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Ergodicity for Generalized Newtonian Fluids

June 12th, 2012

A Motivating Example

Consider the well-known stochastic Navier-Stokes equations

$$
\begin{aligned}
&\partial_t u = \nu \Delta u - (u \cdot \nabla) u - \nabla \pi + \eta \quad \text{in } [0, \infty) \times \mathcal{O}, \\
&\operatorname{div} u = 0, \quad \text{in } [0, \infty) \times \mathcal{O}, \\
&u(0) = u_0, \quad \text{in } \mathcal{O}.
\end{aligned} \tag{1}
$$

Here, $\mathcal{O} = \mathbb{T}^d$, $d \geq 2$, with periodic boundary conditions, η is some Gaussian noise.
Question 1: Is there a unique solution?

- Well-studied, but no uniqueness for $d = 3$.

Question 2: How well does it describe real world phenomena?

- Non-Newtonian behaviour occurs almost everywhere.

A More General Example

Modificate the extra stress tensor appearing in $\Delta = \operatorname{div} \mathbf{E}u$

$$\mathbf{S}(\mathbf{E}u) = \nu \mathbf{E}u \quad \text{to} \quad \mathbf{S}(\mathbf{E}u) = \nu(\mathbf{E}u).$$

Known Results: For so-called power-law fluids

$$\mathbf{S}(\mathbf{E}u) = \nu_0 \left(1 + |\mathbf{E}u|^2\right)^{\frac{p-2}{2}} \mathbf{E}u$$

by Terasawa and Yoshida [2011]. Existence of weak martingale solutions for $p \geq \frac{9}{5}$ and Uniqueness for $p \geq \frac{5}{2}$ in $d = 3$.

Suppose **S** admits a so-called p-potential, i. e.

$$\mathbf{S}(\mathbf{E}u) = \nabla_{d\times d}\Phi(\mathbf{E}u).$$

following Nečas, Růžička and co-authors. Φ is the p-potential defined by:

Definition

Let $p > 1$ and $F \in C^2([0,\infty))$ be convex with $F'(0) = 0$. Define $\Phi(A) = F(|A|)$ and suppose $\exists\,\gamma_1, \gamma_2 > 0$ such that

$$\sum_{ijkl}\left(\partial_{ij}\partial_{kl}\Phi\right)(A)B_{ij}B_{kl} \geq \gamma_1\left(1+|A|^2\right)^{\frac{p-2}{2}}|B|^2,$$

$$\left(\sum_{ijkl}\left(\partial_{ij}\partial_{kls}\Phi\right)(A)^2\right)^{\frac{1}{2}} \leq \gamma_2\left(1+|A|^2\right)^{\frac{p-2}{2}} \quad \text{for all } A, B \in \mathbb{R}^{d\times d}_{\text{sym}}.$$

Additive Gaussian Noise

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The simplest case of an SPDE is a PDE with some additive random forcing η. Typically, one assumes

$$\mathbb{E}[\eta(t, x)\eta(s, y)] = \delta_{t-s}\delta_{x-y},$$

so-called space-time white noise. Rigerous realization by cylindrical Wiener process on a separable Hilbert space H

$$W(t) = \sum_k e_k \beta_k(t) \Rightarrow \mathbb{E}\left[\langle W(t), \phi\rangle\langle W(s), \psi\rangle\right] = (t \wedge s)\langle \phi, \psi\rangle.$$

$W(t) \notin H$, therefore fix $Q \in L(H)$ symmetric, nonnegative with $\operatorname{tr} Q < \infty$. Then $\sqrt{Q}W(t) = \sum_k \sigma_k e_k \beta_k(t)$ and

$$\mathbb{E}\left[\langle \sqrt{Q}W(t), \phi\rangle\langle \sqrt{Q}W(s), \psi\rangle\right] = (t \wedge s)\langle Q\phi, \psi\rangle.$$

An SPDE for Generalized Newtonian Fluids

Consider $\mathcal{D} = \{u \in C^\infty_{\text{per}} : \operatorname{div} u = 0, \int_{\mathbb{T}^d} u(\xi)\mathrm{d}\xi = 0\}$ and the spaces H, V as the closure w. r. t. the L^2- and $H^{1,p}$-norm, respectively.
Furthermore, define the Gelfand triple $V \hookrightarrow H \hookrightarrow V^*$.

$$\begin{aligned} \mathrm{d}u(t) &= \mathcal{A}\big(u(t)\big)\,\mathrm{d}t + \sqrt{Q}\,\mathrm{d}W(t), \\ u(0) &= u_0. \end{aligned} \tag{2}$$

- $\operatorname{tr} Q < \infty$, $u_0 \in H$.
- $\mathcal{A} : V \to V^*$, $u \mapsto \operatorname{div} \mathbf{S}(\mathbf{E}u) + (u \cdot \nabla)u$.
- Use the so-called variational approach.

Properties of the Drift

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There exists $C, \kappa_p, \beta > 0$ such that for all $u, v, w \in V$

- (H1, Hemicontinuity)
  The map $s \mapsto {}_{V^*}\langle \mathcal{A}(u + sv), w \rangle_V$ is continuous in $\mathbb{R}$.
- (H2, Local Monotonicity)

$${}_{V^*}\langle \mathcal{A}(u) - \mathcal{A}(v), u - v \rangle_V \leq (C + \rho(v)) \|u - v\|_H^2$$

  with $\rho : V \to [0, \infty)$ measurable and locally bounded in V.
- (H3, Coercivity)

$${}_{V^*}\langle \mathcal{A}(u), u \rangle_V \leq -\kappa_p \|u\|_V^p.$$

- (H4, Growth)

$$\|\mathcal{A}(u)\|_{V^*}^{\frac{p}{p-1}} \leq C(1 + \|u\|_V^p)(1 + \|u\|_H^\beta).$$

Existence and Uniqueness of Solutions

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Theorem

Suppose **S** *admits a p-potential with* $p \geq 1 + \frac{2d}{d+2}$*. Then (H1)-(H4) are satisfied.*

Theorem

For every fixed initial condition $x \in H$ *and time* $T > 0$*, there exists a solution to equation (2) up to time* T *satisfying*

$$\mathbb{E}\left[\sup_{t \leq T} \|u(t)\|_H^{2+\beta} + \int_0^T \|u(t)\|_V^p \, \mathrm{d}t\right] < \infty.$$

If moreover

$$\rho(v) \leq C(1 + \|v\|_V^p) \Leftrightarrow p \geq 1 + \frac{d}{2},$$

the solution is also unique.

Stationary Distributions

We are interested in stationary distributions μ, i. e. $u_0 \sim \mu$ implies

$$\mathbb{E}\left[f\big(u(t)\big)\right] = \mathbb{E}\left[f\big(u_0\big)\right] \quad \text{for all } t > 0, f \in C_b(H).$$

Formally, $P_t f := \mathbb{E}\left[f\big(u(t)\big)\right]$ defines a semigroup and we are looking for invariant measures μ with the property

$$P_t^* \mu = \mu.$$

The usual existence proof due to Krylov-Bogoliubov is based on the Feller property and the tightness of

$$\mu_x^T(A) = \frac{1}{T}\int_0^T P_t 1_A(x)\,\mathrm{d}t, \quad A \in \mathcal{B}(H).$$

Existence of Invariant Measures

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Theorem

Suppose $p \geq 1 + \frac{d}{2}$, i. e. there exists a unique solution to (2). Then, there also exists an invariant measure for its transition semigroup P_t.

Proof Based on three steps:

- A priori estimates yield boundedness of $\frac{1}{T}\mathbb{E}\left[\int_0^T \|u(t)\|_V^p \, \mathrm{d}t\right]$.
- The embedding $V \hookrightarrow H$ is compact.
- P_t is Feller, since $u(t)$ depends continuously on the initial condition x.

Uniqueness of the Invariant Measure

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Standard way:

- Strong Feller property, i. e. $P_t\big(\mathcal{B}_b(H)\big) \subset C_b(H)$.
- Irreducibility, i. e. for all $t > 0$, $x \in H$ and nonempty, open $A \subseteq H$ holds $P_t 1_A(x) > 0$.

Better suited for infinite dimensional problems, a recent approach by Komorowski, Peszat, Szarek [2010].

- P_t has the e-property, if $(P_t f)_{t\geq 0}$ is equicontinuous at every point $x \in H$ for any $f \in \mathrm{Lip}_b(H)$.
- The process $u(t)$ has at least one asymptotically recurrent state $z \in H$, i. e. for every $\delta > 0$, $x \in H$

$$\liminf_{T\nearrow\infty} \mu_x^T\big(B_\delta(z)\big) > 0.$$

Uniqueness of the Invariant Measure

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Theorem

Suppose $p \geq 1 + \frac{d}{2}$, $\mathrm{tr}(-\Delta)Q < \infty$ and the noise satisfies a certain smallness assumption in terms of amplitudes. Then, there exists a unique invariant measure μ for P_t and moreover

$$\frac{1}{T}\int_0^T P_t^* \nu \, \mathrm{d}t \xrightarrow{T \nearrow \infty} \mu$$

for every $\nu \in \mathcal{M}_1(H)$.

Some Crucial Steps in the Proof

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- The smallness assumption is needed for the e-property, since $\mathcal{A}$ is only locally monotone.
- Consider the deterministic solution $v(t)$ and its asymptotic behaviour.
- Stochastic stability for $u(t)$, i. e.

$$\mathbb{P}\left[\|u(t) - v(t)\|_H < \varepsilon\right] > 0.$$

- It is needed, that $\sqrt{Q}W(t) \in V$, therefore the trace condition.

Conclusion

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- The variational approach yields good results for $p \geq 1 + \frac{2d}{d+2}$ and in particular for $p \geq 1 + \frac{d}{2}$.
- The gap between 2 and $\frac{11}{5}$ for $d = 3$ needs improved a priori estimates.
- Large values of p imply the existence of unique (stationary) solutions.
- Smaller values of p need weaker notions of solutions, which were also studied.