Initial profile for the slow decay of the Navier-Stokes flow in the half-space

Takahiro OKABE

Hirosaki University

11–15 June, 2012
Waseda universtiy

Navier-Stokes equations in $\mathbb{R}^n_+$ $(x_n > 0)$

Let $n \geq 2$.

$$
\text{(N-S)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + u \cdot \nabla u + \nabla p = 0 & \text{in } \mathbb{R}^n_+ \times (0, \infty) \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^n_+ \times (0, \infty) \\ u = 0 & \text{on } \partial\mathbb{R}^n_+ \times (0, \infty) \\ u(0) = a & \text{in } \mathbb{R}^n_+, \end{cases}
$$

$u(x, t) = (u^1(x, t), \ldots, u^n(x, t))$: unknown velocity vector
$p(x, t)$: unknown pressure
$a(x) = (a^1(x), \ldots, a^n(x))$: given initial data

$$
u \cdot \nabla := \sum_{j=1}^{n} u^j \frac{\partial}{\partial x_j} = u^1 \frac{\partial}{\partial x_1} + \cdots + u^n \frac{\partial}{\partial x_n}
$$

Leray's problem: energy decay (1934)

$$\|u(t)\|_{L^2} := \left(\int_{\mathbb{R}^n} |u(x,t)|^2 \, dx \right)^{1/2} \to 0 \ ? \qquad t \to \infty$$

- Masuda first proved this problem $\lim_{t\to\infty} \|u(t)\|_{L^2} = 0$ (1984), provided

  $$\text{(SE)} \qquad \|u(t)\|_{L^2}^2 + 2\int_s^t \|\nabla u(\tau)\|_{L^2}^2 \, d\tau \le \|u(s)\|_{L^2}^2$$

  for a.e. $s > 0$, $s = 0$, for all $t \ge s \ge 0$.
- Fujigaki - Miyakawa (2001, 2002) showed decay rate $\|u(t)\|_{L^2} = O(t^{-\frac{n+2}{4}})$ as $t \to \infty$ in $\mathbb{R}^n$, $\mathbb{R}^n_+$.

Aim of this talk

Behavior of nonlinear Duhamel term

Kajikiya-Miyakawa '86 $\mathbb{R}^n$; Borchers-Miyakawa '88 $\mathbb{R}^n_+$

If $a \in L^2_\sigma(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)$ with $r \approx 1$, then

$$\|u(t) - e^{-tA}a\|_2 = O(t^{-\frac{n+2}{4}}), \quad t \to \infty.$$

Aim

Find initial profile for a so that

$$\|u(t)\|_2 \geq Ct^{-\alpha}, \quad \frac{n}{4} \leq \alpha < \frac{n+2}{4}$$

The linear stokes flow $e^{-tA}a$ is dominant.

Known results; lower bound ($\mathbb{R}^n$)

Schonbek, 1986

$\|u(t)\|_{L(\mathbb{R}^3)} \geq Ct^{-\frac{3}{4}}$, if $a \in L^r(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ with $|\hat{a}(\xi)| \geq C$ near $\xi = 0$

Miyakawa–Schonbek, 2001

$\|u(t)\|_{L(\mathbb{R}^n)} \geq Ct^{-\frac{n}{4}}$, if $\liminf_{r \to 0} \int_{S^{n-1}} |\hat{a}(r, \omega)|^2 \, d\omega > 0$

O., 2009, 2010

$\|u(t)\|_{L^2(\mathbb{R}^n)} \geq Ct^{-\frac{n+2\gamma}{4}}$, $\frac{n}{4} \leq \frac{n+2\gamma}{4} < \frac{n+2}{4}$
if $a \in L^r(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ with $|\hat{a}(\xi)| \geq C|\xi_n|^\gamma$ near $\xi = 0$

Half-space $\mathbb{R}^n_+$: By Ukai's solution formula

(A1) $a(x) = (a^1(x), \dots, a^{n-1}(x), 0)$ $x = (x_1, \dots, x_{n-1}, x_n)$

(A2) $a'(x) = (a^1(x), \dots, a^{n-1}(x)) = a''(x_1, \dots, x_{n-1})\eta(x_n)$

(A3) $\eta(-x_n) = -\eta(x_n)$ with $|\hat{\eta}(\xi_n)| \geq C > 0$ near $\xi_n = 0$

(A4) $|\widehat{a''}(\xi_1, \dots, \xi_{n-1})| \geq C|\xi_{n-1}|^\gamma$ near $(\xi_1, \dots, \xi_{n-1}) = 0$, $\gamma > 0$

Prop. (the Stokes flow in $\mathbb{R}^n_+$)

Let $n \geq 3$. If $a \in L^2_\sigma$ satisfies from (A1) to (A4), then

$$\|e^{-tA}a\|_2 \geq Ct^{-\frac{n+2\gamma}{4}}, \quad t \geq 1.$$

Problem

- $n = 2$? (since $(a^1, a^2) = (0, 0)$)
- Profile of perturbations to n-th component?

Main result

(P1) $a^n(x) = \tilde{a}^n(x')\varphi(x_n)$ with $|\widehat{\tilde{a}^n}(\xi')| \le C|\xi'|^\beta$ near a.e. $\xi' = 0$, and $\varphi \in H^1(\mathbb{R}) \cap L^1(\mathbb{R})$ odd function such that

$$\int_{\mathbb{R}} (1 + |x_n|)|\partial_n \varphi(x_n)| \, dx_n < \infty.$$

(P2) $1 \le \exists i_0 \le n-1$ s.t. $a^{i_0}(x) = \tilde{a}^{i_0}(x')\eta(x_n)$ with $|\widehat{\tilde{a}^{i_0}}(\xi')| \ge C|\xi_{n-1}|^\gamma$ and $|\widehat{\eta}(\xi_n)| \ge C$ near a.e. $\xi = 0$.

Theorem (Navier-Stokes flow)

Let $n \ge 2$. Let $a \in L^2_\sigma \cap L^r$ satisfy (P1) and (P2) for

(i) $1 < r \le 2n/(n+2)$, $0 \le \gamma < 1$,

(ii) $2n/(n+2) < r < 2n/(n+1)$, $0 \le \gamma < 2n/r - n - 1$.

If $\gamma < \beta$ then, $\|u(t)\|_2 \ge Ct^{-\frac{n+2\gamma}{4}}$, $t \gg 1$.

Remark

- (A1),...,(A4) is essential for the lower bound in our method.

$$a(x', x_n) = (a''(x')\eta(x_n), 0), \quad |\widehat{a''}(\xi')| \geq C|\xi'|^\gamma, \quad |\widehat{\eta}(\xi_n)| > C.$$

- Another perturbation (Fujigaki-Miyakawa profile)

$$\int_{\mathbb{R}^n_+} (1 + |y_n|)|a(y)|dy < \infty$$

is available for $n \geq 3$, but not for $n = 2$.

- If $C_1|\xi_{n-1}|^\gamma \leq |\widehat{a}^{i_0}(\xi')| \leq C_2|\xi'|^\gamma$, $C_1 \leq |\widehat{\eta}(\xi_n)| \leq C_2$ near a.e. $\xi = 0$, then

$$C_1 t^{-\frac{n+2\gamma}{4}} \leq \|u(t)\|_2 \leq C_2 t^{-\frac{n+2\gamma}{4}}$$

The proof of our theorem is consist of two part:
the lower bound part + the perturbation part

Stokes flow $e^{-tA}a = v(t) = (v'(t), v^n(t))$ in $\mathbb{R}^n_+$

Ukai's formula 1987

Let $v(t) = (v'(t), v^n(t)) = e^{-tA}a$ be the Stokes flow in $\mathbb{R}^n_+$.

$$v^n(t) = Ue^{-tB}[a^n + S \cdot a'] \quad v'(t) = e^{-tB}[a' - Sa^n] + Sv^n(t).$$

$U \in \mathbf{B}(L^r)$,

$S = (S_1, \ldots, S_{n-1})$:Riesz transform over $\mathbb{R}^{n-1}$ $\sigma(S_j) = \dfrac{-i\xi_j}{|\xi'|}$

$e^{-tB}f = e^{t\Delta}f^*|_{\mathbb{R}^n_+}$, where f^* : odd extension w.r.t. x_n

$$f^*(x', x_n) = \begin{cases} f(x', x_n) & x_n > 0 \\ -f(x', -x_n) & x_n < 0. \end{cases}$$

$x', \xi' \in \mathbb{R}^{n-1}$.

The lower bound part:

(A1) $a(x) = (a^1(x), \dots, a^{n-1}(x), 0)\ x = (x_1, \dots, x_{n-1}, x_n)$

(A2) $a'(x) = (a^1(x), \dots, a^{n-1}(x)) = a''(x_1, \dots, x_{n-1})\eta(x_n)$

(A3) $\eta(-x_n) = -\eta(x_n)$ with $|\hat{\eta}(\xi_n)| \geq C > 0$ near $\xi_n = 0$

(A4) $|\widehat{a''}(\xi_1, \dots, \xi_{n-1})| \geq C|\xi_{n-1}|^\gamma$ near $(\xi_1, \dots, \xi_{n-1}) = 0$

Lemma (the Stokes flow in $\mathbb{R}^n_+$)

Let $n \geq 3$. If $a \in L^2_\sigma$ satisfies from (A1) to (A4), then

$$\|v(t)\|_2 \geq Ct^{-\frac{n+2\gamma}{4}}, \quad t \geq 1.$$

Remark

Conditions (A1),...,(A4) can be generalized as (P2) in the main theorem.

Outline of proof:the lower bound part

$$v^n(t) = Ue^{-tB}[a^n + S\cdot a'] \quad v'(t) = e^{-tB}[a' - Sa^n] + Sv^n(t).$$

$v^n(t) \equiv 0$. Since $a^n = 0$ (A1) and $\mathrm{div}\ a = 0$,

$$\widehat{\mathrm{div}\ a}(\xi', x_n) = \sum_{j=1}^{n-1} \widehat{\partial_j a^j}(\xi', x_n) = \sum_{j=1}^{n-1} i\xi_j \widehat{a^j}(\xi', x_n) = 0$$

$$\widehat{S\cdot a'}(\xi', x_n) = \sum_{j=1}^{n-1} \frac{-i\xi_j}{|\xi'|}\widehat{a^j}(\xi', x_n) = 0$$

$v'(t) = e^{-tB}a' = e^{t\Delta}(a')^*|_{\mathbb{R}^n_+}$

Recall $a'(x', x_n) = a''(x')\eta(x_n)$ (A2), $\eta(-x_n) = -\eta(x_n)$ (A3)

$$(a')^* = a''\eta^* = a''\eta$$

$v'(t) = e^{t\Delta}[a''\eta]$: odd w.r.t x_n.Then we have

$$
\begin{aligned}
\|v(t)\|^2_{L^2(\mathbb{R}^n_+)} &= \|v'(t)\|^2_{L^2(\mathbb{R}^n_+)} = \frac{1}{2}\|v'(t)\|^2_{L^2(\mathbb{R}^n)} \\
&= \frac{1}{2}\int_{\mathbb{R}^n} e^{-2t|\xi|^2}|\widehat{a''}(\xi')\hat{\eta}(\xi_n)|^2\,d\xi \\
&= \frac{1}{2}\Big(\int_{\mathbb{R}^{n-1}} e^{-2t|\xi'|^2}|\widehat{a''}(\xi')|^2 d\xi'\Big)\Big(\int_{-\infty}^{\infty} e^{-2t\xi_n^2}|\hat{\eta}(\xi_n)|^2 d\xi_n\Big) \\
&=: \frac{1}{2}J_1\cdot J_2
\end{aligned}
$$

J_1: Since $|\widehat{a}''(\xi')| \geq C_1|\xi_{n-1}|^\gamma$ near $\xi = 0$ by (A4)

$$J_1 \geq Ct^{-\frac{(n-1)+2\gamma}{2}} \quad t \geq 1.$$

Estimate of J_2

$$|\hat{\eta}(\xi_n)| > C \quad \text{near } \xi_n = 0 \quad (A3).$$

$$\begin{aligned} J_2 &\geq \int_{|\xi_n|\leq\delta} e^{-2t\xi_n^2}|\hat{\eta}(\xi_n)|^2 \, d\xi_n \\ &\geq C^2 \int_{|\xi_n|\leq\delta} e^{-2t\xi_n^2} \, d\xi_n \\ &= \int_0^{\sqrt{t}\delta} e^{-2\rho^2} \frac{d\rho}{\sqrt{t}} \geq Ct^{-\frac{1}{2}} \end{aligned}$$

for $t \geq 1$.
Therefore

$$\|v(t)\|_2^2 \geq CJ_1 \cdot J_2 \geq Ct^{-\frac{(n-1)+2\gamma}{2}} t^{-\frac{1}{2}} = Ct^{-\frac{n+2\gamma}{2}}, \quad t \geq 1$$

Key idea: perturbation part

- In previous result, for $(a^1, \dots, a^{n-1}, 0)$,

$$v^n(t) \equiv 0,$$
$$v'(t) = e^{-tB} a'$$

- Regard $v^n(t)$ as a perturbation, i.e., rapid decay.

However, by Ukai's formula,

$$v^n(t) = Ue^{-tB}[a^n + S \cdot a']$$
$$v'(t) = e^{-tB}[a' - Sa^n] + Sv^n(t).$$

Note that since $\mathrm{div}\ a = 0$

$$\widehat{S \cdot a'}(\xi', x_n) = \frac{-\widehat{\tilde{a}^n}(\xi')\partial_n \varphi(x_n)}{|\xi'|}$$

$v^n(t) = Ue^{-tB}[a^n + S\cdot a'] \quad v'(t) = e^{-tB}[a' - Sa^n] + Sv^n(t).$

Since $a^n(x) = \tilde{a}^n(x')\varphi(x_n)$ and $\varphi \in L^1$: odd

$$\|e^{-tB}a^n\|_{L^2(\mathbb{R}^n_+)} = \frac{1}{2}\|e^{t\Delta}\tilde{a}^n\varphi\|_{L^2(\mathbb{R}^n)} = \frac{1}{2}\|e^{-t|\xi|^2}\widehat{\tilde{a}^n}\widehat{\varphi}\|_{L^2(\mathbb{R}^n)}$$

Since $|\widehat{\tilde{a}^n}(\xi')| \le C|\xi'|^\beta$ near $\xi' = 0$ and $\widehat{\varphi}(0) = 0$,

$\therefore \|e^{-tB}a^n\| \le Ct^{-\frac{n+2\beta}{4}}$. The same way valid for $e^{-tB}Sa^n$.

Since $\widehat{S\cdot a'} = \dfrac{\widehat{\tilde{a}^n}\partial_n\varphi}{|\xi'|}$,

$$\widehat{e^{-tB}S\cdot}\, a'(\xi) = e^{-t|\xi|^2}\frac{\widehat{\tilde{a}^n}(\xi')\widehat{(\partial_n\varphi)^*}(\xi_n)}{|\xi'|}$$

Since $\widehat{(\partial_n\varphi)^*}(0) = 0$ and differentiable at $\xi_n = 0$, i.e.,

$|\widehat{(\partial_n\varphi)^*}(\xi_n)| \le C|\xi_n|$ near $\xi_n = 0$. $\|e^{-tB}S\cdot a'\|_{L^2(\mathbb{R}^n_+)} \le Ct^{-\frac{n+2\beta}{4}}$.