On the $L_p$-$L_q$ maximal regularity of the Neumann-Dirichlet problem for the Stokes equations in an infinite layer

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§0 Contents

1 Problem
   • Stokes equations in an infinite layer

2 Known results

3 Main result
   • $L_p$-$L_q$ maximal regularity for Stokes equations

4 Outline of proof
   • Resolvent Stokes equations
   • Solution formula of Stokes equations
   • Estimates of the solution to Stokes equations
Can you provide further details on the problem described in the image? Specifically, what is the context and the purpose of the mathematical equations shown?
§1 Problem

\(\begin{align*}
\begin{cases}
\partial_t u + u \cdot \nabla u - \mu \Delta u + \nabla \theta = 0, \quad \nabla \cdot u = 0 & \text{in } \Omega(t) \quad t > 0, \\
\partial_t \eta + u' \cdot \nabla' \eta - u_N = 0 & \text{on } \Gamma(t) \quad t > 0, \\
S(u, \theta)n + (c_g \eta - c_\sigma H)n = 0 & \text{on } \Gamma(t) \quad t > 0, \\
u = 0 & \text{on } \Gamma_0, \\
\eta_0 = \eta_0(x'), \quad u_0 = u_0(x). &
\end{cases}
\end{align*}\)

\(\eta = \eta(x', t), \quad x' \in \mathbb{R}^{N-1} : \) the height from the bottom to sea level.
\(c_g > 0: \) the gravity constant, \(c_\sigma > 0: \) a surface tension constant, \(H: \) mean curvature of \(\Gamma(t)\).
\section*{§1 Problem}

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S(u, \theta)n + (c_g \eta - c_\sigma H)n &= 0 \quad \text{on} \quad \Gamma(t) \quad t > 0, \\
\eta_0 &= \eta_0(x'), \quad u_0 = u_0(x).
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\(\eta = \eta(x', t), \ x' \in \mathbb{R}^{N-1}\) : the height from the bottom to sea level.

\(c_g > 0\) : the gravity constant, \(c_\sigma > 0\) : a surface tension constant, \(H\) : mean curvature of \(\Gamma(t)\).
§2 Known results

Results in $L_2$-$L_2$ framework:

- Beale (1980) ⇒ The existence of a unique local time solution in (NS) with $c_\sigma = 0$.
- Beale (1984) ⇒ The existence of a unique global time solution in (NS) with $c_\sigma \neq 0$.
- Beale and Nishida (1985) ⇒ Decay properties for global time solution in (NS) with $c_\sigma \neq 0$.
- Hataya (2009) ⇒ The existence of a unique global time solution in (NS) with $c_\sigma = 0$.

Results in $L_p$-$L_q$ framework:

- Denk, Geissert, Hieber, Saal and Sawada (2011) ⇒ The spin coating process

⇒ The goal of this talk is to show the $L_p$-$L_q$ maximal regularity for (SP) by using resolvent analysis.
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§3 Main result

Theorem ($L_p$-$L_q$ maximal regularity)

Let $1 < p, q < \infty$ and $\gamma_0 > 0$. Then, for any $f \in L_{p,\gamma_0,0}(\mathbb{R}, L_q(\Omega))^N$, $g \in L_{p,\gamma_0,0}(\mathbb{R}, W^1_q(\Omega))^N \cap H_{p,\gamma_0,0}^{1/2}(\mathbb{R}, L_q(\Omega))^N$ (SP) admits a unique solution $(u, \theta)$ such that

$$u \in L_{p,\gamma_0,0}(\mathbb{R}, W^2_q(\Omega))^N \cap W_{p,\gamma_0,0}^1(\mathbb{R}, L_q(\Omega))^N,$$

$$\theta \in L_{p,\gamma_0,0}(\mathbb{R}, W^1_q(\Omega))$$

satisfying with the estimate:

$$\|e^{-\gamma t}(u, \gamma u, \Lambda_{\gamma}^{1/2} \nabla u, \nabla^2 u, \theta, \nabla \theta)\|_{L_p(\mathbb{R}, L_q(\Omega))} \leq C \|e^{-\gamma_0 t}(f, \Lambda_{\gamma}^{1/2} g, \nabla g)\|_{L_p(\mathbb{R}, L_q(\Omega))}$$

for any $\gamma \geq \gamma_0$ with some constant $C$ independent of $\gamma$.

Remark

$$L_{p,\gamma_0,0}(\mathbb{R}, X) = \{ f : \mathbb{R} \rightarrow X \mid \|e^{-\gamma_0 t}f(t)\|_X \in L_p(\mathbb{R}), f(t) = 0 \ (t < 0) \}.$$
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$$u \in L_{p,\gamma_0,0}(\mathbb{R}, W^2_q(\Omega))^N \cap W^{1}_{p,\gamma_0,0}(\mathbb{R}, L_q(\Omega))^N,$$

$$\theta \in L_{p,\gamma_0,0}(\mathbb{R}, W^1_q(\Omega))$$

satisfying with the estimate:

$$\|e^{-\gamma t}(u_t, \gamma u, \Lambda_{\gamma}^\frac{1}{2} \nabla u, \nabla^2 u, \theta, \nabla \theta)\|_{L_{p}(\mathbb{R}, L_q(\Omega))} \leq C \|e^{-\gamma t}(f, \Lambda_{\gamma}^\frac{1}{2} g, \nabla g)\|_{L_{p}(\mathbb{R}, L_q(\Omega))}$$

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$$L_{p,\gamma_0,0}(\mathbb{R}, X) = \left\{ f : \mathbb{R} \to X \mid \|e^{-\gamma_0 t}f(t)\|_X \in L_p(\mathbb{R}), \ f(t) = 0 \ (t < 0) \right\}.$$
§3 Main result

Laplace transform $\mathcal{L}$ and its inverse $\mathcal{L}_\lambda^{-1}$

Let $u(t)$ and $v(\tau)$ be functions defined on $\mathbb{R}$. Then, for $\lambda = \gamma + i\tau$ ($\gamma, \tau \in \mathbb{R}$) Laplace transform and its inverse are defined by

$$\mathcal{L}[u(t)](\lambda) = \int_{-\infty}^{\infty} e^{-\lambda t} u(t) dt, \quad \mathcal{L}_\lambda^{-1}[v(\tau)](t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\lambda t} v(\tau) d\tau.$$ 

We set:

$$(\Lambda_{1/2}^{\gamma} f)(t) = \mathcal{L}_\lambda^{-1}[|\lambda|^{1/2} \mathcal{L}[f](\lambda)](t).$$

Bessel potential space $H^{1/2}_{p,\gamma_0,0}(\mathbb{R}, X)$

Let $1 < p < \infty$ and $\gamma_0 > 0$. We define the following function space:

$$H^{1/2}_{p,\gamma_0,0}(\mathbb{R}, X) = \left\{ f \in L_{p,\gamma_0,0}(\mathbb{R}, X) \left| \|e^{-\gamma t}(\Lambda_{1/2}^{\gamma} f)(t)\|_X \in L_p(\mathbb{R}) (\gamma \geq \gamma_0) \right. \right\}.$$
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§4 Outline of proof

First, we consider the resolvent Stokes equations:

\[
\begin{cases}
\lambda v - \mu \Delta v + \nabla p = f, & \nabla \cdot v = 0 \quad \text{in} \quad \Omega, \\
S(v, p)n = g & \text{on} \quad \Gamma_h, \\
u = 0 & \text{on} \quad \Gamma_0,
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(RP)

where $\mu$ and $S(v, p)$ are same symbols of (SP).
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**Lemma (cf. T.Abe (2004))**

Let $1 < q < \infty$, $0 < \varepsilon < \pi/2$ and $\gamma_0 > 0$. For any $\lambda \in \Sigma_{\varepsilon, \gamma_0}$, there exist the operators $S(\lambda)$ in $\mathcal{B}(L_q(\Omega)^{2N+N^2}, L_q(\Omega)^N)$ and $T(\lambda)$ in $\mathcal{B}(L_q(\Omega)^{2N+N^2}, L_q(\Omega))$ such that

$$v = S(\lambda)(f, |\lambda|^{1/2}g, \nabla g), \quad p = T(\lambda)(f, |\lambda|^{1/2}g, \nabla g),$$

solve (RP) for any $f \in L_q(\Omega)^N$ and $g \in W^{1}_{q}(\Omega)^N$.

We can obtain the solution formula for (SP) by using $S(\lambda)$ and $T(\lambda)$. We set

$$u(t) = \mathcal{L}^{-1}_\lambda [S(\lambda)\mathcal{L}(f, \Lambda_{\gamma}^{1/2}g, \nabla g)](t), \quad \theta(t) = \mathcal{L}^{-1}_\lambda [T(\lambda)\mathcal{L}(f, \Lambda_{\gamma}^{1/2}g, \nabla g)](t).$$

Then, $(u, \theta)$ solve (SP). Next, we estimate the solution $(u, \theta)$ to complete our proof.
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We give some important lemmas and ideas to estimate the solution.
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Key lemma (To estimate the solution of (SP))

Let $1 < p, q < \infty$, $0 < \varepsilon < \pi/2$ and $\gamma_0 > 0$. Let $\Phi_\lambda$ be a $C^1$ function of $\tau \in \mathbb{R} \setminus \{0\}$, where $\lambda = \gamma + i\tau$, with its value in $\mathcal{B}(L^q(\Omega))$. Assume that the sets $\{\Phi_\lambda \mid \lambda \in \Sigma_{\varepsilon, \gamma_0}\}$ and $\{\tau \partial_\tau \Phi_\lambda \mid \lambda \in \Sigma_{\varepsilon, \gamma_0}\}$ are $\mathcal{R}$-bounded families in $\mathcal{B}(L^q(\Omega))$. For $f \in C_0^\infty(\mathbb{R}_+, L^q(\Omega))$, we define the following operator:

$$(\Psi f)(t) = \mathcal{L}^{-1}_\lambda [\Phi_\lambda \mathcal{L}[f](\lambda)](t).$$

Then, there exists a constant $C_{p, q}$ depending on $p, q$ such that

$$\|e^{-\gamma t} \Psi f\|_{L^p(\mathbb{R}, L^q(\Omega))} \leq C_{p, q} M \|e^{-\gamma t} f\|_{L^p(\mathbb{R}, L^q(\Omega))} \quad (f \in L^p(\mathbb{R}_+, L^q(\Omega)))$$

for any $\gamma \geq \gamma_0$ and $M$ is the $\mathcal{R}$-bound of $\{\Phi_\lambda\}_{\lambda \in \Sigma_{\varepsilon, \gamma_0}}$ and $\{\tau \partial_\tau \Phi_\lambda\}_{\lambda \in \Sigma_{\varepsilon, \gamma_0}}$. 
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Then, there exists a constant \( C_{p,q} \) depending on \( p, q \) such that

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\|e^{-\gamma t}\Psi f\|_{L_p(\mathbb{R}, L_q(\Omega))} \leq C_{p,q} M \|e^{-\gamma t}f\|_{L_p(\mathbb{R}, L_q(\Omega))} \quad (f \in L_p(\mathbb{R}^+, L_q(\Omega)))
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for any \( \gamma \geq \gamma_0 \) and \( M \) is the \( \mathcal{R} \)-bound of \( \{\Phi_\lambda\}_{\lambda \in \Sigma_{\varepsilon,\gamma_0}} \) and \( \{\tau \partial_\tau \Phi_\lambda\}_{\lambda \in \Sigma_{\varepsilon,\gamma_0}} \).
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$$(\Psi f)(t) = L^{-1}_\lambda [\Phi_\lambda \mathcal{L}[f](\lambda)](t).$$

Then, there exists a constant $C_{p,q}$ depending on $p, q$ such that

$$\|e^{-\gamma t}\Psi f\|_{L_p(\mathbb{R}, L_q(\Omega))} \leq C_{p,q} M \|e^{-\gamma t}f\|_{L_p(\mathbb{R}, L_q(\Omega))} \quad (f \in L_p(\mathbb{R}^+, L_q(\Omega)))$$

for any $\gamma \geq \gamma_0$ and $M$ is the $\mathcal{R}$-bound of $\{\Phi_\lambda\}_{\lambda \in \Sigma_{\varepsilon, \gamma_0}}$ and $\{\tau \partial_\tau \Phi_\lambda\}_{\lambda \in \Sigma_{\varepsilon, \gamma_0}}$. 
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$$(\Psi f)(t) = \mathcal{L}_\lambda^{-1} [\Phi_\lambda \mathcal{L}[f](\lambda)](t).$$

Then, there exists a constant $C_{p,q}$ depending on $p, q$ such that

$$\|e^{-\gamma t} \Psi f\|_{L_p(\mathbb{R}_+, L_q(\Omega))} \leq C_{p,q} M \|e^{-\gamma t} f\|_{L_p(\mathbb{R}_+, L_q(\Omega))} \quad (f \in L_p(\mathbb{R}_+, L_q(\Omega)))$$

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**Definition (R-boundedness)**

Let $X$ and $Y$ be Banach spaces, and $\| \cdot \|_X$ and $\| \cdot \|_Y$ denote their norms, respectively. A family of operators $\mathcal{T} \subset \mathcal{B}(X, Y)$ is called $\mathcal{R}$-bounded, if there exist a constant $C > 0$ and $p \in [1, \infty)$ such that for $m \in \mathbb{N}$, $\{T_j\}_{j=1}^m \subset \mathcal{T}$, $\{x_j\}_{j=1}^m \subset X$ and for all sequences $\{r_j(u)\}_{j=1}^m$ of independent symmetric, $\{1, -1\}$-valued random variables on $[0, 1]$ there holds the inequality:

$$\left\{ \int_0^1 \left\| \sum_{j=1}^m r_j(u)T_j(x_j) \right\|_Y^p du \right\}^{\frac{1}{p}} \leq C \left\{ \int_0^1 \left\| \sum_{j=1}^m r_j(u)x_j \right\|_X^p du \right\}^{\frac{1}{p}}.$$

The smallest $C$ is called $\mathcal{R}$-bound of $\mathcal{T}$. 
§4 Outline of proof

We give a sufficient condition to prove $R$-boundedness.

**Lemma (Sufficient condition of $R$-boundedness)**

Let $1 \leq q < \infty$, $0 < \varepsilon < \pi/2$ and $\gamma_0 > 0$. Consider a family $\mathcal{T} = \{T_\lambda \mid \lambda \in \Sigma_{\varepsilon, \gamma_0}\}$, which belongs to $\mathcal{B}(L_q(\Omega))$, of kernel operators:

$$(T_\lambda f)(x) = \int_\Omega k_\lambda(x, y)f(y)dy \quad (x \in \Omega, \lambda \in \Sigma_{\varepsilon, \gamma_0}),$$

which are dominated by a kernel $k_0$, i.e.,

$$|k_\lambda(x, y)| \leq k_0(x, y) \quad (a.e. \ x, y \in \Omega, \lambda \in \Sigma_{\varepsilon, \gamma_0}).$$

We set

$$(T_0 f)(x) = \int_\Omega k_0(x, y)f(y)dy \quad (x \in \Omega).$$

If $T_0$ is bounded in $L_q(\Omega)$, then $\mathcal{T}$ is $R$-bounded in $\mathcal{B}(L_q(\Omega))$ whose $R$-bound is bounded by $\|T_0\|$. 
Lemma

Let $1 < q < \infty$, $0 < \varepsilon < \pi/2$ and $\gamma_0 > 0$. Let $S(\lambda)$ and $T(\lambda)$ be the solution operators defined for (RP). Then, for any $d = 0, 1$ and $j, k = 1, \ldots, N$

\[
\{(\tau \partial_\tau)^d (\lambda S(\lambda)) \mid \lambda \in \Sigma_{\varepsilon, \gamma_0}\}, \quad \{(\tau \partial_\tau)^d (\gamma S(\lambda)) \mid \lambda \in \Sigma_{\varepsilon, \gamma_0}\},
\]

\[
\{(\tau \partial_\tau)^d (|\lambda|^{\frac{1}{2}} D_j S(\lambda)) \mid \lambda \in \Sigma_{\varepsilon, \gamma_0}\}, \quad \{(\tau \partial_\tau)^d (D_j D_k S(\lambda)) \mid \lambda \in \Sigma_{\varepsilon, \gamma_0}\}
\]

are $\mathcal{R}$-bounded in $\mathcal{B}(L_q(\Omega)^{2N+N^2}, L_q(\Omega)^N)$ and

\[
\{(\tau \partial_\tau)^d T(\lambda) \mid \lambda \in \Sigma_{\varepsilon, \gamma_0}\}, \quad \{(\tau \partial_\tau)^d (D_k T(\lambda)) \mid \lambda \in \Sigma_{\varepsilon, \gamma_0}\}
\]

are $\mathcal{R}$-bounded in $\mathcal{B}(L_q(\Omega)^{2N+N^2}, L_q(\Omega))$.

Example

\[
u = \mathcal{L}_\lambda^{-1} [S(\lambda) \mathcal{L}(f, \Lambda^{\frac{1}{2}}_{\gamma} g, \nabla g)] \Rightarrow
\]

\[
\partial_t \nu = \mathcal{L}_\lambda^{-1} [\lambda S(\lambda) \mathcal{L}(f, \Lambda^{\frac{1}{2}}_{\gamma} g, \nabla g)], \quad \nabla^2 \nu = \mathcal{L}_\lambda^{-1} [\nabla^2 S(\lambda) \mathcal{L}(f, \Lambda^{\frac{1}{2}}_{\gamma} g, \nabla g)].
\]
§4 Outline of proof

**Lemma**

Let $1 < q < \infty$, $0 < \varepsilon < \pi/2$ and $\gamma_0 > 0$. Let $S(\lambda)$ and $T(\lambda)$ be the solution operators defined for (RP). Then, for any $d = 0, 1$ and $j, k = 1, \ldots, N$

\[
\{(\tau \partial_\tau)^d(\lambda S(\lambda)) \mid \lambda \in \Sigma_{\varepsilon, \gamma_0}\}, \quad \{(\tau \partial_\tau)^d(\gamma S(\lambda)) \mid \lambda \in \Sigma_{\varepsilon, \gamma_0}\},
\]

\[
\{(\tau \partial_\tau)^d(|\lambda|^{1/2} D_j S(\lambda)) \mid \lambda \in \Sigma_{\varepsilon, \gamma_0}\}, \quad \{(\tau \partial_\tau)^d(D_j D_k S(\lambda)) \mid \lambda \in \Sigma_{\varepsilon, \gamma_0}\}
\]

are $R$-bounded in $B(L_q(\Omega)^{2N+N^2}, L_q(\Omega)^N)$ and

\[
\{(\tau \partial_\tau)^dT(\lambda) \mid \lambda \in \Sigma_{\varepsilon, \gamma_0}\}, \quad \{(\tau \partial_\tau)^d(D_k T(\lambda)) \mid \lambda \in \Sigma_{\varepsilon, \gamma_0}\}
\]

are $R$-bounded in $B(L_q(\Omega)^{2N+N^2}, L_q(\Omega))$.

**Example**

\[
u = L^{-1}_\lambda[S(\lambda)L(f, \Lambda^\frac{1}{\gamma} g, \nabla g)] \
\partial_t u = L^{-1}_\lambda[\lambda S(\lambda)L(f, \Lambda^\frac{1}{\gamma} g, \nabla g)] , \quad \nabla^2 u = L^{-1}_\lambda[\nabla^2 S(\lambda)L(f, \Lambda^\frac{1}{\gamma} g, \nabla g)].
\]
§4 Outline of proof

**Lemma**

Let $1 < q < \infty$, $0 < \varepsilon < \pi/2$ and $\gamma_0 > 0$. Let $S(\lambda)$ and $T(\lambda)$ be the solution operators defined for (RP). Then, for any $d = 0, 1$ and $j, k = 1, \ldots, N$

$$
\{(\tau \partial_\tau)^d (\lambda S(\lambda)) \mid \lambda \in \Sigma_{\varepsilon, \gamma_0}\}, \quad \{(\tau \partial_\tau)^d (\gamma S(\lambda)) \mid \lambda \in \Sigma_{\varepsilon, \gamma_0}\},
$$

$$
\{(\tau \partial_\tau)^d (|\lambda|^{\frac{1}{2}} D_j S(\lambda)) \mid \lambda \in \Sigma_{\varepsilon, \gamma_0}\}, \quad \{(\tau \partial_\tau)^d (D_j D_k S(\lambda)) \mid \lambda \in \Sigma_{\varepsilon, \gamma_0}\}
$$

are $R$-bounded in $\mathcal{B}(L_q(\Omega)^{2N+N^2}, L_q(\Omega)\mathcal{N})$ and

$$
\{(\tau \partial_\tau)^d T(\lambda) \mid \lambda \in \Sigma_{\varepsilon, \gamma_0}\}, \quad \{(\tau \partial_\tau)^d (D_k T(\lambda)) \mid \lambda \in \Sigma_{\varepsilon, \gamma_0}\}
$$

are $R$-bounded in $\mathcal{B}(L_q(\Omega)^{2N+N^2}, L_q(\Omega))$.

**Example**

$$
u = \mathcal{L}_\lambda^{-1}[S(\lambda)\mathcal{L}(f, \Lambda_\gamma^\frac{1}{2} g, \nabla g)] \Rightarrow \partial_t u = \mathcal{L}_\lambda^{-1}[\lambda S(\lambda)\mathcal{L}(f, \Lambda_\gamma^\frac{1}{2} g, \nabla g)], \quad \nabla^2 u = \mathcal{L}_\lambda^{-1}[\nabla^2 S(\lambda)\mathcal{L}(f, \Lambda_\gamma^\frac{1}{2} g, \nabla g)].$$
§4 Outline of proof

**Theorem (\(L_p-L_q\) maximal regularity)**

Let \(1 < p, q < \infty\) and \(\gamma_0 > 0\). Then, for any \(f \in L_{p,\gamma_0,0}(\mathbb{R}, L_q(\Omega))^N\), \(g \in L_{p,\gamma_0,0}(\mathbb{R}, W^{1}_q(\Omega))^N \cap H^{1/2}_{p,\gamma_0,0}(\mathbb{R}, L_q(\Omega))^N\) (SP) admits a unique solution \((u, \theta)\) such that

\[
\begin{align*}
    u &\in L_{p,\gamma_0,0}(\mathbb{R}, W^{2}_q(\Omega))^N \cap W^{1}_{p,\gamma_0,0}(\mathbb{R}, L_q(\Omega))^N, \\
    \theta &\in L_{p,\gamma_0,0}(\mathbb{R}, W^{1}_q(\Omega))
\end{align*}
\]

satisfying with the estimate:

\[
\|e^{-\gamma t}(u_t, \gamma u, \Lambda^\frac{1}{\gamma} \nabla u, \nabla^2 u, \theta, \nabla \theta)\|_{L_p(\mathbb{R}, L_q(\Omega))} \leq C\|e^{-\gamma t}(f, \Lambda^\frac{1}{\gamma} g, \nabla g)\|_{L_p(\mathbb{R}, L_q(\Omega))}
\]

for any \(\gamma \geq \gamma_0\) with some constant \(C\) independent of \(\gamma\).
\[ V_{1,\ell}^j(t, x) = - \sum_{n=1}^{2} \mathcal{L}_\lambda^{-1} \left[ \int_0^h \mathcal{F}^{-1}_\xi \left[ \varphi_h(y_N) \left( \frac{m_{1,\ell} \lambda^{(0)-Adn}}{B^2} \right) e^{-A(d_{\ell}(x_N)+dn(y_N))} \right] f_N(\lambda, \xi', y_N) dy_N \right] \]

\[ + \sum_{n=1}^{2} \mathcal{L}_\lambda^{-1} \left[ \int_0^h \mathcal{F}^{-1}_\xi \left[ \varphi_h(y_N) \left( \frac{A}{B} \frac{m_{1,\ell} \lambda^{(0)-Bdn}}{B^2} \right) e^{-Ad\ell(x_N)} e^{-Bdn(y_N)} \right] f_N(\lambda, \xi', y_N) dy_N \right] \]

\[ - \sum_{n=1}^{2} \mathcal{L}_\lambda^{-1} \left[ \int_0^h \mathcal{F}^{-1}_\xi \left[ \varphi_h(y_N) \left( \frac{m_{1,\ell+1} \lambda^{(0)-Adn}}{B^2} \right) e^{-Ad\ell(x_N)+dn(y_N)} \right] f_N(\lambda, \xi', y_N) dy_N \right] \]

\[ + \sum_{n=1}^{2} \mathcal{L}_\lambda^{-1} \left[ \int_0^h \mathcal{F}^{-1}_\xi \left[ \varphi_h(y_N) \left( \frac{A}{B} \frac{m_{1,\ell+1} \lambda^{(0)-Bdn}}{B^2} \right) e^{-B(d_{\ell}(x_N)+dn(y_N))} \right] f_N(\lambda, \xi', y_N) dy_N \right] \]

\[ - \sum_{n=1}^{2} (-1)^n \mathcal{L}_\lambda^{-1} \left[ \int_0^h \mathcal{F}^{-1}_\xi \left[ \varphi_h(y_N) \left( \frac{i \lambda^{(0)-Adn}}{A} \right) \right] f_N(\lambda, \xi', y_N) \right] \]

\[ + \sum_{n=1}^{2} (-1)^n \mathcal{L}_\lambda^{-1} \left[ \int_0^h \mathcal{F}^{-1}_\xi \left[ \varphi_h(y_N) \left( \frac{i \lambda^{(0)-Bdn}}{A} \right) \right] f_N(\lambda, \xi', y_N) \right] \]

\[ - \sum_{n=1}^{2} (-1)^n \mathcal{L}_\lambda^{-1} \left[ \int_0^h \mathcal{F}^{-1}_\xi \left[ \varphi_h(y_N) \left( \frac{i \lambda^{(0)-Adn}}{A} \right) \right] f_N(\lambda, \xi', y_N) \right] \]

\[ + \sum_{n=1}^{2} (-1)^n \mathcal{L}_\lambda^{-1} \left[ \int_0^h \mathcal{F}^{-1}_\xi \left[ \varphi_h(y_N) \left( \frac{i \lambda^{(0)-Bdn}}{A} \right) \right] f_N(\lambda, \xi', y_N) \right] \]