$

subject to some boundary condition.

- $u = (u_1, \ldots, u_N)$: unknown velocity field, $N \geq 2$: dimension
- $P = P(\rho)$: pressure. $P'(\rho) > 0$. ρ : unknown density
- $S(u, P) = 2\mu_1 D(u) + (\mu_2 \operatorname{div} u - P)I$: stress tensor
  $D(u) = \{\nabla u + (\nabla u)^{\mathrm{T}}\}/2$: deformation tensor
- μ_1, μ_2 : the first and second viscosity coefficient.
  $\mu_1 > 0,\ \mu_1 + \mu_2 > 0.$

Known results

- the generation of analytic semigroup

  · Strömer (1987), Shibata and Tanaka (2004)
  $\Rightarrow$ the generation of analytic semigroup for the compressible viscous fluid in a general domain with Dirichlet boundary condition
  (e.g. half-space, bounded domain, exterior domain)

- the maximal regularity

  · Solonnikov (1965)
  $\Rightarrow$ the maximal regularity for the general parabolic equations in a general domain with uniform Lopatinski-Shapiro condition
  · Kakizawa (2011)
  $\Rightarrow$ the maximal regularity of the linearized initial boundary value problem for the compressible viscous fluid in bounded domain with Navier boundary condition

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Known result and motivation

Enomoto and Shibata (2012, preprint)⇒

· the generation of analytic semigroup and the maximal regularity by the $\mathcal{R}$-boundedness of Stokes operator

· a local in time unique existence theorem in a general domain with Dirichlet boundary condition for the initial boundary value problem for the compressible viscous fluid flow

The difference between "result of Enomoto and Shibata" and "our result" is the following:

	domain	boundary condition
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Linearized problem

We consider the linearized problem of (NP) in the half-space with slip boundary condition.

$$
\text{(P)} \quad \begin{cases}
\dfrac{\partial \rho}{\partial t} + \gamma \operatorname{div} u = f & \text{in } \mathbb{R}^N_+, \ t > 0, \\
\dfrac{\partial u}{\partial t} - \alpha \Delta u - \beta \nabla \operatorname{div} u + \gamma \nabla \rho = g & \text{in } \mathbb{R}^N_+, \ t > 0, \\
S(u, \rho)\nu|_{\tan} = h|_{\tan} & \text{on } \mathbb{R}^N_0, \ t > 0, \\
u_N = 0 & \text{on } \mathbb{R}^N_0, \ t > 0, \\
(\rho, u)|_{t=0} = (0, 0).
\end{cases}
$$

- $\mathbb{R}^N_+ = \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N \mid x_N > 0\}$,
  $\mathbb{R}^N_0 = \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N \mid x_N = 0\}$.
- α, β, γ : constant. $\alpha, \gamma > 0$, $\alpha + \beta > 0$.
- $\nu = (0, \ldots, 0, -1)$: unit outer normal field on $\mathbb{R}^N_0$
- $S(u, \rho)\nu|_{\tan}, h|_{\tan}$: tangential part of $S(u, \rho)\nu, h$

Resolvent problem

The corresponding resolvent problem:

$$
\text{(RP)} \quad \begin{cases} \lambda\rho + \gamma \operatorname{div} u = f & \text{in } \mathbb{R}^N_+, \\ \lambda u - \alpha\Delta u - \beta\nabla \operatorname{div} u + \gamma\nabla\rho = g & \text{in } \mathbb{R}^N_+, \\ S(u,\rho)\nu|_{\tan} = h|_{\tan} & \text{on } \mathbb{R}^N_0, \\ u_N = 0 & \text{on } \mathbb{R}^N_0. \end{cases}
$$

In order to show the generation of analytic semigroup, we define a linear operator $\mathcal{A}$ by

$$
\mathcal{A}(\rho,u) = (-\gamma \operatorname{div} u, \alpha\Delta u + \beta\nabla \operatorname{div} u - \gamma\nabla\rho) \text{ for } (\rho,u) \in \mathcal{D}(\mathcal{A}),
$$

$$
\mathcal{D}(\mathcal{A}) = \{(\rho,u) \in W^{1,2}_q(\mathbb{R}^N_+) \mid S(u,\rho)\nu|_{\tan} = 0, u_N = 0 \text{ on } \mathbb{R}^N_0\},
$$

where we set $W^{m,l}_q(\mathbb{R}^N_+) = \{(f,g) \mid f \in W^m_q(\mathbb{R}^N_+), g \in W^l_q(\mathbb{R}^N_+)^N\}$.

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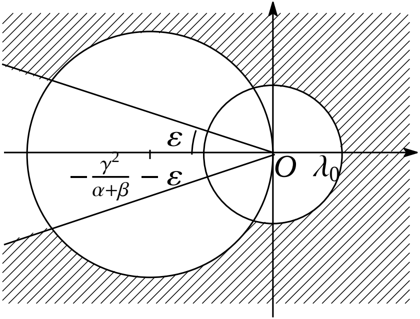
Let $0 < \varepsilon < \pi/2, \lambda_0 > 0$. we set

$$\Lambda_{\varepsilon,\lambda_0} = \Sigma_{\varepsilon,\lambda_0} \cap K_\varepsilon,$$

$\Sigma_{\varepsilon,\lambda_0}$ and K_ε is the set defined by

$$\Sigma_{\varepsilon,\lambda_0} = \left\{\lambda \in \mathbb{C} \setminus \{0\} \;\middle|\; |\arg\lambda| \le \pi - \varepsilon, |\lambda| \ge \lambda_0\right\},$$

$$K_\varepsilon = \left\{\lambda \in \mathbb{C} \;\middle|\; \left(\mathrm{Re}\,\lambda + \frac{\gamma^2}{\alpha+\beta} + \varepsilon\right)^2 + (\mathrm{Im}\,\lambda)^2 \ge \left(\frac{\gamma^2}{\alpha+\beta} + \varepsilon\right)^2\right\}.$$



Aim and key point

aim

- $\mathcal{A}$ generates an analytic semigroup $\{T(t)\}_{t\geq 0}$ on $W_q^{1,0}(\mathbb{R}_+^N)$.
- the maximal L_p-L_q regularity for (P).

By the method due to Enomoto and Shibata, we see that it is sufficient to prove $\mathcal{R}$-boundedness of the solution operator to resolvent problem.

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$\mathcal{R}$-boundedness

Definition ($\mathcal{R}$ – *boundedness*)

A family of operators $\mathcal{T} \subset \mathcal{L}(X, Y)$ is called $\mathcal{R}$-bounded on $\mathcal{L}(X, Y)$, if there exist constants $C > 0$ and $p \in [1, \infty)$ such that for each $m \in \mathbb{N}$, $T_j \in \mathcal{T}$, $f_j \in X$ ($j = 1, \ldots, m$) for all sequences $\{r_j(u)\}_{j=1}^m$ of independent, symmetric, $\{-1, 1\}$-valued random variables on $[0, 1]$, there holds the inequality :

$$\int_0^1 \Big\| \sum_{j=1}^m r_j(u) T_j f_j \Big\|_Y^p du \le C \int_0^1 \Big\| \sum_{j=1}^m r_j(u) f_j \Big\|_X^p du.$$

Remark

The smallest such C is called $\mathcal{R}$-bound of $\mathcal{T}$ on $\mathcal{L}(X, Y)$, which is denoted by $\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T})$.

For any Banach spaces X and Y, $\mathcal{L}(X, Y)$ denotes the set of all bounded linear operators from X into Y. $\mathcal{L}(X) = \mathcal{L}(X, X)$.

Main theorem

Let $0 < \varepsilon < \pi/2,\ 1 < q < \infty$. Then, there exist a $\lambda_0 > 0$ depending on ε, q, N and an operator $R(\lambda) \in \mathcal{L}(W^{1,0}_q(\mathbb{R}^N_+) \times L_q(\mathbb{R}^N_+)^{N^2-1}, W^{1,2}_q(\mathbb{R}^N_+))$ such that the following two assertions hold :

(i) For any $(f, g) \in W^{1,0}_q(\mathbb{R}^N_+),\ h \in W^1_q(\mathbb{R}^N_+)^{N-1}$ and $\lambda \in \Lambda_{\varepsilon,\lambda_0}$, $(\rho, u) = R(\lambda)(f, g, \nabla h, |\lambda|^{\frac{1}{2}} h) \in W^{1,2}_q(\mathbb{R}^N_+)$ solves the equations (RP) uniquely.

(ii) There exist $\gamma_0 > 0$ such that

$$\mathcal{R}_{\mathcal{L}(W^{1,0}_q(\mathbb{R}^N_+)\times L_q(\mathbb{R}^N_+)^{N^2-1},\ W^{1,0}_q(\mathbb{R}^N_+))}(\{\lambda R(\lambda) \mid \lambda \in \Lambda_{\varepsilon,\lambda_0}\}) \le \gamma_0,$$

$$\mathcal{R}_{\mathcal{L}(W^{1,0}_q(\mathbb{R}^N_+)\times L_q(\mathbb{R}^N_+)^{N^2-1},\ L_q(\mathbb{R}^N_+)^{N^2})}(\{|\lambda|^{\frac{1}{2}} \nabla P_v R(\lambda) \mid \lambda \in \Lambda_{\varepsilon,\lambda_0}\}) \le \gamma_0,$$

$$\mathcal{R}_{\mathcal{L}(W^{1,0}_q(\mathbb{R}^N_+)\times L_q(\mathbb{R}^N_+)^{N^2-1},\ L_q(\mathbb{R}^N_+)^{N^3})}(\{\nabla^2 P_v R(\lambda) \mid \lambda \in \Lambda_{\varepsilon,\lambda_0}\}) \le \gamma_0,$$

where we set $P_v R(\lambda)(f, g, \nabla h, |\lambda|^{\frac{1}{2}} h) = u$.

The generation of analytic semigroup

Since the definition of $\mathcal{R}$-boundedness with $m = 1$ implies the usual boundedness, it follows from main theorem (ii) that

$$|\lambda| \|(\rho, u)\|_{W_q^{1,0}(\mathbb{R}^n_+)} + |\lambda|^{\frac{1}{2}} \|\nabla u\|_{L_q(\mathbb{R}^n_+)} + \|\nabla^2 u\|_{L_q(\mathbb{R}^n_+)} \leq C(\|(f, g)\|_{W_q^{1,0}(\mathbb{R}^n_+)} + \|(|\lambda|^{\frac{1}{2}} h, \nabla h)\|_{L_q(\mathbb{R}^n_+)}).$$

Theorem

Let $1 < q < \infty$.Then, the operator $\mathcal{A}$ generates an analytic semigroup $\{T(t)\}_{t \geq 0}$ on $W_q^{1,0}(\mathbb{R}^n_+)$.

The maximal L_p-L_q regularity

Applying the main theorem (ii) and Weis' operator valued Fourier multiplier theorem, we obtain the following theorem.

Theorem

Let $1 < p, q < \infty$. Then, there exists a constant $\gamma_1 > 0$ such that the following two assertions hold:

(i) For any $f \in L_{p,\gamma_1,0}(\mathbb{R}, W_q^1(\mathbb{R}_+^N))$, $g \in L_{p,\gamma_1,0}(\mathbb{R}, L_q(\mathbb{R}_+^N)^N)$ and $h \in L_{p,\gamma_1,0}(\mathbb{R}, W_q^1(\mathbb{R}_+^N)^N) \cap H_{p,\gamma_1,0}^{\frac{1}{2}}(\mathbb{R}, L_q(\mathbb{R}_+^N)^N)$, the problem (P) admits a unique solution (ρ, u) satisfying

$$\rho \in W_{p,\gamma_1,0}^1(\mathbb{R}, W_q^1(\mathbb{R}_+^N)),$$
$$u \in L_{p,\gamma_1,0}(\mathbb{R}, W_q^2(\mathbb{R}_+^N)^N) \cap W_{p,\gamma_1,0}^1(\mathbb{R}, L_q(\mathbb{R}_+^N)^N)$$

(ii) $$\|e^{-\gamma t}(\rho_t, \gamma\rho)\|_{L_p(\mathbb{R}, W_q^1(\mathbb{R}_+^N))} + \|e^{-\gamma t}(u_t, \gamma u, \Lambda_\gamma^{\frac{1}{2}}\nabla u, \nabla^2 u)\|_{L_p(\mathbb{R}, L_q(\mathbb{R}_+^N))}$$
$$\leq C\|e^{-\gamma t}(f, \nabla f, \Lambda_\gamma^{\frac{1}{2}} h, \nabla h, g)\|_{L_p(\mathbb{R}, L_q(\mathbb{R}_+^N))} \quad (\forall \gamma \geq \gamma_1).$$

Functional space

For a Banach space X, we set

$$
\begin{aligned}
L_{p,\gamma_1}(\mathbb{R}, X) &= \{ f(t) \in L_{p,loc}(\mathbb{R}, X) \mid e^{-\gamma_1 t} f(t) \in L_p(\mathbb{R}, X) \}, \\
L_{p,\gamma_1,0}(\mathbb{R}, X) &= \{ f(t) \in L_{p,\gamma_1}(\mathbb{R}, X) \mid f(t) = 0, \quad t < 0 \}, \\
W^m_{p,\gamma_1}(\mathbb{R}, X) &= \{ f(t) \in L_{p,\gamma_1}(\mathbb{R}, X) \mid e^{-\gamma_1 t} D_t^j f(t) \in L_p(\mathbb{R}, X), \\
&\qquad\qquad\qquad\qquad\qquad\qquad j = 1, 2, \ldots, m \}, \\
W^m_{p,\gamma_1,0}(\mathbb{R}, X) &= W^m_{p,\gamma_1}(\mathbb{R}, X) \cap L_{p,\gamma_1,0}(\mathbb{R}, X).
\end{aligned}
$$

Laplace transform $\mathcal{L}$ and its inverse $\mathcal{L}_\lambda^{-1}$

$$\mathcal{L}[f](\lambda) = \int_{-\infty}^{\infty} e^{-\lambda t} f(t)dt, \ \mathcal{L}_\lambda^{-1}[g](t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\lambda t} g(\tau)d\tau. \ (\lambda = \gamma + i\tau)$$

We set

$$\Lambda_\gamma^{\frac{1}{2}}[f](t) = \mathcal{L}_\lambda^{-1}[|\lambda|^{\frac{1}{2}}\mathcal{L}[f](\lambda)](t).$$

Bessel potential space $H_{p,\gamma_1}^{\frac{1}{2}}$

$$H_{p,\gamma_1}^{\frac{1}{2}}(\mathbb{R}, X) = \{ f : \mathbb{R} \to X \mid e^{-\gamma t}\Lambda_\gamma^{\frac{1}{2}}[f](t) \in L_p(\mathbb{R}, X), \ \forall \gamma \geq \gamma_1 \},$$

$$H_{p,\gamma_1,0}^{\frac{1}{2}}(\mathbb{R}, X) = \{ f \in H_{p,\gamma_1}^{\frac{1}{2}}(\mathbb{R}, X) \mid f(t) = 0, \quad t < 0 \}.$$

Outline of proof in $\mathbb{R}^N$

The resolvent problem in $\mathbb{R}^N$

$$\begin{cases} \lambda\rho + \gamma \operatorname{div} u = f & \text{in } \mathbb{R}^N, \\ \lambda u - \alpha\Delta u - \beta\nabla \operatorname{div} u + \gamma\nabla\rho = g & \text{in } \mathbb{R}^N. \end{cases}$$

Setting $\rho = \lambda^{-1}(f - \gamma \operatorname{div} u)$ and $\eta_\lambda = \beta + \gamma^2\lambda^{-1}$, we convert upper equation into the equation:

$$\lambda u - \alpha\Delta u - \eta_\lambda\nabla \operatorname{div} u = g - \gamma\lambda^{-1}\nabla f =: f \quad \text{in } \mathbb{R}^N.$$

The solution formula is

$$u_j = \frac{1}{\alpha}\sum_{k=1}^{N}\mathcal{F}_\xi^{-1}\left[\frac{\delta_{jk} - \xi_j\xi_k|\xi|^{-2}}{\alpha^{-1}\lambda + |\xi|^2}\hat{f}_k\right](x) + \frac{1}{\alpha+\eta_\lambda}\sum_{k=1}^{N}\mathcal{F}_\xi^{-1}\left[\frac{\xi_j\xi_k|\xi|^{-2}}{(\alpha+\eta_\lambda)^{-1}\lambda + |\xi|^2}\hat{f}_k\right](x).$$

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Lemma(key of proof of $\mathcal{R}$-boundedness)

Let $1 < q < \infty$. Let $m(\lambda, \xi) \in C^\infty(\mathbb{R}^N \setminus \{0\})$ be a function such that for any multi-index $\beta \in \mathbb{N}_0^N$ ($\mathbb{N}_0 = \mathbb{N} \cup \{0\}$) there exist constants C, d_1 and d_2 satisfying the following estimates:

$$\text{(i) } |D_\xi^\beta m(\lambda, \xi)| \leq C_{\beta,\varepsilon,\lambda_0}(|\lambda|^{\frac{1}{2}} + |\xi|)^2 |\xi|^{-|\beta|},$$

$$\text{(ii) } d_1(|\lambda|^{\frac{1}{2}} + |\xi|)^2 \leq ||\xi|^2 + p(\lambda)| \leq d_2(|\lambda|^{\frac{1}{2}} + |\xi|)^2$$

for any $(\lambda, \xi') \in \Lambda_{\varepsilon,\lambda_0} \times (\mathbb{R}^{N-1} \setminus \{0\})$.
Then, let $K(\lambda)$ be an operator defined by:

$$[K(\lambda)f](x) = \mathcal{F}_\xi^{-1}[k(\lambda, \xi)\hat{f}(\xi)](x), \quad k(\lambda, \xi) = \frac{m(\lambda, \xi)}{|\xi|^2 + p(\lambda)}$$

$\{K(\lambda) | \lambda \in \Lambda_{\varepsilon,\lambda_0}\}$ is $\mathcal{R}$-bounded on $\mathcal{L}(L_q(\mathbb{R}^N))$.

$$u_j = \frac{1}{\alpha}\sum_{k=1}^N \mathcal{F}_\xi^{-1}\left[\frac{\delta_{jk} - \xi_j\xi_k|\xi|^{-2}}{\alpha^{-1}\lambda + |\xi|^2}\hat{f}_k\right](x) + \frac{1}{\alpha + \eta_\lambda}\sum_{k=1}^N \mathcal{F}_\xi^{-1}\left[\frac{\xi_j\xi_k|\xi|^{-2}}{(\alpha + \eta_\lambda)^{-1}\lambda + |\xi|^2}\hat{f}_k\right](x).$$

Lemma(key of proof of $\mathcal{R}$-boundedness)

Let $1 < q < \infty$. Let $m(\lambda, \xi) \in C^\infty(\mathbb{R}^N \setminus \{0\})$ be a function such that for any multi-index $\beta \in \mathbb{N}_0^N$ ($\mathbb{N}_0 = \mathbb{N} \cup \{0\}$) there exist constants C, d_1 and d_2 satisfying the following estimates:

$$\text{(i) } |D_\xi^\beta m(\lambda, \xi)| \leq C_{\beta,\varepsilon,\lambda_0}(|\lambda|^{\frac{1}{2}} + |\xi|)^2 |\xi|^{-|\beta|},$$

$$\text{(ii) } d_1(|\lambda|^{\frac{1}{2}} + |\xi|)^2 \leq ||\xi|^2 + p(\lambda)| \leq d_2(|\lambda|^{\frac{1}{2}} + |\xi|)^2$$

for any $(\lambda, \xi') \in \Lambda_{\varepsilon,\lambda_0} \times (\mathbb{R}^{N-1} \setminus \{0\})$.
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Outline of proof in $\mathbb{R}^N_+$

The resolvent problem in $\mathbb{R}^N_+$:

$$\begin{cases} \lambda\rho + \gamma\,\mathrm{div}\,u = f & \text{in } \mathbb{R}^N_+, \\ \lambda u - \alpha\Delta u - \beta\nabla\,\mathrm{div}\,u + \gamma\nabla\rho = g & \text{in } \mathbb{R}^N_+, \\ S(u,\rho)\nu|_{\tan} = h|_{\tan} & \text{on } \mathbb{R}^N_0, \\ u_N = 0 & \text{on } \mathbb{R}^N_0, \end{cases}$$

$$\Longleftrightarrow \begin{cases} \lambda u - \alpha\Delta u - \eta_\lambda\nabla\,\mathrm{div}\,u = f & \text{in } \mathbb{R}^N_+, \\ \alpha(D_N u_j + D_j u_N) = -h_j & \text{on } \mathbb{R}^N_0, (j = 1, \ldots, N-1) \\ u_N = 0 & \text{on } \mathbb{R}^N_0, \end{cases}$$

where $\eta_\lambda = \beta + \gamma^2\lambda^{-1}$.

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where $\eta_\lambda = \beta + \gamma^2\lambda^{-1}$.

Let $F = (f_1^e, \ldots, f_{N-1}^e, f_N^o)$ be extension of f, where

$$f_j^e(x) = \begin{cases} f_j(x) \ (x_N > 0), \\ f_j(x', -x_N) \ (x_N < 0). \end{cases} \qquad f_N^o(x) = \begin{cases} f_N(x) \ (x_N > 0), \\ -f_N(x', -x_N) \ (x_N < 0). \end{cases}$$

By the result in $\mathbb{R}^N$, there exists U satisfying the following equations:

$$\lambda U - \alpha \Delta U - \eta_\lambda \nabla \operatorname{div} U = F \quad \text{in } \mathbb{R}^N_+, \qquad U_N = 0 \quad \text{on } \mathbb{R}^N_0,$$

We set $u = U + v$, then v satisfies the equations:

$$\begin{cases} \lambda v - \alpha \Delta v - \eta_\lambda \nabla \operatorname{div} v = 0 & \text{in } \mathbb{R}^N_+, \\ \alpha(D_N v_j + D_j v_N) = -l_j & \text{on } \mathbb{R}^N_0 \ (j = 1, \ldots, N-1), \\ v_N = 0 & \text{on } \mathbb{R}^N_0, \end{cases}$$

where $l_j = h_j + \alpha(D_N U_j + D_j U_N)$.

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where $l_j = h_j + \alpha(D_N U_j + D_j U_N)$.

$$
\begin{aligned}
v_j(x) = &-\sum_{k=1}^{N-1}\int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[\frac{i\xi_j\eta_\lambda}{K}\frac{A}{|\xi'|}|\xi'|\mathcal{M}(x_N+y_N)\mathcal{F}[D_k l_k](\xi',y_N)\right](x')dy_N \\
&+\sum_{k=1}^{N-1}\int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[\frac{i\xi_j\eta_\lambda}{K}\frac{i\xi_k}{|\xi'|}|\xi'|\mathcal{M}(x_N+y_N)\mathcal{F}[D_N l_k](\xi',y_N)\right](x')dy_N \\
&-\sum_{k=1}^{N-1}\int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[\frac{i\xi_j\eta_\lambda}{KB}\frac{i\xi_k}{|\xi'|}|\xi'|e^{-B(x_N+y_N)}\mathcal{F}[D_N l_k](\xi',y_N)\right](x')dy_N \\
&+\frac{1}{\alpha^2}\int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[\frac{\lambda|\lambda|^{-1}}{B^2}|\lambda|^{\frac{1}{2}}e^{-B(x_N+y_N)}\mathcal{F}[|\lambda|^{\frac{1}{2}}l_j](\xi',y_N)\right](x')dy_N \\
&+\cdots,
\end{aligned}
$$

where

$A = \sqrt{|\xi'|^2+(\alpha+\eta_\lambda)^{-1}\lambda}$, $B = \sqrt{|\xi'|^2+\alpha^{-1}\lambda}$, $(\xi' = (\xi_1,\ldots,\xi_{N-1}))$

$\mathcal{M}(x_N) = \dfrac{e^{-Ax_N}-e^{-Bx_N}}{A-B}$, $K = \alpha(\alpha+\eta_\lambda)A(A+B)$.

Lemma(key of proof of $\mathcal{R}$-boundedness)

Let $0 < \varepsilon < \pi/2,\ 1 < q < \infty$ and $\lambda_0 \geq 0$. Let k_i be functions such that for any multi-index $\beta' \in \mathbb{N}_0^{N-1}$ there exist constants C_i such that

$$|D_{\xi'}^{\beta'} k_1(\lambda, \xi')| \leq C_1(|\lambda|^{\frac{1}{2}} + |\xi'|)^{-|\beta'|},\ |D_{\xi'}^{\beta'} k_2(\lambda, \xi')| \leq C_2|\xi'|^{-|\beta'|},$$
$$|D_{\xi'}^{\beta'} k_3(\lambda, \xi')| \leq C_3(|\lambda|^{\frac{1}{2}} + |\xi'|)|\xi'|^{-|\beta'|}$$

for any $(\lambda, \xi') \in \Lambda_{\varepsilon,\lambda_0} \times (\mathbb{R}^{n-1} \setminus \{0\})$. Let $K_i(\lambda)$ $(i = 1, 2, 3)$ be operator defined by

$$[K_1(\lambda)h](x) = \int_0^\infty \mathcal{F}_{\xi'}^{-1}[k_1(\lambda, \xi')|\lambda|^{\frac{1}{2}} e^{-B(x_N+y_N)} \hat{h}(\xi', y_N)](x')dy_N,$$
$$[K_2(\lambda)h](x) = \int_0^\infty \mathcal{F}_{\xi'}^{-1}[k_2(\lambda, \xi')|\xi'| e^{-B(x_N+y_N)} \hat{h}(\xi', y_N)](x')dy_N,$$
$$[K_3(\lambda)h](x) = \int_0^\infty \mathcal{F}_{\xi'}^{-1}[k_3(\lambda, \xi')|\xi'| \mathcal{M}(x_N + y_N) \hat{h}(\xi', y_N)](x')dy_N.$$

Then, $K_i(\lambda)$is $\mathcal{R}$-bounded on $\mathcal{L}(L_q(\mathbb{R}_+^N))$.

Main theorem

Let $0 < \varepsilon < \pi/2,\ 1 < q < \infty$. Then, there exist a $\lambda_0 > 0$ depending on ε, q, N and an operator $R(\lambda) \in \mathcal{L}(W_q^{1,0}(\mathbb{R}_+^N) \times L_q(\mathbb{R}_+^N)^{N^2-1}, W_q^{1,2}(\mathbb{R}_+^N))$ such that the following two assertions hold :

(i) For any $(f, g) \in W_q^{1,0}(\mathbb{R}_+^N),\ h \in W_q^1(\mathbb{R}_+^N)^{N-1}$ and $\lambda \in \Lambda_{\varepsilon,\lambda_0}$, $(\rho, u) = R(\lambda)(f, g, \nabla h, |\lambda|^{\frac{1}{2}} h) \in W_q^{1,2}(\mathbb{R}_+^N)$ solves the equations (RP) uniquely.

(ii) There exist $\gamma_0 > 0$ such that

$$\mathcal{R}_{\mathcal{L}(W_q^{1,0}(\mathbb{R}_+^N)\times L_q(\mathbb{R}_+^N)^{N^2-1},\ W_q^{1,0}(\mathbb{R}_+^N))}(\{\lambda R(\lambda) \mid \lambda \in \Lambda_{\varepsilon,\lambda_0}\}) \le \gamma_0,$$

$$\mathcal{R}_{\mathcal{L}(W_q^{1,0}(\mathbb{R}_+^N)\times L_q(\mathbb{R}_+^N)^{N^2-1},\ L_q(\mathbb{R}_+^N)^{N^2})}(\{|\lambda|^{\frac{1}{2}} \nabla P_v R(\lambda) \mid \lambda \in \Lambda_{\varepsilon,\lambda_0}\}) \le \gamma_0,$$

$$\mathcal{R}_{\mathcal{L}(W_q^{1,0}(\mathbb{R}_+^N)\times L_q(\mathbb{R}_+^N)^{N^2-1},\ L_q(\mathbb{R}_+^N)^{N^3})}(\{\nabla^2 P_v R(\lambda) \mid \lambda \in \Lambda_{\varepsilon,\lambda_0}\}) \le \gamma_0,$$

Future work

- next step:case of general domain

  We can prove the same assertion as half space case by means of a perturbation method.

Theorem (operator-valued Fourier multiplier theorem)

Let X and Y be two UMD Banach spaces and $1 < p < \infty$. Let M be a function in $C^1(\mathbb{R} \setminus \{0\}, \mathcal{L}(X,Y))$ such that

$$\mathcal{R}_{\mathcal{L}(X,Y)}(\{M(\tau) \mid \tau \in \mathbb{R} \setminus \{0\}\}) = \kappa_0 < \infty,$$
$$\mathcal{R}_{\mathcal{L}(X,Y)}(\{\tau M'(\tau) \mid \tau \in \mathbb{R} \setminus \{0\}\}) = \kappa_1 < \infty.$$

If we define the operator $T_M : \mathcal{F}^{-1}\mathcal{D}(\mathbb{R}, X) \to \mathcal{S}'(\mathbb{R}, Y)$ by the formula:

$$T_M\phi = \mathcal{F}^{-1}[M\mathcal{F}[\phi]], \ (\mathcal{F}[\phi] \in \mathcal{D}(\mathbb{R}, X)).$$

Then, the operator T_M is extended to a bounded linear operator from $L_p(\mathbb{R}, X)$ into $L_p(\mathbb{R}, Y)$. Moreover, denoting this extension by T_M, we have

$$\|T_M\|_{\mathcal{L}(L_P(\mathbb{R},X),L_P(\mathbb{R},Y))} \leq C(\kappa_0 + \kappa_1)$$

for some constant $C > 0$ depending on p, X and Y.