Uniqueness of steady Navier-Stokes flows in exterior domains

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Ω: Exterior domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$

$$
\text{(NSt)} \begin{cases} -\Delta u + u \cdot \nabla u + \nabla p = \text{div } F & \text{in } \Omega, \\ \text{div } u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u(x) \to 0 & \text{as } |x| \to \infty. \end{cases}
$$

$u = (u_1, u_2, u_3)$: unknown velocity vector

p: unknown pressure

div F: given external force

Kozono-Yamazaki (1998)

(1) $F \in L_{3/2,\infty}(\Omega)$, $\|F\|_{L_{3/2,\infty}(\Omega)} \le \exists\delta$

$\Rightarrow \exists$ **solution** $\{u,p\} \in \dot{H}^1_{3/2,\infty}(\Omega) \times L_{3/2,\infty}(\Omega)$ **of (NSt)**

(2) $\{u,p\}, \{v,q\} \in \dot{H}^1_{3/2,\infty}(\Omega) \times L_{3/2,\infty}(\Omega)$: **solutions of (NSt)**

$\|u\|_{L_{3,\infty}(\Omega)}, \|v\|_{L_{3,\infty}(\Omega)} \le \exists\widetilde{\delta} \Rightarrow \{u,p\} = \{v,q\}$

$(\dot{H}^1_{3/2,\infty}(\Omega) \subset L_{3,\infty}(\Omega))$

Question

Either u **or** v **is not small in** $L_{3,\infty}(\Omega) \Rightarrow$ **Uniqueness?**

Function spaces: $1 < p < \infty$ **and** $1 < q \leq \infty$

- $L_{p,q}(\Omega)$: **Lorentz space with norm** $\|\cdot\|_{p,q}$
  $(L_{p,q}(\Omega) = (L_{p_0}(\Omega), L_{p_1}(\Omega))_{\theta,q} \qquad (1/p = (1-\theta)/p_0 + \theta/p_1))$

- $\dot{H}^1_p(\Omega)$: **the completion of** $C_0^\infty(\Omega)$ **in the norm** $\|\nabla \cdot\|_p$

- $\dot{H}^1_{p,q}(\Omega) := (\dot{H}^1_{p_0}(\Omega), \dot{H}^1_{p_1}(\Omega))_{\theta,q}$ **with norm** $\|\nabla \cdot\|_{p,q}$
  $(0 < \theta < 1,\ 1/p = (1-\theta)/p_0 + \theta/p_1)$

- $\dot{H}^{-1}_{p,q}(\Omega) := \dot{H}^1_{p',q'}(\Omega)^*$

L_p theory

- Nonlinearity of (NSt) $\Rightarrow$ Seek a solution $u \in \dot{H}^1_{3/2}(\Omega)$

- $u \in \dot{H}^1_{3/2}(\Omega)$ is a solution

$$\Rightarrow \int_{\partial\Omega} (T[u,p] + F) \cdot \nu \, dS = 0$$

$$(T[u,p] = (\partial_i u_j + \partial_j u_i - \delta_{ij} p)_{i,j=1}^3)$$

- Best decay rates: $|u(x)| = O(|x|^{-1})$, $|\nabla u(x)| = O(|x|^{-2})$

Known results

- Galdi (1994), Miyakawa (1995), Kozono-Yamazaki (1999)
  (in the Leray class $\dot{H}^1_2(\Omega)$)

- Taniuchi (2009) (restricted to the stationary problem)
  $u, v \in L_{3,\infty}(\Omega)$: solutions of (NSt)
  $\|u\|_{3,\infty} \leq \exists\delta$ and $u, v \in L_{6,2}(\Omega) \Rightarrow u = v$

- N. (2012)
  $\{u, p\}, \{v, q\} \in \dot{H}^1_{3/2,\infty}(\Omega) \times L_{3/2,\infty}(\Omega)$: solutions of (NSt)
  $\|u\|_{3,\infty} \leq \exists\widetilde{\delta}$ and $u, v \in L_r(\Omega)$ for some $r > 3$
  $\Rightarrow \{u, p\} = \{v, q\}$

Main result

Theorem. *Suppose that* $\{u,p\},\{v,q\} \in \dot{H}^1_{3/2,\infty}(\Omega) \times L_{3/2,\infty}(\Omega)$ *are solutions of (NSt). For every* $r > 3$ *there exists a constant* $\delta = \delta(r) > 0$ *such that if*

$$v \in L_r(\Omega)$$

and

$$\|u\|_{3,\infty} \le \delta,$$

then $\{u,p\} = \{v,q\}$*.*

For the proof, we consider

$$(W)\begin{cases} -\Delta w + w\cdot\nabla u + v\cdot\nabla w + \nabla\pi = 0 & \text{in } \Omega, \\ \text{div } w = 0 & \text{in } \Omega, \end{cases}$$

$(w := u - v, \pi := p - q)$ and its dual equation

$$(D)\begin{cases} -\Delta\psi - \sum_{i=1}^{3} u_i \nabla\psi_i - v\cdot\nabla\psi + \nabla\chi = f & \text{in } \Omega, \\ \text{div } \psi = 0 & \text{in } \Omega. \end{cases}$$

- The relation

$$\text{Solvability of } (D) \longleftrightarrow \text{Uniqueness of } (W)$$

Take $w \in \dot{H}^1_{p,q}(\Omega)$ as a test function in

$$(\nabla\psi, \nabla\varphi) - (\sum_{i=1}^{3} u_i \nabla\psi_i, \varphi) - (v \cdot \nabla\psi, \varphi) - (\chi, \text{div } \varphi) = (f, \varphi)$$

and $\psi \in \dot{H}^1_{p',q'}(\Omega)$ in

$$(\nabla w, \nabla\widetilde{\varphi}) - (\sum_{i=1}^{3} u_i \nabla\widetilde{\varphi}_i, w) - (v \cdot \nabla\widetilde{\varphi}, w) - (\pi, \text{div } \widetilde{\varphi}) = 0$$

$\Rightarrow \quad (f, w) = 0 \quad$ for $\forall f \in \dot{H}^{-1}_{p',q'}(\Omega)$

- Difficulty
  $C_0^\infty(\Omega)$ is not dense in $\dot{H}^1_{3/2,\infty}(\Omega)$

Outline of the proof

1. Establish the regularity theory for the perturbed Stokes equations

$$(W') \qquad -\Delta w + w\cdot\nabla u + \nabla\pi = f, \qquad \text{div } w = 0$$

and

$$(D') \qquad -\Delta\psi - \sum_{i=1}^{3} u_i \nabla\psi_i + \nabla\chi = \widetilde{f}, \qquad \text{div } \psi = 0$$

2. (W): Show $w \in \dot{H}^1_{p,q}(\Omega)$ ($C_0^\infty(\Omega)$ is dense in $\dot{H}^1_{p,q}(\Omega)$)

(D): Employ the bootstrap argument for the solution of (D) (obtained within the L_2-framework)
$\Rightarrow$ Deduce the desired regularity $\psi \in \dot{H}^1_{p',q'}(\Omega)$

Lemma 1. *Assume for each* $i = 0, 1$

$$\text{(i) } (p_i, q_i) = (3/2, \infty) \quad \textit{or} \quad \text{(ii) } 3/2 < p_i < 3,\ 1 < q_i \le \infty.$$

$\Rightarrow \forall f \in \dot{H}^{-1}_{p_0,q_0}(\Omega) \cap \dot{H}^{-1}_{p_1,q_1}(\Omega);$ $\exists$*unique solution*

$$\{u, p\} \in (\dot{H}^{1}_{p_0,q_0}(\Omega) \cap \dot{H}^{1}_{p_1,q_1}(\Omega)) \times (L_{p_0,q_0}(\Omega) \cap L_{p_1,q_1}(\Omega))$$

of the Stokes equation

$$\begin{cases} -\Delta u + \nabla p = f & \textit{in } \Omega, \\ \textit{div } u = 0 & \textit{in } \Omega, \end{cases}$$

with the estimate

$$\|\nabla u\|_{L_{p_0,q_0} \cap L_{p_1,q_1}} + \|p\|_{L_{p_0,q_0} \cap L_{p_1,q_1}} \le C_{p_0,q_0,p_1,q_1} \|f\|_{\dot{H}^{-1}_{p_0,q_0} \cap \dot{H}^{-1}_{p_1,q_1}}$$

Lemma 2. *Let* $u \in L_{3,\infty}(\Omega)$ *and assume for each* $i = 0, 1$

(i) $(p_i, q_i) = (3/2, \infty)$ *or* (ii) $3/2 < p_i < 3,\ 1 < q_i \leq \infty.$

Suppose $\{w, \pi\} \in \dot{H}^1_{p_0,q_0}(\Omega) \times L_{p_0,q_0}(\Omega)$ *is a solution of*

$$(W') \qquad -\Delta w + w \cdot \nabla u + \nabla \pi = f, \quad \text{div } w = 0$$

for $f \in \dot{H}^{-1}_{p_0,q_0}(\Omega) \cap \dot{H}^{-1}_{p_1,q_1}(\Omega)$ *and*

$$\|u\|_{3,\infty} \leq \exists \delta = \delta(p_0, q_0, p_1, q_1)$$

$\Rightarrow w \in \dot{H}^1_{p_0,q_0}(\Omega) \cap \dot{H}^1_{p_1,q_1}(\Omega)$ *and* $\pi \in L_{p_0,q_0}(\Omega) \cap L_{p_1,q_1}(\Omega)$

- **The same result holds for a solution of** (D') **if we exclude the condition** (i) $(p_i, q_i) = (3/2, \infty)$.