On the L_p-L_q maximal regularity of the Neumann-Dirichlet problem for the Stokes equations in an infinite layer

Hirokazu Saito

Waseda University

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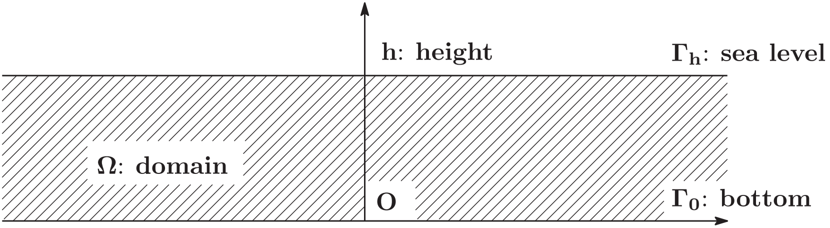
§0 Contents

§1 Problem

(SP)
$$\begin{cases} \partial_t u - \mu\Delta u + \nabla\theta = f, \quad \nabla\cdot u = 0 & \text{in} \quad \Omega\times(0,\infty), \\ S(u,\theta)\mathbf{n} = g & \text{on} \quad \Gamma_h\times(0,\infty), \\ u = 0 & \text{on} \quad \Gamma_0\times(0,\infty), \\ u|_{t=0} = 0 & \text{in} \quad \Omega. \end{cases}$$

$\mu > 0$: a coefficient of viscosity, f, g: given functions.
$u = (u_1, \ldots, u_N)$: velocity, θ: pressure: unknown.
$S(u,\theta) = -\theta I + \mu[\{\nabla u + (\nabla u)^T\}]$: stress tensor, I: $N\times N$ identity matrix. $\Omega \subset \mathbf{R}^N$ ($N \geq 2$):



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\text{(NS)} \quad \begin{cases} \partial_t u + u\cdot\nabla u - \mu\Delta u + \nabla\theta = 0, \quad \nabla\cdot u = 0 & \text{in} \quad \Omega(t) \quad t>0, \\ \partial_t \eta + u'\cdot\nabla'\eta - u_N = 0 & \text{on} \quad \Gamma(t) \quad t>0, \\ S(u,\theta)\mathbf{n} + (c_g\eta - c_\sigma H)\mathbf{n} = 0 & \text{on} \quad \Gamma(t) \quad t>0, \\ u = 0 & \text{on} \quad \Gamma_0, \\ \eta_0 = \eta_0(x'), \quad u_0 = u_0(x). \end{cases}
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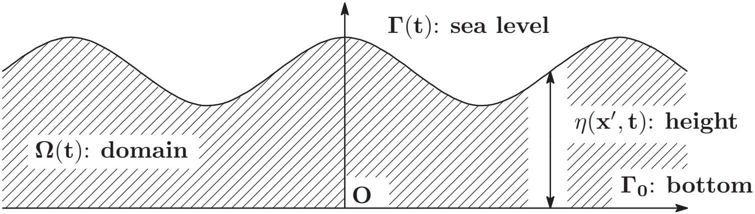
$\eta = \eta(x', t)$, $x' \in \mathbf{R}^{N-1}$: the height from the bottom to sea level.
$c_g > 0$: the gravity constant, $c_\sigma > 0$: a surface tension constant,
H: mean curvature of $\Gamma(t)$.



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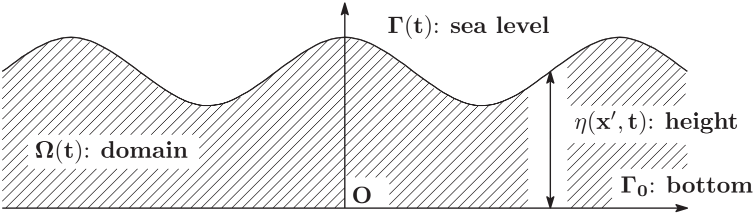

§2 Known results

Results in L_2-L_2 framework:

- Beale (1980) ⇒ The existence of a unique local time solution in (NS) with $c_\sigma = 0$.
- Beale (1984) ⇒ The existence of a unique global time solution in (NS) with $c_\sigma \neq 0$.
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- Abe (2004) ⇒ Resolvent problem corresponding to (SP).
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§3 Main result

Theorem (L_p-L_q maximal regularity)

Let $1 < p, q < \infty$ and $\gamma_0 > 0$. Then, for any $f \in L_{p,\gamma_0,0}(\mathbf{R}, L_q(\Omega))^N$, $g \in L_{p,\gamma_0,0}(\mathbf{R}, W_q^1(\Omega))^N \cap H_{p,\gamma_0,0}^{1/2}(\mathbf{R}, L_q(\Omega))^N$ (SP) admits a unique solution (u, θ) such that

$$u \in L_{p,\gamma_0,0}(\mathbf{R}, W_q^2(\Omega))^N \cap W_{p,\gamma_0,0}^1(\mathbf{R}, L_q(\Omega))^N,$$
$$\theta \in L_{p,\gamma_0,0}(\mathbf{R}, W_q^1(\Omega))$$

satisfying with the estimate:

$$\|e^{-\gamma t}(u_t, \gamma u, \Lambda_\gamma^{\frac{1}{2}}\nabla u, \nabla^2 u, \theta, \nabla\theta)\|_{L_p(\mathbf{R}, L_q(\Omega))} \leq C\|e^{-\gamma t}(f, \Lambda_\gamma^{\frac{1}{2}} g, \nabla g)\|_{L_p(\mathbf{R}, L_q(\Omega))}$$

for any $\gamma \geq \gamma_0$ with some constant C independent of γ.

Remark

$$L_{p,\gamma_0,0}(\mathbf{R}, X) = \left\{ f : \mathbf{R} \to X \;\middle|\; \|e^{-\gamma_0 t} f(t)\|_X \in L_p(\mathbf{R}),\ f(t) = 0\ (t < 0) \right\}.$$

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$$u \in L_{p,\gamma_0,0}(\mathbf{R}, W_q^2(\Omega))^N \cap W^1_{p,\gamma_0,0}(\mathbf{R}, L_q(\Omega))^N,$$
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Laplace transform $\mathcal{L}$ and its inverse $\mathcal{L}_\lambda^{-1}$

Let $u(t)$ and $v(\tau)$ be functions defined on $\mathbf{R}$. Then, for $\lambda = \gamma + i\tau$ $(\gamma, \tau \in \mathbf{R})$ Laplace transform and its inverse are defined by

$$\mathcal{L}[u(t)](\lambda) = \int_{-\infty}^{\infty} e^{-\lambda t} u(t)dt, \quad \mathcal{L}_\lambda^{-1}[v(\tau)](t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\lambda t} v(\tau)d\tau.$$

We set:

$$(\Lambda_\gamma^{\frac{1}{2}} f)(t) = \mathcal{L}_\lambda^{-1}[|\lambda|^{\frac{1}{2}} \mathcal{L}[f](\lambda)](t).$$

Bessel potential space $H^{1/2}_{p,\gamma_0,0}(\mathbf{R}, X)$

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First, we consider the resolvent Stokes equations:

$$\text{(RP)} \qquad \begin{cases} \lambda v - \mu \Delta v + \nabla p = f, \quad \nabla \cdot v = 0 & \text{in} \quad \Omega, \\ S(v,p)\mathbf{n} = g & \text{on} \quad \Gamma_h, \\ u = 0 & \text{on} \quad \Gamma_0, \end{cases}$$

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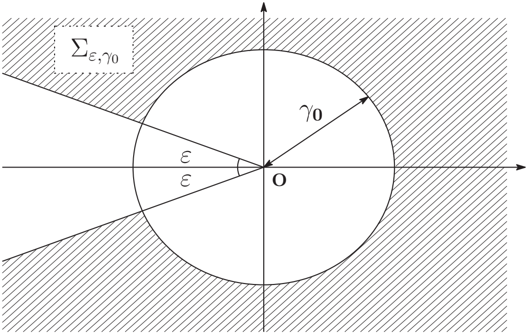

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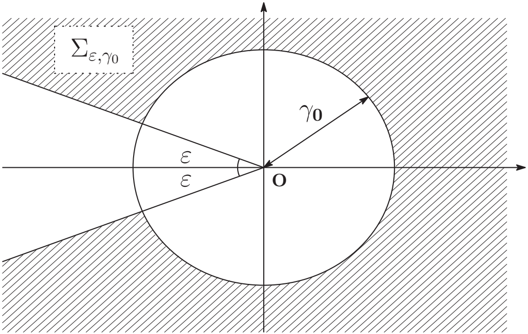

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Lemma (cf. T.Abe (2004))

Let $1 < q < \infty$, $0 < \varepsilon < \pi/2$ and $\gamma_0 > 0$. For any $\lambda \in \Sigma_{\varepsilon,\gamma_0}$, there exist the operators $\mathcal{S}(\lambda)$ in $\mathcal{B}(L_q(\Omega)^{2N+N^2}, L_q(\Omega)^N)$ and $\mathcal{T}(\lambda)$ in $\mathcal{B}(L_q(\Omega)^{2N+N^2}, L_q(\Omega))$ such that

$$v = \mathcal{S}(\lambda)(f, |\lambda|^{\frac{1}{2}}g, \nabla g), \quad p = \mathcal{T}(\lambda)(f, |\lambda|^{\frac{1}{2}}g, \nabla g),$$

solve (RP) for any $f \in L_q(\Omega)^N$ and $g \in W_q^1(\Omega)^N$.

We can obtain the solution formula for (SP) by using $\mathcal{S}(\lambda)$ and $\mathcal{T}(\lambda)$. We set

$$u(t) = \mathcal{L}_\lambda^{-1}[\mathcal{S}(\lambda)\mathcal{L}(f, \Lambda_\gamma^{\frac{1}{2}}g, \nabla g)](t), \;\; \theta(t) = \mathcal{L}_\lambda^{-1}[\mathcal{T}(\lambda)\mathcal{L}(f, \Lambda_\gamma^{\frac{1}{2}}g, \nabla g)](t).$$

Then, (u, θ) solve (SP). Next, we estimate the solution (u, θ) to complete our proof.

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Key lemma (To estimate the solution of (SP))

Let $1 < p, q < \infty$, $0 < \varepsilon < \pi/2$ and $\gamma_0 > 0$. Let Φ_λ be a C^1 function of $\tau \in \mathbf{R} \setminus \{0\}$, where $\lambda = \gamma + i\tau$, with its value in $\mathcal{B}(L_q(\Omega))$. Assume that the sets $\{\Phi_\lambda \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}$ and $\{\tau\partial_\tau\Phi_\lambda \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}$ are $\mathcal{R}$-bounded families in $\mathcal{B}(L_q(\Omega))$. For $f \in C_0^\infty(\mathbf{R}_+, L_q(\Omega))$, we define the following operator:

$$(\Psi f)(t) = \mathcal{L}_\lambda^{-1}[\Phi_\lambda \mathcal{L}[f](\lambda)](t).$$

Then, there exists a constant $C_{p,q}$ depending on p, q such that

$$||e^{-\gamma t}\Psi f||_{L_p(\mathbf{R},L_q(\Omega))} \leq C_{p,q}M||e^{-\gamma t}f||_{L_p(\mathbf{R},L_q(\Omega))} \quad (f \in L_p(\mathbf{R}_+, L_q(\Omega)))$$

for any $\gamma \geq \gamma_0$ and M is the $\mathcal{R}$-bound of $\{\Phi_\lambda\}_{\lambda\in\Sigma_{\varepsilon,\gamma_0}}$ and $\{\tau\partial_\tau\Phi_\lambda\}_{\lambda\in\Sigma_{\varepsilon,\gamma_0}}$.

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§4 Outline of proof

Definition ($\mathcal{R}$-boundedness)

Let X and Y be Banach spaces, and $\|\cdot\|_X$ and $\|\cdot\|_Y$ denote their norms, respectively. A family of operators $\mathcal{T} \subset \mathcal{B}(X, Y)$ is called $\mathcal{R}$-bounded, if there exist a constant $C > 0$ and $p \in [1, \infty)$ such that for $m \in \mathbf{N}$, $\{T_j\}_{j=1}^m \subset \mathcal{T}$, $\{x_j\}_{j=1}^m \subset X$ and for all sequences $\{r_j(u)\}_{j=1}^m$ of independent symmetric, $\{1, -1\}$-valued random variables on $[0, 1]$ there holds the inequality:

$$\Big\{ \int_0^1 \Big\| \sum_{j=1}^m r_j(u) T_j(x_j) \Big\|_Y^p du \Big\}^{\frac{1}{p}} \leq C \Big\{ \int_0^1 \Big\| \sum_{j=1}^m r_j(u) x_j \Big\|_X^p du \Big\}^{\frac{1}{p}}.$$

The smallest C is called $\mathcal{R}$-bound of $\mathcal{T}$.

§4 Outline of proof

We give a sufficient condition to prove $\mathcal{R}$-boundedness.

Lemma (Sufficient condition of $\mathcal{R}$-boundedness)

Let $1 \le q < \infty, 0 < \varepsilon < \pi/2$ and $\gamma_0 > 0$. Consider a family $\mathcal{T} = \{T_\lambda \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}$, which belongs to $\mathcal{B}(L_q(\Omega))$, of kernel operators:

$$(T_\lambda f)(x) = \int_\Omega k_\lambda(x,y)f(y)dy \quad (x \in \Omega, \lambda \in \Sigma_{\varepsilon,\gamma_0}),$$

which are dominated by a kernel k_0, *i.e.*,

$$|k_\lambda(x,y)| \le k_0(x,y) \qquad (a.e.\ x,y \in \Omega, \lambda \in \Sigma_{\varepsilon,\gamma_0}).$$

We set

$$(T_0 f)(x) = \int_\Omega k_0(x,y)f(y)dy \qquad (x \in \Omega).$$

If T_0 is bounded in $L_q(\Omega)$, then $\mathcal{T}$ is $\mathcal{R}$-bounded in $\mathcal{B}(L_q(\Omega))$ whose $\mathcal{R}$-bound is bounded by $\|T_0\|$.

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Lemma

Let $1 < q < \infty, 0 < \varepsilon < \pi/2$ and $\gamma_0 > 0$. Let $\mathcal{S}(\lambda)$ and $\mathcal{T}(\lambda)$ be the solution operators defined for (RP). Then, for any $d = 0, 1$ and $j, k = 1, \dots, N$

$$\{(\tau\partial_\tau)^d(\lambda\mathcal{S}(\lambda)) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}, \qquad \{(\tau\partial_\tau)^d(\gamma\mathcal{S}(\lambda)) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\},$$

$$\{(\tau\partial_\tau)^d(|\lambda|^{\frac{1}{2}}D_j\mathcal{S}(\lambda)) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}, \quad \{(\tau\partial_\tau)^d(D_jD_k\mathcal{S}(\lambda)) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}$$

are $\mathcal{R}$-bounded in $\mathcal{B}(L_q(\Omega)^{2N+N^2}, L_q(\Omega)^N)$ and

$$\{(\tau\partial_\tau)^d\mathcal{T}(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}, \quad \{(\tau\partial_\tau)^d(D_k\mathcal{T}(\lambda)) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}$$

are $\mathcal{R}$-bounded in $\mathcal{B}(L_q(\Omega)^{2N+N^2}, L_q(\Omega))$.

Example

$u = \mathcal{L}_\lambda^{-1}[\mathcal{S}(\lambda)\mathcal{L}(f, \Lambda_\gamma^{\frac{1}{2}}g, \nabla g)] \Rightarrow$

$\partial_t u = \mathcal{L}_\lambda^{-1}[\lambda\mathcal{S}(\lambda)\mathcal{L}(f, \Lambda_\gamma^{\frac{1}{2}}g, \nabla g)]$, $\nabla^2 u = \mathcal{L}_\lambda^{-1}[\nabla^2\mathcal{S}(\lambda)\mathcal{L}(f, \Lambda_\gamma^{\frac{1}{2}}g, \nabla g)]$.

§4 Outline of proof

Lemma

Let $1 < q < \infty, 0 < \varepsilon < \pi/2$ and $\gamma_0 > 0$. Let $\mathcal{S}(\lambda)$ and $\mathcal{T}(\lambda)$ be the solution operators defined for (RP). Then, for any $d = 0, 1$ and $j, k = 1, \dots, N$

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§4 Outline of proof

Lemma

Let $1 < q < \infty, 0 < \varepsilon < \pi/2$ and $\gamma_0 > 0$. Let $\mathcal{S}(\lambda)$ and $\mathcal{T}(\lambda)$ be the solution operators defined for (RP). Then, for any $d = 0, 1$ and $j, k = 1, \ldots, N$

$$\{(\tau\partial_\tau)^d(\lambda\mathcal{S}(\lambda)) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}, \qquad \{(\tau\partial_\tau)^d(\gamma\mathcal{S}(\lambda)) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\},$$

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are $\mathcal{R}$-bounded in $\mathcal{B}(L_q(\Omega)^{2N+N^2}, L_q(\Omega)^N)$ and

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Example

$u = \mathcal{L}_\lambda^{-1}[\mathcal{S}(\lambda)\mathcal{L}(f, \Lambda_\gamma^{\frac{1}{2}}g, \nabla g)] \Rightarrow$
$\partial_t u = \mathcal{L}_\lambda^{-1}[\lambda\mathcal{S}(\lambda)\mathcal{L}(f, \Lambda_\gamma^{\frac{1}{2}}g, \nabla g)]$, $\nabla^2 u = \mathcal{L}_\lambda^{-1}[\nabla^2\mathcal{S}(\lambda)\mathcal{L}(f, \Lambda_\gamma^{\frac{1}{2}}g, \nabla g)]$.

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Theorem (L_p-L_q maximal regularity)

Let $1 < p, q < \infty$ and $\gamma_0 > 0$. Then, for any $f \in L_{p,\gamma_0,0}(\mathbf{R}, L_q(\Omega))^N$, $g \in L_{p,\gamma_0,0}(\mathbf{R}, W_q^1(\Omega))^N \cap H^{1/2}_{p,\gamma_0,0}(\mathbf{R}, L_q(\Omega))^N$ (SP) admits a unique solution (u, θ) such that

$$u \in L_{p,\gamma_0,0}(\mathbf{R}, W_q^2(\Omega))^N \cap W^1_{p,\gamma_0,0}(\mathbf{R}, L_q(\Omega))^N,$$

$$\theta \in L_{p,\gamma_0,0}(\mathbf{R}, W_q^1(\Omega))$$

satisfying with the estimate:

$$\|e^{-\gamma t}(u_t, \gamma u, \Lambda_\gamma^{\frac{1}{2}}\nabla u, \nabla^2 u, \theta, \nabla\theta)\|_{L_p(\mathbf{R}, L_q(\Omega))} \leq C\|e^{-\gamma t}(f, \Lambda_\gamma^{\frac{1}{2}} g, \nabla g)\|_{L_p(\mathbf{R}, L_q(\Omega))}$$

for any $\gamma \geq \gamma_0$ with some constant C independent of γ.

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$$
\begin{aligned}
V^j_{1,\ell}(t,x) = &- \sum_{n=1}^{2} \mathcal{L}_\lambda^{-1}\Big[\int_0^h \mathcal{F}_{\xi'}^{-1}[\varphi_h(y_N)(\frac{m^j_{1,\ell} e^{-Ad_n(0)}}{B^2})Ae^{-A(d_\ell(x_N)+d_n(y_N))}\widehat{f_N}(\lambda,\xi',y_N)]dy_N\Big] \\
&+ \sum_{n=1}^{2} \mathcal{L}_\lambda^{-1}\Big[\int_0^h \mathcal{F}_{\xi'}^{-1}[\varphi_h(y_N)(\frac{A}{B})(\frac{m^j_{1,\ell} e^{-Bd_n(0)}}{B^2})Ae^{-Ad_\ell(x_N)}e^{-Bd_n(y_N)}\widehat{f_N}(\lambda,\xi',y_N)]dy_N\Big] \\
&- \sum_{n=1}^{2} \mathcal{L}_\lambda^{-1}\Big[\int_0^h \mathcal{F}_{\xi'}^{-1}[\varphi_h(y_N)(\frac{m^j_{1,\ell+2} e^{-Ad_n(0)}}{B^2})Ae^{-Bd_\ell(x_N)}e^{-Ad_n(y_N)}\widehat{f_N}(\lambda,\xi',y_N)]dy_N\Big] \\
&+ \sum_{n=1}^{2} \mathcal{L}_\lambda^{-1}\Big[\int_0^h \mathcal{F}_{\xi'}^{-1}[\varphi_h(y_N)(\frac{A}{B})(\frac{m^j_{1,\ell+2} e^{-Bd_n(0)}}{B^2})Ae^{-B(d_\ell(x_N)+d_n(y_N))}\widehat{f_N}(\lambda,\xi',y_N)]dy_N\Big] \\
&- \sum_{n=1}^{2} (-1)^n \mathcal{L}_\lambda^{-1}\Big[\int_0^h \mathcal{F}_{\xi'}^{-1}[\varphi_h(y_N)(\frac{i\xi'}{A})(\frac{m^j_{1,\ell} e^{-Ad_n(0)}}{B^2})Ae^{-A(d_\ell(x_N)+d_n(y_N))}\cdot\widehat{f'}(\lambda,\xi',y_N)]dy_N\Big] \\
&+ \sum_{n=1}^{2} (-1)^n \mathcal{L}_\lambda^{-1}\Big[\int_0^h \mathcal{F}_{\xi'}^{-1}[\varphi_h(y_N)(\frac{i\xi'}{A})(\frac{m^j_{1,\ell} e^{-Bd_n(0)}}{B^2})Ae^{-Ad_\ell(x_N)}e^{-Bd_n(y_N)}\cdot\widehat{f'}(\lambda,\xi',y_N)]dy_N\Big] \\
&- \sum_{n=1}^{2} (-1)^n \mathcal{L}_\lambda^{-1}\Big[\int_0^h \mathcal{F}_{\xi'}^{-1}[\varphi_h(y_N)(\frac{i\xi'}{A})(\frac{m^j_{1,\ell+2} e^{-Ad_n(0)}}{B^2})Ae^{-Bd_\ell(x_N)}e^{-Ad_n(y_N)}\cdot\widehat{f'}(\lambda,\xi',y_N)]dy_N\Big] \\
&+ \sum_{n=1}^{2} (-1)^n \mathcal{L}_\lambda^{-1}\Big[\int_0^h \mathcal{F}_{\xi'}^{-1}[\varphi_h(y_N)(\frac{i\xi'}{A})(\frac{m^j_{1,\ell+2} e^{-Bd_n(0)}}{B^2})Ae^{-B(d_\ell(x_N)+d_n(y_N))}\cdot\widehat{f'}(\lambda,\xi',y_N)]dy_N\Big]
\end{aligned}
$$