L_p-theory for a class of viscoelastic fluids with a free surface

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Equation of motion

- Motion of a fluid is described by

$$\rho(\partial_t u + u.\nabla u) = \operatorname{div} S \qquad \text{(conservation of momentum)}$$
$$\operatorname{div} u = 0 \qquad \text{(conservation of mass)}$$

(ρ: constant density, u: velocity field, S: stress tensor)

- Newtonian fluid
  - $S(u,\pi) = 2\alpha Eu - \pi I, \quad \alpha > 0, \quad Eu = \frac{1}{2}(\nabla u + \nabla u^T)$
    (π: pressure)
  - $\curvearrowright$ Navier-Stokes equations

$$\rho(\partial_t u + u.\nabla u) = \alpha\Delta u - \nabla\pi, \quad \operatorname{div} u = 0$$

  - Examples: water, air, ...

Examples of non-Newtonian fluids

- Generalized Newtonian fluids:
  - $S(u,\pi) = 2\alpha(|Eu|^2)Eu - \pi I$,
    $\alpha\colon [0,\infty) \to [0,\infty)$, $\quad |Eu|^2 = \sum_{j,k=1}^{n} (Eu)_{j,k}^2$
  - $\curvearrowright$ generalized Navier-Stokes equations

$$\rho(\partial_t u + u.\nabla u) = \operatorname{div} 2\alpha(|Eu|^2)Eu - \nabla\pi, \quad \operatorname{div} u = 0$$

  - Example: ketchup, toothpaste, . . .
- Generalized viscoelastic fluids:
  - $S(u,\pi,\tau) = 2\alpha(|Eu|^2)Eu - \pi I + \mu(\tau), \alpha\colon [0,\infty) \to [0,\infty), \quad \mu\colon \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$
    τ is given by

$$\partial_t\tau + u.\nabla\tau = g(\nabla u,\tau), \quad g\colon \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$$

  - $\curvearrowright$

$$\rho(\partial_t u + u.\nabla u) = \operatorname{div} 2\alpha(|Eu|^2)Eu - \nabla\pi + \operatorname{div}\mu(\tau), \quad \operatorname{div} u = 0,$$
$$\partial_t\tau + u.\nabla\tau = g(\nabla u,\tau)$$

  - Examples: paints, blood . . .

One phase-flow without surface tension

$S(u,\pi,\tau) = 2\alpha(|Eu|^2)Eu - \pi I + \mu(\tau), \quad \partial_t\tau + u.\nabla\tau = g(\nabla u,\tau)$

- $\Omega(t)$ is a unknown domain with outer normal $\nu(t)$
- $V(t)$ is the normal velocity of $\partial\Omega(t)$

Considered problem:

$$\left.\begin{array}{rcll} \rho(\partial_t u + u.\nabla u) & = & \operatorname{div} S(u,\pi,\tau) & (0,T)\times\Omega(t) \\ \operatorname{div} u & = & 0 & (0,T)\times\Omega(t) \\ \partial_t\tau + u.\nabla\tau & = & g(\nabla u,\tau) & (0,T)\times\Omega(t) \\ S(u,\pi,\tau)\nu & = & 0 & (0,T)\times\partial\Omega(t) \\ V & = & u\cdot\nu & (0,T)\times\partial\Omega(t) \\ u(0) & = & u_0 & \Omega(0) \\ \tau(0) & = & \tau_0 & \Omega(0) \\ \Omega(0) & = & \Omega_0 & \end{array}\right\} \quad (1)$$

Aim:

- Find unique solution $\Omega(t)$ and $(u,\pi,\tau)(t)$ of (1)

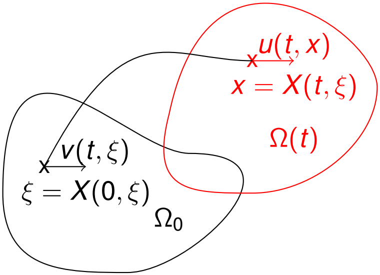
Lagrangian coordinates

- $x \in \Omega(t)$: Eulerian coordinate
  $\xi \in \Omega_0$: Lagrangian coordinate
- Transformation between x and ξ:

$$x = X(t,\xi) = \xi + \int_0^t v(s,\xi)ds,$$

  where $v(t,\xi) = u(t, X(t,\xi))$
- Diffeomorphism $X(t,\cdot)\colon \Omega_0 \to \Omega(t)$
- Some properties
  - $(\partial_t u + u.\nabla u)(t, X(t,\xi)) = \partial_t v(t,\xi)$
  - $\exists B \in C^\infty(\mathbb{R}^{n\times n}, \mathbb{R}^{n\times n})$ with $B(0) = 0$ and

$$(\nabla u)(t, X(t,\xi)) = \nabla v(t,\xi) + B\Big(\int_0^t \nabla v(s,\xi)ds\Big)\nabla v(t,\xi)$$



Formulation in Lagrangian coordinates

$S(v, \theta, \eta) = 2\alpha(|Ev|^2)Ev - \theta I + \mu(\eta)$

- Transformed unknowns: $(v, \theta, \eta)(t, \xi) = (u, \pi, \tau)(t, X(t, \xi))$
- Transformed system:

$$\left.\begin{array}{rcll} \rho\partial_t v &=& \operatorname{div} S(v,\theta,\eta) + F(v,\theta,\eta) & (0,T)\times\Omega_0 \\ \operatorname{div} v &=& F_d(v) & (0,T)\times\Omega_0 \\ \partial_t \eta &=& g(\nabla v, \eta) + G(v,\eta) & (0,T)\times\Omega_0 \\ S(v,\theta,\eta)\nu_0 &=& H(v,\theta,\eta) & (0,T)\times\partial\Omega_0 \\ v(0) &=& u_0 & \Omega_0 \\ \eta(0) &=& \tau_0 & \Omega_0 \end{array}\right\} \quad (2)$$

- Advantages of the Lagrangian formulation:
  - Problem is on the time independent domain Ω_0
  - The transport term $u.\nabla\tau$ vanishes due to

$$(\partial_t \tau + u.\nabla\tau)(t, X(t,\xi)) = \partial_t \eta(t, \xi)$$

  - The boundary of $\Omega(t)$ is given via

$$\partial\Omega(t) = \{X(t,\xi) \colon \xi \in \partial\Omega_0\}$$

Formulation in Lagrangian coordinates

- Nonlinearities:

$$F(v,\theta,\eta) = \operatorname{Div} 2\Big[(\alpha(|\mathcal{E}(v)|^2) - \alpha(|Ev|^2))Ev\Big] + \operatorname{Div}\Big[\alpha(|\mathcal{E}(v)|^2)V_2\Big(\int_0^t \nabla v ds\Big)\nabla v\Big]$$
$$+ \operatorname{Div}\Big[V_3\Big(\int_0^t \nabla v ds\Big)\mu(\eta)\Big] + V_4\Big(\int_0^t \nabla v ds\Big)\nabla\pi$$
$$F_d(v) = -V_5\Big(\int_0^t \nabla v ds\Big)\nabla v = -\operatorname{div}\Big[V_6\Big(\int_0^t \nabla v ds\Big)v\Big]$$
$$G(v,\eta) = g\Big(\nabla v + V_7\Big(\int_0^t \nabla v ds\Big),\eta\Big) - g(\nabla v,\eta)$$
$$H(v,\theta,\eta) = 2\Big[(\alpha(|\mathcal{E}(v)|^2) - \alpha(|Ev|^2))Ev\Big]\nu_0 + \Big[\alpha(|\mathcal{E}(v)|^2)V_2\Big(\int_0^t \nabla v ds\Big)\nabla v\Big]\nu_0$$
$$+ \Big[V_3\Big(\int_0^t \nabla v ds\Big)\mu(\eta)\Big]\nu_0 + V_4\Big(\int_0^t \nabla v ds\Big)\pi\nu_0$$

where

$$\mathcal{E}(v) = Ev + V_1\Big(\int_0^t \nabla v ds\Big)\nabla v \quad \text{with} \quad \mathcal{E}(v)(t,\xi) = (Eu)(t, X(t,\xi))$$

and V_j are smooth functions with $V_j(0) = 0$

Theorem

Let

- $n \in \mathbb{N}$, $n \geq 2$, $n + 2 < p < \infty$
- $\Omega_0 \subset \mathbb{R}^n$ *a domain with a compact boundary*
- α, μ, *and g sufficiently smooth with*

$$\alpha(s) > 0, \quad \alpha(s) + 2s\alpha'(s) > 0, \quad s \geq 0 \quad \textit{und} \quad \mu(0) = g(0,0) = 0$$

- $(u_0, \tau_0) \in W_p^{2-2/p}(\Omega) \times H_p^1(\Omega)$, *satisfy the compatibility conditions*

$$\operatorname{div} u_0 = 0 \quad \textit{and} \quad [2\alpha(|Eu_0|^2)Eu_0\nu_0 + \mu(\tau_0)\nu_0]_{tan} = 0$$

Then

- *there exists* $T > 0$ *and a unique solution of* (2) *in*

$$\begin{aligned}
v &\in H_p^1(0, T; L_p(\Omega_0)) \cap L_p(0, T; H_p^2(\Omega_0)) \\
\eta &\in W_\infty^1(0, T; L_p(\Omega_0)) \cap L_\infty(0, T; H_p^1(\Omega_0)) \\
\theta &\in L_p(0, T; \widehat{H}_p^1(\Omega_0)) \\
\theta|_{\Gamma_0} &\in W_p^{1/2-1/(2p)}(0, T; L_p(\Gamma_0)) \cap L_p(0, T; W_p^{1-1/p}(\Gamma_0))
\end{aligned}$$

Literature

- Viscoelastic Fluid on a time independent domains
    - GUILLOPÉ, SAUT (1990), FERNÁNDEZ-CARA, GUILLÉN, ORTEGA (1998), LIONS, MASMOUDI (2000), VOROTNIKOV, ZVYAGION (2004), GEISSERT, GÖTZ, N. (2012), ...
- Newtonian fluid with a free surface
    - Eulerian coordinates and Hanzawa transformation
        - BEALE (1984), PRÜSS, SIMONETT (2009-2011) DENK, GEISSERT, HIEBER, SAAL, SAWADA (2011), ...
    - Lagrangian coordinates
        - SOLONNIKOV (1977-2003), BEALE (1981), TANI, TANAKA (1995), TANI (1996), SHIBATA, SHIMIZU (2007, 2011), ...
- Generalized Newtonian fluid with a free surface
    - PLOTNIKOV (1993): two-phase flow with surface tension, weak-theory
    - ABELS (2007): two-phase flow with and without surface tension, weak-theory
- Viscoelastic fluid with a free surface
    - Lagrangian coordinates
        - LE MEUR (2010): Oldroyd-B fluid with surface tension, L_2-theory

Sketch of the proof

- Introduce a quasilinear operator $\mathcal{A}$ (generalized Stokes operator) and coressponding Neumann boundary operator $\mathcal{B}$ via

$$\mathcal{A}(v_*)v = \Sigma_{l,m}\mathcal{A}^{l,m}(Ev_*)\partial_l\partial_m v$$
$$\mathcal{B}(v_*)(v,\theta) = \Sigma_{l,m}\mathcal{A}^{l,m}(Ev_*)\nu_{0,l}\partial_m v - \theta\nu_0$$

with

$$\mathcal{A}(v)v = 2\,\mathrm{Div}\,\alpha(|Ev|^2)Ev$$
$$\mathcal{B}(v)(v,\theta) = 2\alpha(|Ev|^2)Ev\nu_0 + 4\alpha'(|Ev|^2)|Ev|^2 Ev\nu_0 - \theta\nu_0$$

- Rewrite (2) equivalently:

$$\left.\begin{array}{rcll} \rho\partial_t v + \mathcal{A}(v)v + \nabla\theta &=& \mathrm{Div}\,\mu(\tau) + F(v,\theta,\eta) & (0,T)\times\Omega_0 \\ \mathrm{div}\,v &=& F_d(v) & (0,T)\times\Omega_0 \\ \partial_t\eta &=& g(\nabla v,\eta) + G(v,\eta) & (0,T)\times\Omega_0 \\ \mathcal{B}(v)(v,\theta) &=& H(v,\theta,\eta) + H_1(v) - \mu(\tau)\nu_0 & (0,T)\times\partial\Omega_0 \\ v(0) &=& u_0 & \Omega_0 \\ \eta(0) &=& \tau_0 & \Omega_0 \end{array}\right\} \quad (3)$$

where

$$H_1(v) = 4\alpha'(|Ev|^2)|Ev|^2 Ev\nu_0$$

Sketch of the proof

- Introduce (v_*, θ_*, η_*) with $(v_*, \eta_*)(0) = (u_0, \tau_0)$
- Set $(w, \psi, \zeta) = (v, \theta, \eta) - (v_*, \theta_*, \eta_*)$
- Rewrite (3) equivalently as fixed point problem of the map $\Phi(\bar{w}, \bar{\psi}, \bar{\zeta}) = (w, \psi, \zeta)$, where (w, ψ, ζ) is solution of

$$\left.\begin{array}{rcll} \rho\partial_t w + \mathcal{A}(v_*)w + \nabla\psi & = & \bar{F}(\bar{w}, \bar{\psi}, \bar{\zeta}) & (0,T)\times\Omega_0 \\ \operatorname{div} w & = & \bar{F}_d(\bar{w}) & (0,T)\times\Omega_0 \\ \partial_t\zeta & = & \bar{G}(\bar{w}, \bar{\zeta}) & (0,T)\times\Omega_0 \\ \mathcal{B}(v_*)(w,\zeta) & = & \bar{H}(\bar{w}, \bar{\psi}, \bar{\zeta}) & (0,T)\times\partial\Omega_0 \\ w(0) & = & 0 & \Omega_0 \\ \zeta(0) & = & 0 & \Omega_0 \end{array}\right\} \quad (4)$$

  Φ is well defined:
  - Problem is not coupled
    - Maximal regularity of generalized Stokes eq. (BOTHE, PRÜSS ('07))
    - $\zeta(t) = \int_0^t \bar{G}(\bar{w}, \bar{\zeta})(s)ds$
  - Mapping properties of $\bar{F}$, $\bar{F}_d$, $\bar{G}$, $\bar{H}$
- Unique fixed point by the contraction mapping principle
  - Key: $(\bar{F}, \bar{F}_d, \bar{G}, \bar{H})$ and $D(\bar{F}, \bar{F}_d, \bar{G}, \bar{H})$ are small, provided T is small

Sketch of the proof

$$
\begin{aligned}
{}_0\mathbb{E} &:= {}_0\mathbb{E}_u \times {}_0\mathbb{E}_\pi \times {}_0\mathbb{E}_\tau \\
{}_0\mathbb{E}_u &:= {}_0H^1_p(0,T;L_p(\Omega_0)) \cap L_p(0,T;H^2_p(\Omega_0)) \\
{}_0\mathbb{E}_\pi &:= \{\psi \in L_p(0,T;\widehat{H}^1_p(\Omega_0))\colon \psi|_{\partial\Omega_0} \in {}_0W_p^{\frac{1}{2}-\frac{1}{2p}}(0,T;L_p(\partial\Omega_0)) \cap L_p(0,T;W_p^{1-\frac{1}{p}}(\partial\Omega_0))\} \\
{}_0\mathbb{E}_\tau &:= H^1_r(0,T;L_p(\Omega)) \cap L_\infty(0,T;H^1_p(\Omega)) \\
{}_0\mathbb{F} &:= \mathbb{F}_f \times {}_0\mathbb{F}_d \times \mathbb{G} \times {}_0\mathbb{H} \\
\mathbb{F}_f &:= L_p(0,T;L_p(\Omega_0)) \\
{}_0\mathbb{F}_d &:= {}_0H^1_p(0,T;\widehat{H}^{-1}_p(\Omega)) \cap L_p(0,T;H^1_p(\Omega)) \\
{}_0\mathbb{G} &:= L_r(0,T;L_p(\Omega_0)) \cap L_1(0,T;H^1_p(\Omega)), \quad r > p \\
{}_0\mathbb{H} &= {}_0W_p^{\frac{1}{2}-\frac{1}{2p}}(0,T;L_p(\partial\Omega_0)) \cap L_p(0,T;W_p^{1-\frac{1}{p}}(\partial\Omega_0))
\end{aligned}
$$

Lemma

For $R_0, T_0 > 0$, there exists $\mathfrak{R}\colon R_+ \to \mathbb{R}_+$ and $C > 0$ such that for all $0 < R < R_0$, $0 < T < T_0$, $(w,\psi,\zeta) \in B_{{}_0\mathbb{E}}(0,R)$ with

$$\|N(z)\|_{{}_0\mathbb{F}} \le CR^2 + \mathfrak{R}(T) \quad \text{and} \quad \|DN(z)\|_{\mathcal{L}({}_0\mathbb{E},{}_0\mathbb{F})} \le CR + \mathfrak{R}(T)$$

THANK YOU FOR YOUR ATTENTION