Analysis of characteristics finite difference schemes for convection-diffusion problems

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Contents

- Background.
- Definition of a discrete L^2-norm for FD functions.
- A discrete L^2-estimate for a composite function.
- An application of the discrete L^2-estimate to a first order characteristics finite difference scheme.
- A second order characteristics FD scheme.
- Stability and convergence results.
- An advantage of characteristics FD schemes (Numerical integration).
- Conclusion.

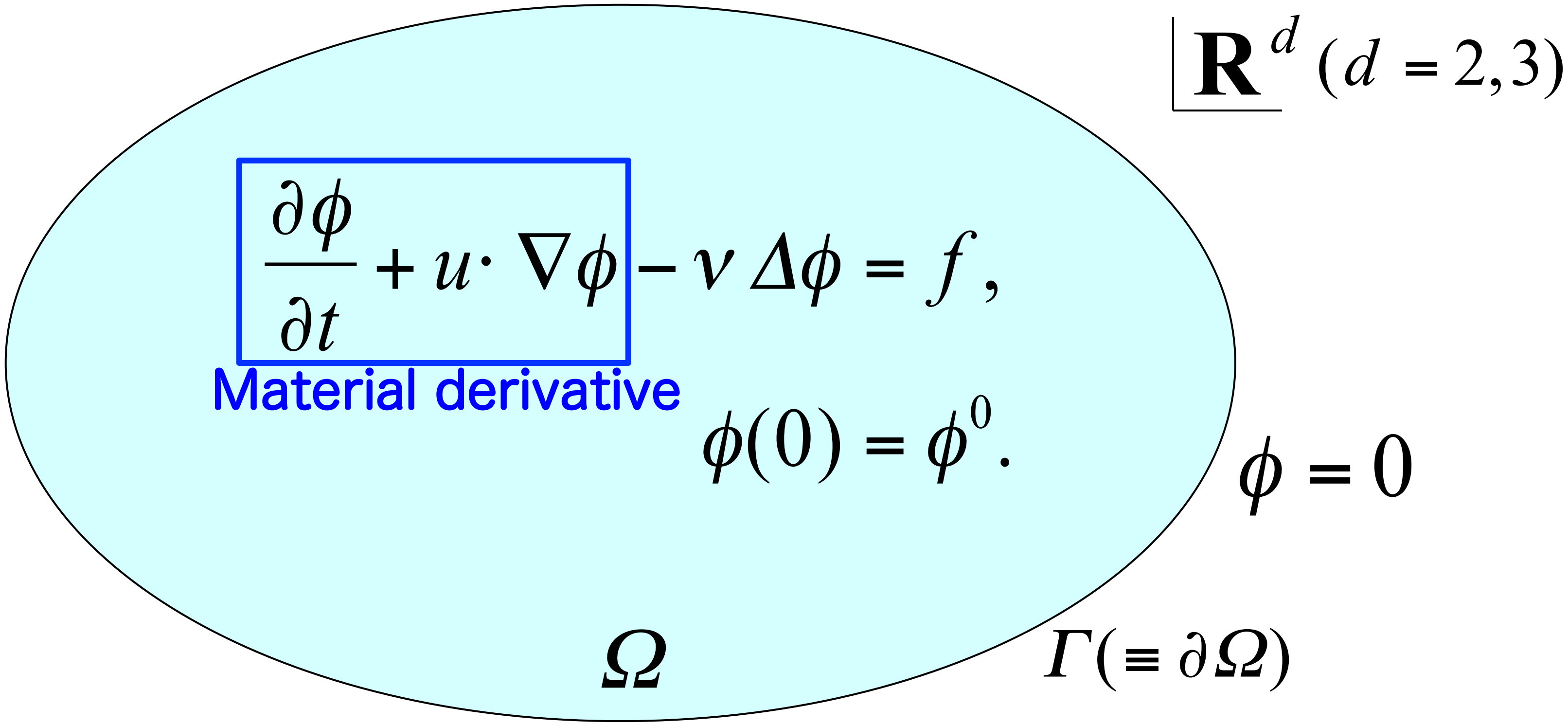
Convection-diffusion problems

Find $\phi : \Omega \times (0,T) \to \mathbf{R}$ s.t.

$$\boxed{\frac{\partial \phi}{\partial t} + u \cdot \nabla \phi} - \nu \Delta \phi = f,$$

Material derivative

$$\phi(0) = \phi^0.$$



$u : \Omega \times (0,T) \to \mathbf{R}^d$, $f : \Omega \times (0,T) \to \mathbf{R}$, and $\phi^0 : \Omega \to \mathbf{R}$, are given.

$u|_\Gamma = 0.$

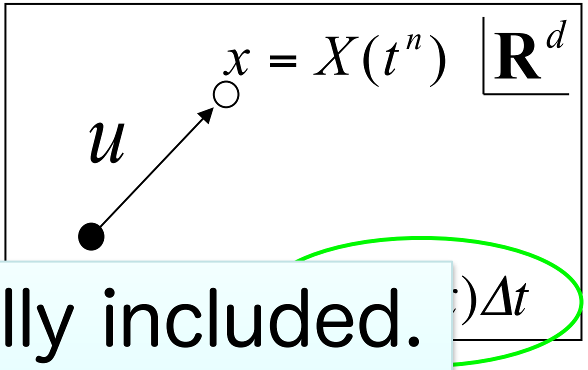
A basic idea of characteristics schemes

$\Omega \subset \mathbf{R}^d \ (d = 2, 3)$. $u : \Omega \times (0,T) \to \mathbf{R}^d$: given, $\phi : \Omega \times (0,T) \to \mathbf{R}$: unknown, $t^n \equiv n\Delta t$.

Material derivative : $\dfrac{D\phi}{Dt} \equiv \left(\dfrac{\partial}{\partial t} + u \cdot \nabla \right) \phi$

is discretized as follows.

Let $X(\cdot\,; x) : (0,T) \to \mathbf{R}^d$ be the sol. of the ODE;



1. Upwind technique is naturally included.

2. The coefficient matrix is symmetric.

The idea can be combined with FDM, FEM, FVM, and so on.

$(x)\Delta t.$

$\approx \dfrac{\quad}{\Delta t} \equiv \dfrac{\varphi^n(x) - \varphi^{n-1} \circ X_1^n(x)}{\Delta t}$

$A\vec{x} = \vec{b}$

non-symmetric part goes to RHS vector

Background

Based on the maximum principle

Numerical integration is required

Method	Analysis: L^∞	Analysis: L^2
Characteristics FEM	Pironneau and Tabata, 2010	○
Characteristics FDM	○	x → ○

- For FDM the L^∞-analysis based on the maximum principle is usually employed.

- Can we give *a discrete L^2-estimate* for a characteristics FD scheme ?

- Combining the method of characteristics with FDM, we do not need integration.

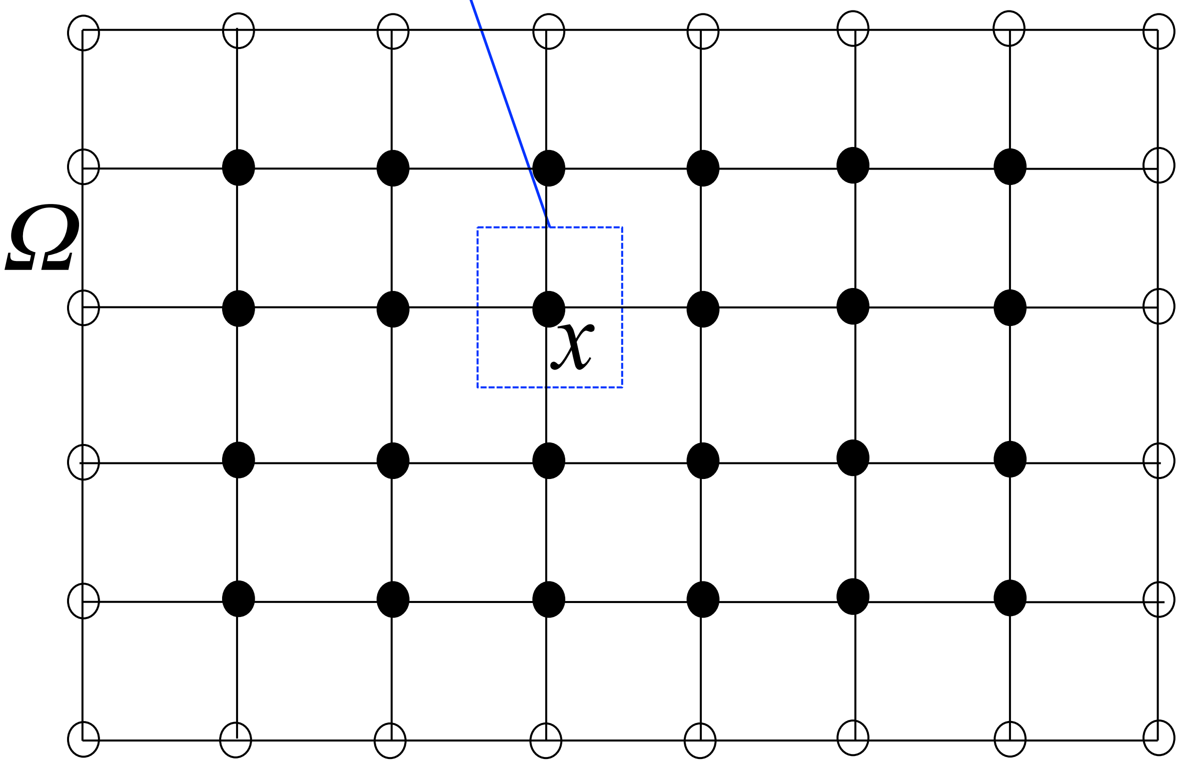
Definition of discrete L^2-norm for FD functions

For $\phi_h, \psi_h \in V_h$,

$$(\phi_h, \psi_h)_{\bar{\Omega}_h} \approx \int_\Omega \phi_h \psi_h dx.$$

$$(\phi_h, \psi_h)_{\bar{\Omega}_h} \equiv h^2 \sum_{x \in \bar{\Omega}_h} \phi_h(x) \psi_h(x)$$: discrete L^2-inner product.

$$\|\phi_h\|_{l^2(\bar{\Omega}_h)} \equiv (\phi_h, \phi_h)_{\bar{\Omega}_h}^{1/2}$$: discrete L^2-norm.



$$V_h \equiv \{ v_h : \bar{\Omega}_h \to \mathbf{R} \}.$$

- ● Ω_h
- ○ Γ_h

$$\bar{\Omega}_h \equiv \Omega_h \cup \Gamma_h$$

Proposition.

$w \in C^1(\bar{\Omega})^2,\ w|_\Gamma = 0.\ \ 0 < \delta < 1/\|w\|_{W^{1,\infty}(\Omega)}.$

$\exists C_1 > 0$ s.t. $W_0\delta \le C_1 h.$

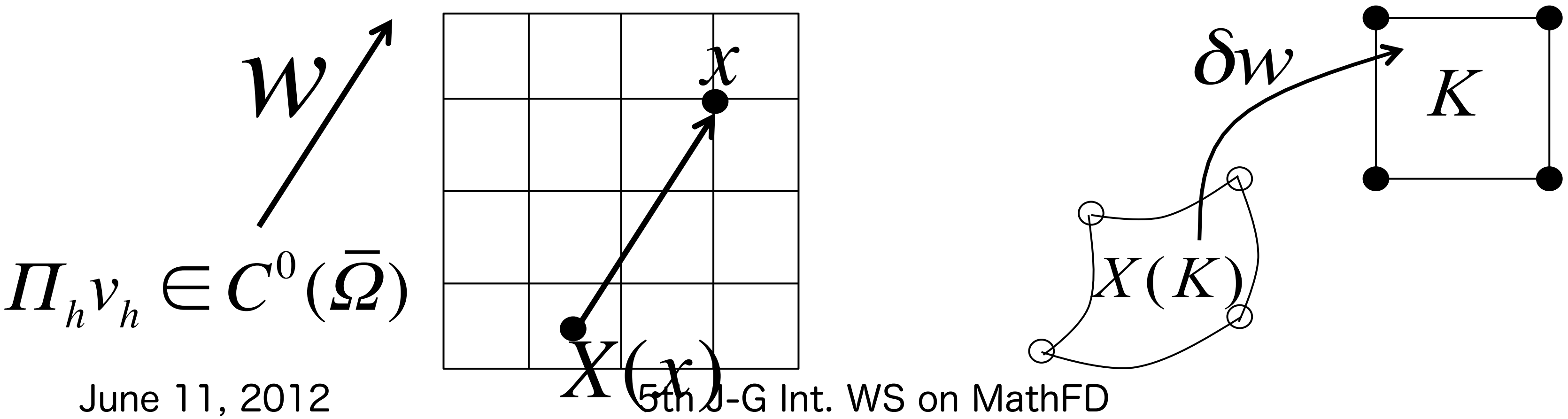
$X(x) \equiv x - \delta w(x).$

$\Rightarrow$

$\exists c_1(w) > 0$ s.t. $\|(\Pi_h v_h)\circ X\|_{l^2(\bar{\Omega}_h)} \le (1 + c_1\delta)\|v_h\|_{l^2(\bar{\Omega}_h)}\ \ (\forall v_h \in V_h)$

$\Pi_h : V_h \to C^0(\bar{\Omega})$, the bilinear interpolation operator,

$W_0 \equiv \max\{|w_i(x)|;\ x \in \bar{\Omega},\ i = 1, 2\}.$



A first order characteristics FD scheme

$$\frac{\partial \phi}{\partial t} + u \cdot \nabla \phi = \frac{d}{dt}\phi(X(t),t)$$

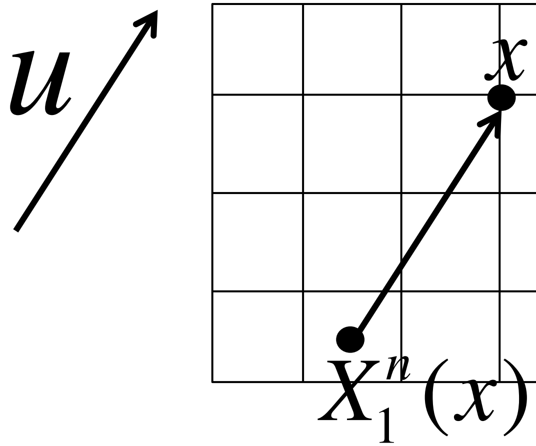
Find $\left\{\phi_h^n\right\}_{n=1}^{N_T} \subset V_h$ s.t.

$$\frac{\phi_h^n - (\Pi_h \phi_h^{n-1}) \circ X_1^n}{\Delta t}(x) - \nu \Delta_h \phi_h^n(x) = f^n(x), \quad \forall x \in \Omega_h,$$

$$\approx \frac{\partial \phi}{\partial t} + u \cdot \nabla \phi$$

$$\phi_h^n(x) = 0, \forall x \in \Gamma_h,$$

$$\phi_h^0(x) = \phi^0(x) \; (x \in \bar{\Omega}_h),$$



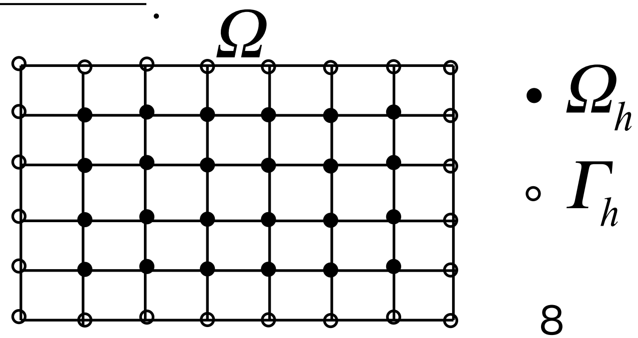
where $X_1^n(x) \equiv x - u^n(x)\Delta t,$

$$x_{i,j} \equiv (ih, \; jh)^T,$$

$$\Delta_h \psi_h(x_{i,j}) \equiv \frac{\psi_h(x_{i+1,j}) + \psi_h(x_{i-1,j}) + \psi_h(x_{i,j+1}) + \psi_h(x_{i,j-1}) - 4\psi_h(x_{i,j})}{h^2}.$$

Symmetric matrix

$$A\vec{x} = \vec{b}$$



Lemma.

Let v_h be FD func. defined on $\bar{\Omega}_h$.

some assumptions including $\Delta t \le Ch$.

$\Rightarrow$

$$\left\|(\Pi_h v_h)\circ X_1^n\right\|_{l^2(\bar{\Omega}_h)} \le (1+c\Delta t)\left\|v_h\right\|_{l^2(\bar{\Omega}_h)}.$$

$$\begin{cases} \delta \leftarrow \Delta t, \\ w \leftarrow u^n \ (X \leftarrow X_1^n). \end{cases}$$

Remark. This yields stability and convergence: $O(h+\Delta t)$ results in a sense of the discrete L^2-norm.

Stability of the characteristics FD scheme

$$\frac{\phi_h^n - (\Pi_h \phi_h^{n-1}) \circ X_1^n}{\Delta t}(x) - \nu \Delta_h \phi_h^n(x) = f^n(x), \forall x \in \Omega_h,$$

Multiplying $h^2 \phi_h^n(x)$, and summing for all $x \in \Omega_h$, we have

$$\left(\frac{\phi_h^n - (\Pi_h \phi_h^{n-1}) \circ X_1^n}{\Delta t}, \phi_h^n \right)_{\bar{\Omega}_h} - \nu \left(\Delta_h \phi_h^n, \phi_h^n \right)_{\bar{\Omega}_h} = \left(f^n, \phi_h^n \right)_{\bar{\Omega}_h}$$

$$(a-b)a = \frac{1}{2}(a^2 - b^2) + \frac{1}{2}(a-b)^2$$

$$\frac{1}{2\Delta t}\left(\|\phi_h^n\|_{l^2(\bar{\Omega}_h)}^2 - \|(\Pi_h \phi_h^{n-1}) \circ X_1^n\|_{l^2(\bar{\Omega}_h)}^2 \right) + \nu \left(\nabla_h \phi_h^n, \nabla_h \phi_h^n \right)_{\bar{\Omega}_h^{(\frac{1}{2},0)} \times \bar{\Omega}_h^{(0,\frac{1}{2})}} \le \frac{1}{2}\left(\|f^n\|_{l^2(\bar{\Omega}_h)}^2 + \|\phi_h^n\|_{l^2(\bar{\Omega}_h)}^2 \right)$$

By $\|(\Pi_h v_h) \circ X_1^n\|_{l^2(\bar{\Omega}_h)} \le (1 + c\Delta t)\|v_h\|_{l^2(\bar{\Omega}_h)} \quad (\forall v_h \in V_h)$,

$$\frac{1}{2\Delta t}\left(\|\phi_h^n\|_{l^2(\bar{\Omega}_h)}^2 - \|\phi_h^{n-1}\|_{l^2(\bar{\Omega}_h)}^2 \right) + \nu \|\nabla_h \phi_h^n\|_{h^1(\bar{\Omega}_h)}^2 \le c\left(\|\phi_h^n\|_{l^2(\bar{\Omega}_h)}^2 + \|\phi_h^{n-1}\|_{l^2(\bar{\Omega}_h)}^2 \right) + \frac{1}{2}\|f^n\|_{l^2(\bar{\Omega}_h)}^2$$

By discrete Gronwall's lemma,

$$\max_{n=0,\cdots,N_T} \|\phi_h^n\|_{l^2(\bar{\Omega}_h)} + \sqrt{\nu}\left\{ \Delta t \sum_{n=1}^{N_T} \|\nabla_h \phi_h^n\|_{h^1(\bar{\Omega}_h)}^2 \right\}^{1/2} \le c\left[\|\phi_h^0\|_{l^2(\bar{\Omega}_h)} + \left\{ \Delta t \sum_{n=1}^{N_T} \|f^n\|_{l^2(\bar{\Omega}_h)}^2 \right\}^{1/2} \right]$$

$$\equiv \|f\|_{l^2(l^2)}$$

A second order characteristics FD scheme

N-Rui-Tabata, submitted

Find $\left\{\phi_h^n\right\}_{n=1}^{N_T} \subset V_{h0}$ s.t.

$$\frac{\phi_h^n - (\Pi_h \phi_h^{n-1}) \circ X_2^n}{\Delta t}(x) - \frac{\nu}{2}\left\{\Delta_h \phi_h^n + \tilde{\Delta}_h^{(n)} \phi_h^{n-1}\right\}(x)$$

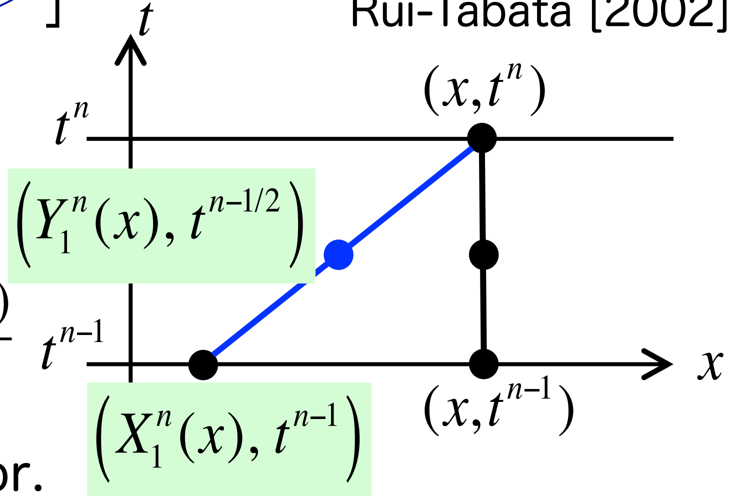
$$- \frac{\nu \Delta t}{2}\left[\sum_{i=1}^{2}\left\{(D^i u_i^n)\Delta_{h,i}\right\} + (D^2 u_1^n + D^1 u_2^n)\nabla_{(2h)1}\nabla_{(2h)2}\right]\phi_h^{n-1}(x)$$

$$= \frac{1}{2}\left\{f^n + f^{n-1} \circ X_1^n\right\}(x), \quad \forall x \in \Omega_h,$$

$$\phi_h^0(x) = \phi^0(x) \; (x \in \bar{\Omega}_h),$$

$\approx \Delta \phi^{n-1} \circ X_1^n(x)$

A second order FEM: Rui-Tabata [2002]



$$Y_1^n(x) \equiv \frac{x + X_1^n(x)}{2}$$

$D^i \equiv \partial / \partial x_i$.

$\Pi_h : V_h \to C^0(\bar{\Omega})$, the bilinear interpolation operator.

Key: $\Delta\phi \circ X_1^n(x) = \nabla\left(\nabla\phi \circ X_1^n\right)(x) + \Delta t\left\{\sum_{i=1}^{2} D^i u_i D^{ii}\phi + \sum_{i,j=1,\, i\neq j}^{2} D^i u_j D^{ij}\phi\right\}(x) + O(\Delta t^2).$

Stability and convergence results

Theorem1 (stability).

$$\Delta t \le C_1(u)h$$

$$\Rightarrow$$

$$\|\phi_h\|_{l^\infty(l^2)} + \sqrt{\nu\Delta t}\,|\phi_h|_{l^\infty(h^1)} + \sqrt{\nu}\,|\phi_h|_{l^2(h^{1'})} \le c(u,f,\phi^0).$$

Theorem2 (error estimate).

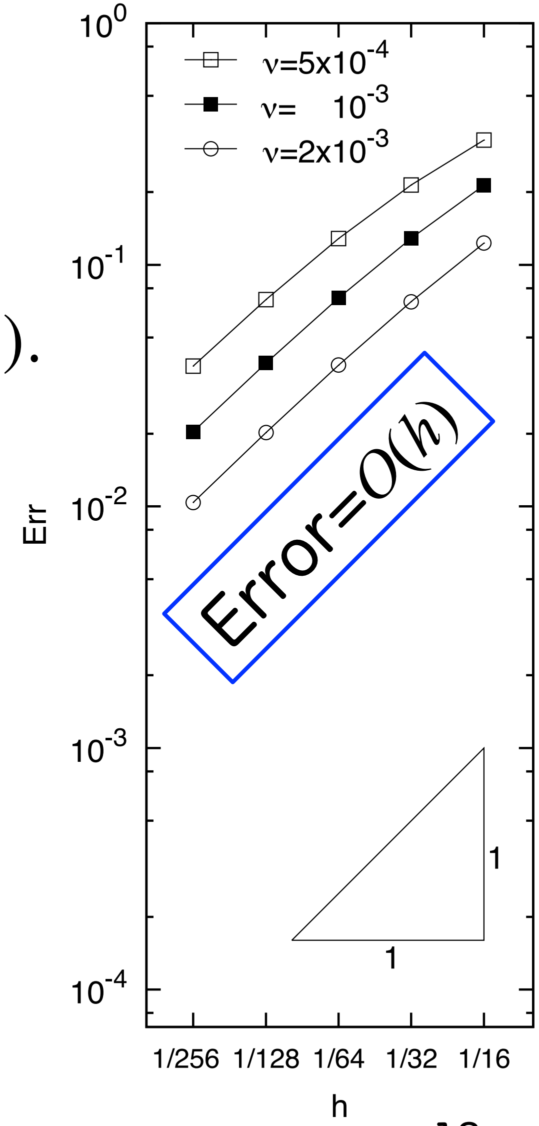
$$\Delta t \le C_1(u)h$$

$$\Rightarrow$$

$$\|\phi_h - \phi\|_{l^\infty(l^2)} + \sqrt{\nu\Delta t}\,|\phi_h - \phi|_{l^\infty(h^1)} + \sqrt{\nu}\,|\phi_h - \phi|_{l^2(h^{1'})}$$

$$\le c'(u,\phi)(\Delta t^2 + h)$$

$$= c''(u,\phi)h.$$



A characteristics FD scheme of second order in both time and space

Find $\{\phi_h^n\}_{n=1}^{N_T} \subset V_{h0}$ s.t.

$$\frac{\phi_h^n - (\Pi_h^{(2)}\phi_h^{n-1})\circ X_2^n}{\Delta t}(x) - \frac{\nu}{2}\left\{\Delta_h \phi_h^n + \tilde{\Delta}_h^{(n),(2)}\phi_h^{n-1}\right\}(x)$$

$$-\frac{\nu \Delta t}{2}\left[\sum_{i=1}^{2}\left\{(D^i u_i^n)\Delta_{h,i}\right\} + (D^2 u_1^n + D^1 u_2^n)\nabla_{(2h)1}\nabla_{(2h)2}\right]\phi_h^{n-1}(x)$$

$$= \frac{1}{2}\left\{f^n + f^{n-1}\circ X_1^n\right\}(x), \ \forall x \in \Omega_h,$$

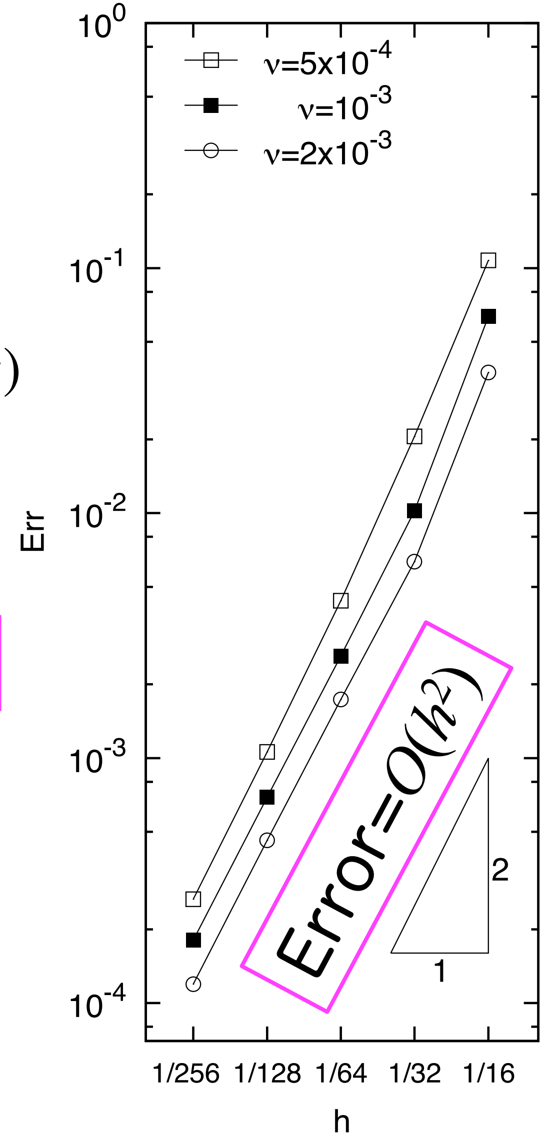
$$\phi_h^0(x) = \phi^0(x)\ (x \in \bar{\Omega}_h),$$

Consistency: $O(\Delta t^2 + h^2)$

$\Pi_h^{(2)} : V_h \to C^0(\bar{\Omega})$, the Q2 (polynomial of degree two) interpolation operator.

$$\left\{\tilde{\Delta}_h^{(n),(2)}\phi\right\}(x) = \nabla_{h1}\left[\left\{\Pi_h^{(1/2,0),(2)}(\nabla_{h1}\phi)\right\}\circ X_1^n\right](x)$$

$$+\nabla_{h2}\left[\left\{\Pi_h^{(0,1/2),(2)}(\nabla_{h2}\phi)\right\}\circ X_1^n\right](x).$$



Lemma: $f:[-1,1]\to\mathbf{R}.$

(1) $\frac{1}{2}\{f(1)+f(-1)\}-f(0)=\frac{1}{2}\int_0^1 ds_1\int_{-s_1}^{s_1} f''(s_2)\,ds_2 \quad (f\in C^2[-1,1]).$

(2) $\frac{1}{2}\{f(1)-f(-1)\}-f'(0)=\frac{1}{2}\int_0^1 ds_1\int_0^{s_1} ds_2\int_{-s_2}^{s_2} f'''(s_3)\,ds_3 \quad (f\in C^3[-1,1]).$

(3) $\{f(1)-2f(0)+f(-1)\}-f''(0)$

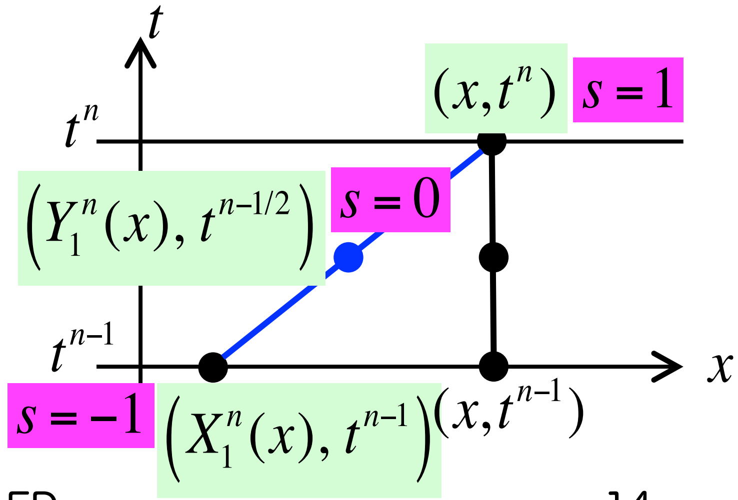
$$=\int_0^1 ds_1\int_0^{s_1} ds_2\int_0^{s_2} ds_3\int_{-s_3}^{s_3} f^{(4)}(s_4)\,ds_4 \quad (f\in C^4[-1,1]).$$

$$f(s)=F\left(x+(s-1)u^n(x)\frac{\Delta t}{2},\ t^{n-1/2}+s\frac{\Delta t}{2}\right)$$

$\Rightarrow$

$$f(1)=F(x,t^n),\ f(-1)=F\left(x-u^n(x)\Delta t,\ t^{n-1}\right),$$

$$f(0)=F\left(x-u^n(x)\frac{\Delta t}{2},\ t^{n-1/2}\right).$$



FEM: $\left\| \phi_h \circ X_1^n \right\|_{L^2(\Omega)} \le (1 + c\Delta t) \left\| \phi_h \right\|_{L^2(\Omega)}$,

where $X_1^n(x) = x - u^n(x)\Delta t$.

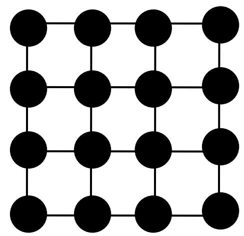
Exact integration is assumed.

Lemma (CFDM): Let ϕ_h be FD func. defined on $\bar{\Omega}_h$.

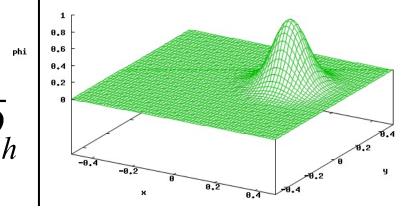
some assumptions including $\Delta t \le Ch$.

$\Rightarrow$

$$\left\| (\Pi_h \phi_h) \circ X_1^n \right\|_{l^2(\bar{\Omega}_h)} \le (1 + c\Delta t) \left\| \phi_h \right\|_{l^2(\bar{\Omega}_h)}.$$



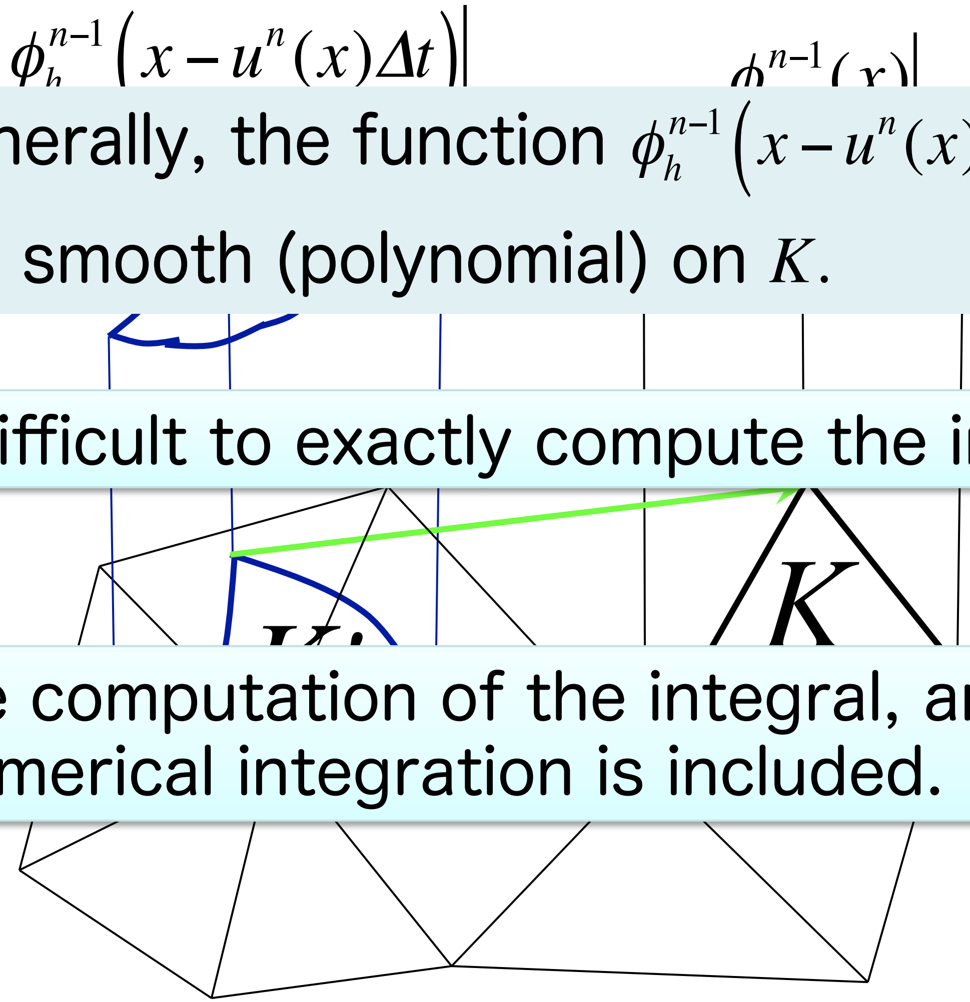
$\bar{\Omega}_h$



No integration.

Integral of a composite function

$$\left(\phi_h^{n-1}\circ X_1^n, \psi_h\right) = \sum_K \int_K \phi_h^{n-1}\left(x - u^n(x)\Delta t\right)\psi_h(x)\,dx$$



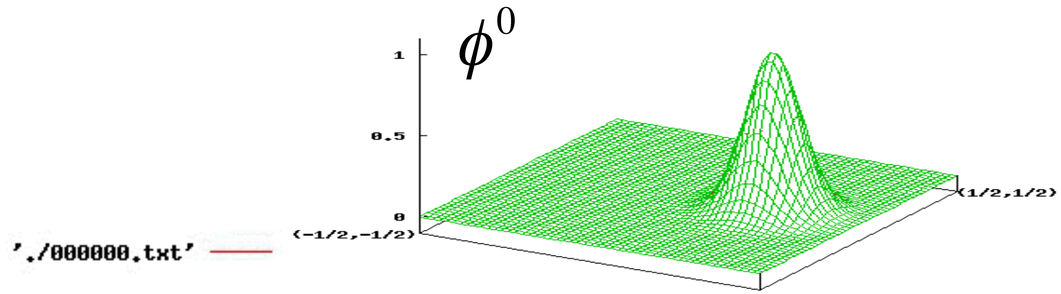
Generally, the function $\phi_h^{n-1}\left(x - u^n(x)\Delta t\right)$ is not smooth (polynomial) on K.

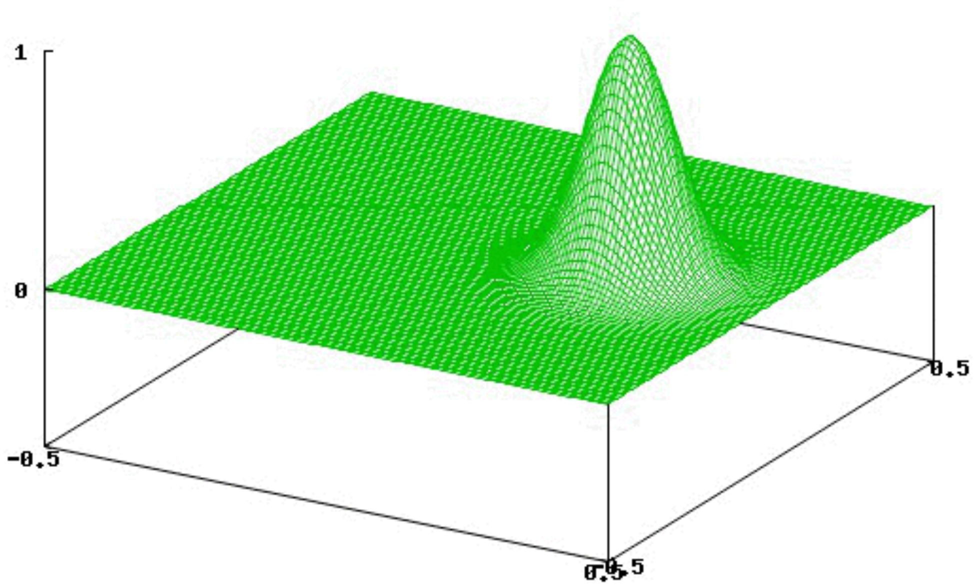
⇨ It is difficult to exactly compute the integral.

⇨ In the computation of the integral, an error of numerical integration is included.

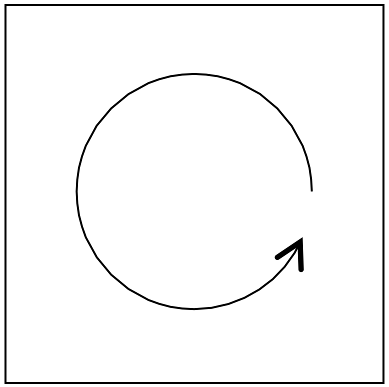
Rotating hill problem (Pe=2,000)

Exact solution:



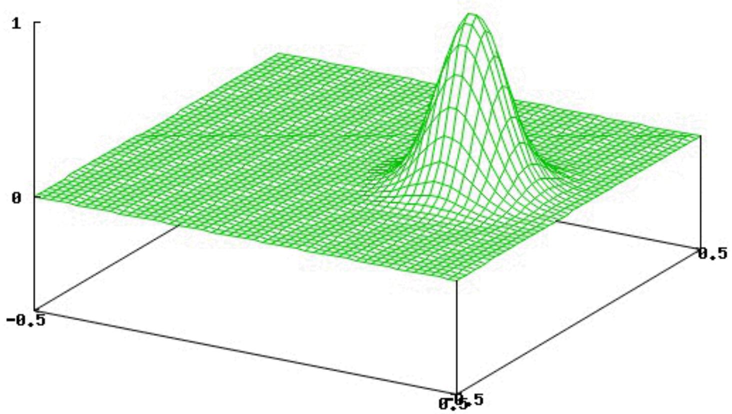


$$f = 0, \quad u(x,t) = (-x_2, x_1)^T.$$

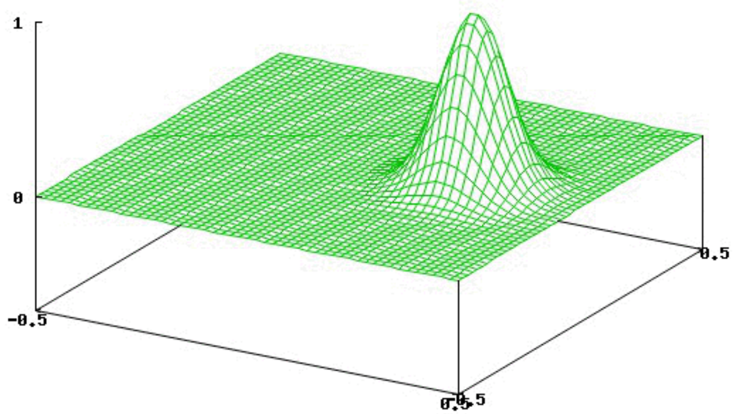


Characteristics FEM (Numerical Integration: deg.=2)

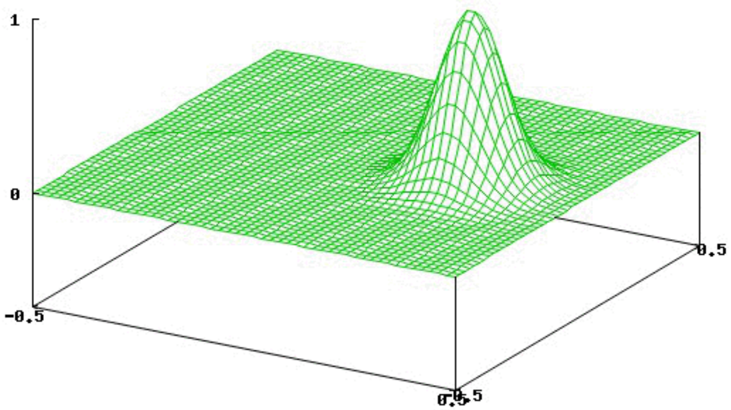
$\Delta t = 1/32$

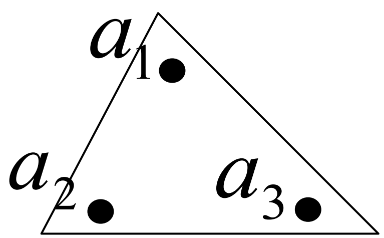


$\Delta t = 1/64$



$\Delta t = 1/48$

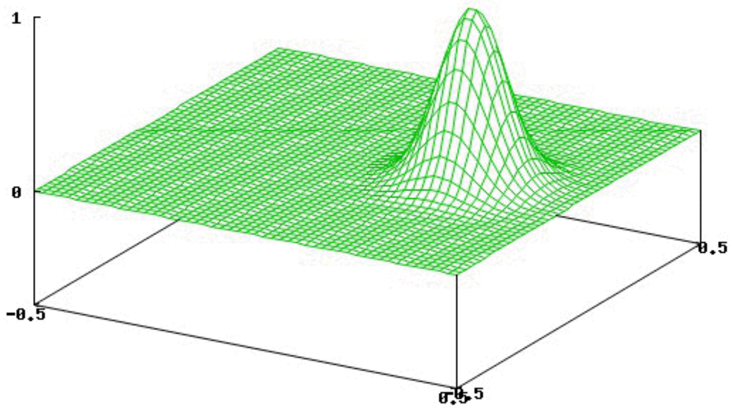




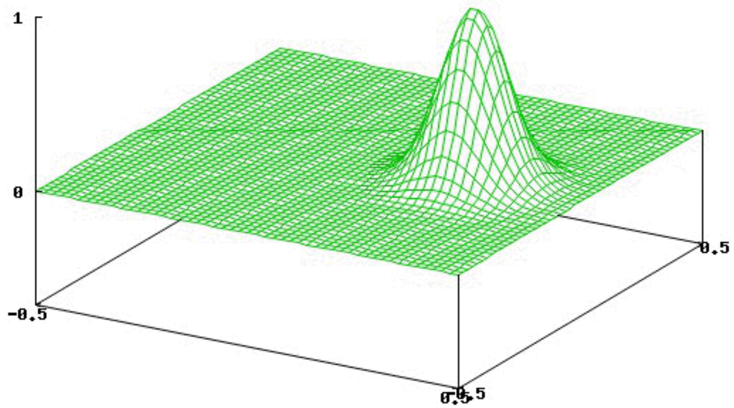
$h = 1/128$

Characteristics FEM (Numerical Integration: deg.=1)

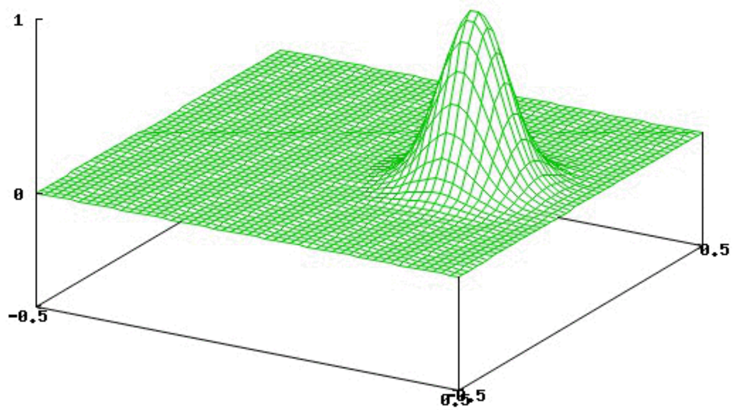
$\Delta t = 1/32$

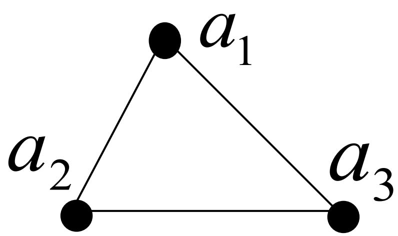


$\Delta t = 1/64$



$\Delta t = 1/48$

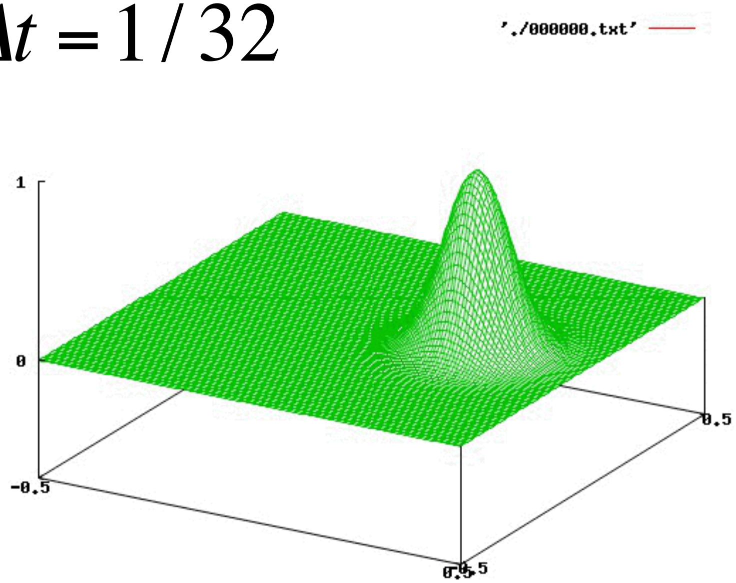




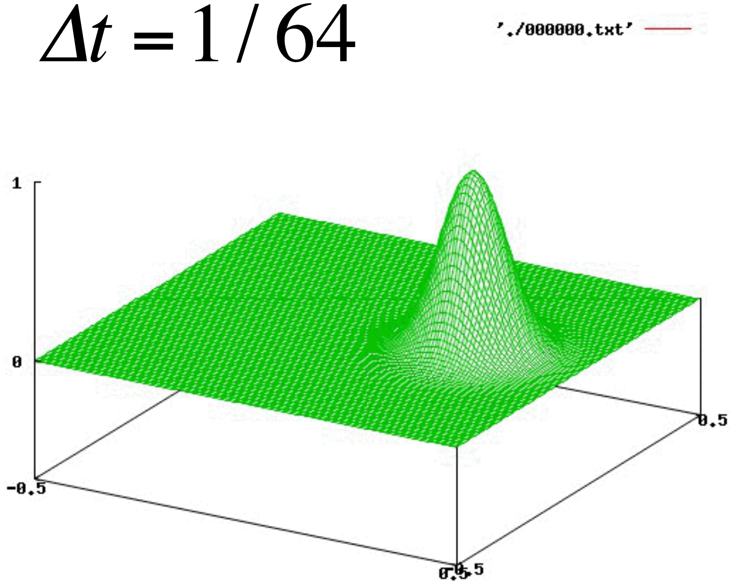
$h = 1/128$

Characteristics FDM

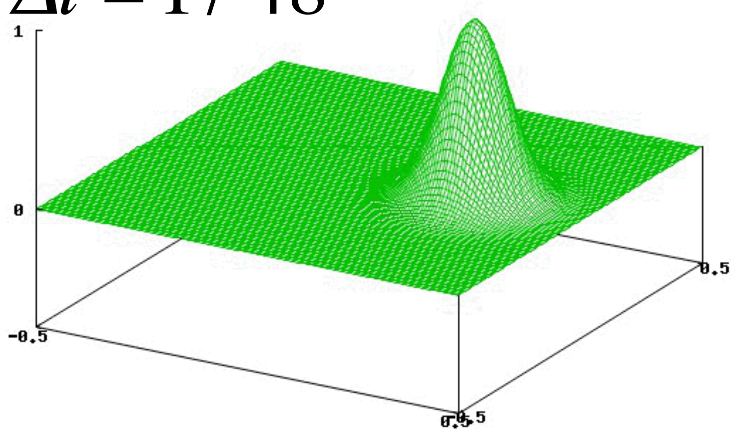
$\Delta t = 1/32$



$\Delta t = 1/64$



$\Delta t = 1/48$



$h = 1/128$

Conclusions

- A characteristics FD scheme of second order in time for convection-diffusion problems.
- We have established a discrete L^2-theory.
- We have applied it to the second order characteristics FD scheme.
- Stability and error estimate.
- In the case of FDM, there is no numerical integration.

For a preprint on the talk, please visit my webpage: http://scheme.hn/

Thank you for your attention.

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