Simulation of stationary fractional Ornstein-Uhlenbeck process and application to turbulence

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Motivation

Assume that the measurable random velocity field

$$v : \Omega \times \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}^2$$

on $(\Omega, \mathcal{F}, \mathbb{P})$ is

- time stationary,
- space homogeneous,
- (local) isotropic.

Phenomenological approach

- by Kolmogorov (1941) in 3D,
- by Kraichnan, Leith, Batchelor (1967-1969) in 2D

yields

Motivation

phenomenological correspondence between

- **second-order structure function**

$$\mathbb{E}\left(\left|v(x+r,t)-v(x,t)\right|^2\right) = C|r|^{\alpha-1} \tag{1}$$

- and **energy spectrum** $E(\cdot)$ of the velocity field v

$$E(k) = \widetilde{C}k^{-\alpha}, \tag{2}$$

where $C, \widetilde{C} > 0$ are some constants, $1 < \alpha < 3$, $r \in \mathbb{R}^2$ and $k > 0$ in the *inertial subrange*.

Special case: $\alpha = 5/3$

- (1): *Kolmogorov's two-thirds law*
- (2): *Kolmogorov's five-thirds law* (or *Kolmogorov energy spectrum*)

Motivation

Connection to fractional Brownian motion (fBm): *Taylor's frozen turbulence hypothesis*(TH)(1938)

Spatial pattern of turbulent motion is unchanged as it is advected by a constant (in space and time) mean velocity $\overline{V}$, $|\overline{V}| := \left(\sum_i \overline{V}_i^2\right)^{\frac{1}{2}}$, *let us say along the* $\overline{x}$ *axis.*

Mathematically, TH says that for any *scalar–valued fluid–mechanics variable* ξ (e.g. v_i) we have

$$\frac{\partial \xi}{\partial t} = -|\overline{V}|\frac{\partial \xi}{\partial \overline{x}}. \tag{3}$$

TH enables us to express the statistical characteristics

- space differences $v(x+r,t) - v(x,t)$ in terms of
- time differences $v(x,t) - v(x,t+s)$ corresponding to a fixed time t.

Motivation

Indeed, (3) implies $v_i(x, t+s) = v_i(x - \overline{V}s, t)$ and therefore by (1) we deduce

$$\mathbb{E}\left(|v(x,t) - v(x,t+s)|^2\right) = C|\overline{V}|^{\alpha-1}s^{\alpha-1} \qquad (4)$$

in the inertial subrange along the time axis.

Note that due to our derivation the properties (2) and (4) are closely related to each other!

Now comparing (4) with the statistical property

- $\mathbb{E}\left(|\beta^H_{t+s} - \beta^H_t|^2\right) = s^{2H}$ fBm $(\beta^H_t)_{t\in\mathbb{R}}$ with Hurst parameter $H \in (0,1)$

indicates that it is reasonable to model the random velocity field v with noise driven by a fBm.

Motivation

In view of this motivation

- Shao (1993/1994),
- Sreenivasan et al. (1993),
- Papanicolaou and Solna (2003)

used finite-dimensional fBm in their models for the velocity field of turbulent flows with special interest in the case $H = \frac{1}{3}$.

Besides, Flandoli (2002).

Short background on fBm $(\beta_t^H)_{t\geq 0}$:

- same definition as standard Bm, but

$$\mathbb{E}(\beta_t^H \beta_s^H) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}), \ \ s,t \geq 0,$$

  with *Hurst parameter* $H \in (0,1)$,
- $H = 1/2$: standard Bm,
- stationary increments, self-similar, $\mathbb{P}$-a.s. Hölder continuous with exponent < H,
- $H \neq 1/2$: not a Markov process, not a semimartingale.

The Model

We are concerned with the following system of equations in non-dimensional form:

$$v(x,t) = \nabla^{\perp}\psi(x,t) = \Big(\frac{\partial\psi}{\partial x_2}(x,t), -\frac{\partial\psi}{\partial x_1}(x,t)\Big), \quad x \in [0,1]^2,\ t \geq 0 \tag{M1}$$

$$d\psi_t = \nu A\psi_t dt + \nu^H Q^{\frac{1}{2}} dB_t^H, \quad \psi_0 \in V,\ t \geq 0. \tag{M2}$$

Assumption A:

(i) $\nu > 0$,

(ii) $\big(V := \{f \in L_2^{per}([0,1]^2) |\ \int_{[0,1]^2} f(x)dx = 0\}, \langle\cdot,\cdot\rangle_V = \langle\cdot,\cdot\rangle_{L_2([0,1]^2)}\big)$ is the separable Hilbert space with orthonormal basis (ONB) $\big(e_k(\cdot)\big)_{k\in K} := \big(e^{i\langle k,\cdot\rangle}\big)_{k\in K}$ where $k \in K := 2\pi\mathbb{Z}^2 \backslash \{(0,0)\}$ and i denotes the imaginary unit,

(iii) $A : \mathcal{D}(A) \subset V \to V$ is a linear operator such that there is a strictly positive sequence $\big(\alpha_k\big)_{k\in K} \subset [c,\infty)$ with $c > 0$, $\alpha_k = \alpha_{-k}$, $Ae_k = -\alpha_k e_k$ and $\alpha_k \to \infty$ for $|k| \to \infty$,

The Model

(iv) $Q^{\frac{1}{2}}: V \to V$ is a bounded linear operator such that there is a positive sequence $\left(\sqrt{\lambda_k}\right)_{k \in K} \subset [0, \infty)$ with $\sqrt{\lambda_k} = \sqrt{\lambda_{-k}}$ and $Q^{\frac{1}{2}} e_k = \sqrt{\lambda_k} e_k$.

(iiv) $\left(B_t^H\right)_{t \geq 0}$ is an infinite–dimensional fractional Brownian motion in V with Hurst parameter $H \in (0, 1)$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ by the formal series

$$B^H(t) = \sum_{k \in K} \beta_k^H(t) e_k, \quad t \in \mathbb{R},$$

where $\left((\beta_k^H(t))_{t \in \mathbb{R}}, k \in K\right)$ is a sequence of complex–valued and normalized fractional Brownian motions, each with the same fixed Hurst parameter $H \in (0, 1)$, i.e. $\beta_k^H = \frac{1}{\sqrt{2}} Re(\beta_k^H) + i \frac{1}{\sqrt{2}} Im(\beta_k^H)$, where $Re(\beta_k^H)$ and $Im(\beta_k^H)$ are independent real–valued and normalized fractional Brownian motions on $\mathbb{R}$, and different β_k^H are independent except $\beta_{-k}^H = (\beta_k^H)^*$.

Mainly interested in the special case when A is the Laplace operator Δ with periodic boundary conditions. Then $\alpha_k = |k|^2$, $k \in K$, and $\mathcal{D}(\Delta) = W^{2,2}([0,1]^2) \cap V$.

Theorem

Suppose Assumption A holds and assume that there is $\epsilon > 0$ such that

$$\sum_{k \in K} \lambda_k \alpha_k^{2(\epsilon - H)} < \infty.$$

Then there exists a unique ergodic mild solution ψ to equation (M2) given by

$$\psi(t) = \sum_{k \in K} \sqrt{\lambda_k} \nu^H \int_{-\infty}^{t} e^{-(t-u)\nu\alpha_k} d\beta_k^H(u) e_k, \ t \in \mathbb{R}.$$

Theorem

Suppose Assumption A holds. Further, assume that there is $m \in \mathbb{N}_0$ and $\gamma \in (0,1)$ such that

$$\sum_{k \in K} \lambda_k \alpha_k^{2\gamma - 2H} |k|^{2m} < \infty \quad \text{and} \quad \sum_{k \in K} \lambda_k \alpha_k^{-2H} |k|^{2m+2\gamma} < \infty.$$

Then there is a unique ergodic mild solution ψ to equation (M2). Further, for all $\delta \in \mathbb{N}_0^2$ with $|\delta| \leq m$ there is a version of $D^\delta \psi$ (again denoted by $D^\delta \psi$) such that

$$D^\delta \psi \in C^\epsilon\left([0,1]^2 \times \mathbb{R}\right)$$

$\mathbb{P}$-a.s. for any $\epsilon \in (0, \min\{\gamma, H\})$. In particular, $D^\delta \psi$ is real-valued.

Statistical properties

Set $A = \Delta$ and thereby $\alpha_k = |k|^2$, $k \in K$.

The *autocovariance function* $R : [0,1]^2 \times [0,1]^2 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{2\times 2}$ of v is given by

$$
\begin{aligned}
R(x,y,t,s) &= \big(\mathbb{E}\big(v_i(x,t)v_j(y,s)\big)\big)_{1\le i,j\le 2} \\
&= \sum_{k\in K} \begin{pmatrix} k_2^2 & -k_2k_1 \\ -k_1k_2 & k_1^2 \end{pmatrix} \lambda_k |k|^{-4H} \delta_k(t-s;H,\nu)\ e_k(x-y),
\end{aligned}
$$

where we set

$$
\delta_k(t-s;H,\nu) := \frac{\Gamma(2H+1)\sin(\pi H)}{\pi} \int_0^\infty \cos((t-s)\nu|k|^2 z)\frac{|z|}{1+z^2}\,dz.
$$

Therefore, v is a *stationary*, *homogeneous* and incompressible mean zero Gaussian random field.

To ensure *isotropy*, we set $\lambda_k := \zeta(|k|)$, $k \in K$, for a suitable positive function $\zeta : [0,\infty) \to [0,\infty)$.

Statistical properties

Recall our main motivation to use fractional noise:

$$\mathbb{E}\left(|v(x,t)-v(x,s)|^2\right) \sim C\ |t-s|^{2H}$$

for $t, s \geq 0$, $x \in [0,1]^2$ and some constant $C > 0$.

Theorem

Suppose Assumption A holds and that there is $m \in \mathbb{N}$ and $\epsilon > 0$ such that $\sum_{k \in K} \lambda_k |k|^{2m+\epsilon} < \infty$.

Then $v = \nabla^{\perp}\psi \in C\left(\mathbb{R}, C^{m-1}([0,1]^2, \mathbb{R}^2)\right)$ $\mathbb{P}$-a.s.. Further, there is a constant $C(H,\nu) > 0$ such that for any $t, s \in \mathbb{R}$ and $x \in [0,1]^2$ we have

$$\mathbb{E}\left(|v(x,t)-v(x,s)|^2\right) \leq C(H,\nu)|t-s|^{2H}$$

and for a fixed $T > 0$ there is a constant $C(H,\nu,T) > 0$ such that for any $t, s \in [-T, T]$ and $x \in [0,1]^2$ we have

$$C(H,\nu,T)|t-s|^{2H} \leq \mathbb{E}\left(|v(x,t)-v(x,s)|^2\right).$$

Statistical properties

Example: To match the Kolmogorov spectrum, i.e.

$$E(|k|) \sim C|k|^{-5/3}$$

and therefore due to Taylor's frozen turbulence hypothesis

$$\mathbb{E}\left(|v(x,t) - v(x,s)|^2\right) \sim C\ |t-s|^{2/3}$$

we set

- $\lambda_k \propto |k|^{-\frac{14}{3}+4H}$
- $H = 1/3$

and by the last theorem $v \in C(\mathbb{R}, C([0,1], \mathbb{R}^2))$ $\mathbb{P}$-a.s..

Simulation

Suppose that Assumption A with

- $A = \triangle$ and thereby $\alpha_k = |k|^2$, $k \in K = 2\pi\mathbb{Z}^2 \setminus \{(0,0)\}$ and
- $\lambda_k = \begin{cases} \xi(|k|) & \text{if } |k| \leq 2\pi R \\ 0 & \text{else} \end{cases}, k \in K,$

for

- a fixed $R \in \mathbb{N}$ and
- a *suitable function* $\xi : [0, \infty) \to [0, \infty)$

to obtain an isotropic random velocity field v with a desired energy spectrum. We have

- $v \in C(\mathbb{R}, C^\infty([0,1]^2, \mathbb{R}^2))$ $\mathbb{P}$-a.s. and
- $v(x,t) = \sum_{\substack{k \in K, \\ |k| \leq 2\pi R}} i\binom{k_2}{-k_1}\hat{\psi}_k(t)e^{i<k,x>}, \quad x \in [0,1]^2, t \in \mathbb{R},$

Simulation

where

$$
\begin{aligned}
\hat{\psi}_k(t) &:= \sqrt{\lambda_k}\nu^H \int_{-\infty}^{t} e^{-(t-u)\nu\alpha_{\boldsymbol{k}}} d\beta_k^H(u) \\
&= \sqrt{\frac{\lambda_k}{2}}\nu^H \int_{-\infty}^{t} e^{-(t-u)\nu\alpha_{\boldsymbol{k}}} dRe(\beta_k^H)(u) + i\sqrt{\frac{\lambda_k}{2}}\nu^H \int_{-\infty}^{t} e^{-(t-u)\nu\alpha_{\boldsymbol{k}}} dIm(\beta_k^H)(u) \\
&=: \hat{\psi}_{k,Re}(t) + i\hat{\psi}_{k,Im}(t).
\end{aligned}
$$

Question:

- How to simulate the real-valued stationary fractional Ornstein-Uhlenbeck (fOU) processes $\hat{\psi}_{k,Re}$, $\hat{\psi}_{k,Im}$?

Then (not in this talk) we can evaluate the series

$$
v_1(x,t) = \sum_{\substack{\boldsymbol{k}\in\boldsymbol{K}, \\ |\boldsymbol{k}|\le 2\pi\boldsymbol{R}}} ik_2\hat{\psi}_k(t)e^{i<k,x>},
$$

$$
v_2(x,t) = \sum_{\substack{\boldsymbol{k}\in\boldsymbol{K}, \\ |\boldsymbol{k}|\le 2\pi\boldsymbol{R}}} (-i)k_1\hat{\psi}_k(t)e^{i<k,x>}
$$

efficiently using fast Fourier transform (FFT) algorithm.

Simulation

Let $\lambda, \alpha > 0$. To simulate the real-valued stationary fOU process

$$X(t) = \sqrt{\lambda} \int_{-\infty}^{t} e^{-(t-u)\alpha} d\beta^H(u) \quad , t \in \mathbb{R},$$

we introduce methods based on the **covariance function** of $(X(t))_{t \in \mathbb{R}}$

$$\begin{aligned} r(s) &:= \mathbb{E}\big(X(0)X(s)\big) \\ &= \frac{\lambda}{\alpha^{2H}} \frac{\Gamma(2H+1)\sin(\pi H)}{\pi} \int_0^\infty \cos(s\alpha x) \frac{x^{1-2H}}{1+x^2} dx \\ &= \frac{\lambda}{2\alpha^{2H}} \cosh(\alpha s)\Gamma(1+2H) - \frac{\lambda}{2}|s|^{2H} {}_1F_2\Big(1; H+\frac{1}{2}, H+1; \frac{\alpha^2 s^2}{4}\Big), \end{aligned}$$

and the (not normalized) **spectral density** $f : D_H \subset \mathbb{R} \to \mathbb{R}$

$$x \mapsto f(x) = \frac{\lambda}{\alpha^{2H}} \frac{\Gamma(2H+1)\sin(\pi H)}{2\pi} \frac{|x|^{1-2H}}{1+x^2}$$

where $\Gamma(\cdot)$ is the Gamma function, $\cosh(\cdot)$ the hyperbolic cosine, ${}_1F_2$ the generalized hypergeometric function ${}_1F_2$ and $D_H = \mathbb{R} \setminus \{0\}$ if $H > 1/2$ and $D_H = \mathbb{R}$ else.

Remark: $X(t)_{t \in \mathbb{R}}$ **short rate dependent**, i.e. $\sum_{n \in \mathbb{N}} |r(n)| < \infty$, for $H \leq 1/2$ and **long range dependent**, i.e. $\sum_{n \in \mathbb{N}} |r(n)| = \infty$, for $H > 1/2$.

Standard Cholesky method

We fix $\triangle t > 0$, $n \in \mathbb{N}$ and set $r_n := r(n\triangle t)$ and $X_n := X_{n\triangle t}$. Further, we denote by

$$C_n := (c_{ij})_{i,j=0,1,\dots,n} := \begin{pmatrix} r_0 & r_1 & \cdots & r_n \\ r_1 & r_0 & \cdots & r_{n-1} \\ \vdots & & \ddots & \vdots \\ r_n & r_{n-1} & \cdots & r_0 \end{pmatrix}$$

the covariance matrix of $\mathbf{X}(n) := (X_0, X_1, \dots, X_n)'$.

C_n admits a *Cholesky decomposition*

$$C_n = G_n G_n',$$

where $G_n := (g_{ij})_{i,j=0,1,\dots,n}$ is square lower triangular.

Let$Z_0, Z_1, \dots, Z_n$ be iid standard normal random variables. Define the $\mathbf{Y}(n) := (Y_0, Y_1, \dots, Y_n)'$ by

$$Y_i := \sum_{k=0}^{i} g_{ik} Z_k \quad , i = 0, 1, \dots, n.$$

We have $\mathbf{Y}(n) \stackrel{d}{=} \mathbf{X}(n)$!

Durbin-Levinson method

Idea: Generate recursively X_{n+1} given $X_0, X_1, \dots, X_n$.

We have

$$X_{n+1}|(X_0, X_1, \dots, X_n) \sim \mathcal{N}(\mu_{n+1}, \sigma^2_{n+1}),$$

where

$$\mu_{n+1} := (C_n^{-1} J_n r_{1:n+1})'(X_0, X_1, \dots, X_n)', \quad \mu_0 := 0,$$

$$\sigma^2_{n+1} := r_0 - (J_n r_{1:n+1})' C_n^{-1} J_n r_{1:n+1}, \quad \sigma^2_0 := r_0,$$

$J_n = (j_{lk})_{l,k=0,1,\dots,n}$ denotes the *exchange matrix* with ones on the antidiagonal and $r_{1:n+1} := (r_1, r_2, \dots, r_{n+1})'$.

The *Durbin-Levinson (DB) algorithm* computes recursively the Cholesky decomposition

$$C_n^{-1} = L_n' D_n^{-1} L_n,$$

together with σ^2_n and $C_n^{-1} J_n r_{1:n+1}$ and with that also μ_n, where $L_n = (l_{ij})_{i,j=0,1,\dots,n}$ is unit lower triangular and $D_n = (d_{ij})_{ij=0,1,\dots,n}$ is diagonal with $d_{nn} = \sigma^2_n$.

Advantage: DB algorithm is an *exact* method with complexity of $\mathcal{O}(n^2)$.

Spectral approximate method

We have for $p \in \{0, 1, \dots, n\}$

$$X_p \overset{d}{=} \int_0^\pi \sqrt{\frac{f_{\Delta t}(u)}{\pi}} \cos(pu) dW_1(u) - \int_0^\pi \sqrt{\frac{f_{\Delta t}(u)}{\pi}} \sin(pu) dW_2(u) =: Y_p,$$

where $f_{\Delta t}(x) := f(x/\Delta t)/\Delta t$ and W_1, W_2 are two real-valued independent standard Brownian motions.

Let $l \in \mathbb{N}$, set $s_k = \pi k/l$ for $k = 0, 1, \dots, l-1$ define the simple function

$$\xi_p^{(l)}(u) := \sqrt{\frac{f_{\Delta t}(s_1)}{\pi}} \cos(ps_1) \mathbf{1}_{\{0\}}(u) + \sum_{k=0}^{l-1} \sqrt{\frac{f_{\Delta t}(s_{k+1})}{\pi}} \cos(ps_{k+1}) \mathbf{1}_{(s_k, s_{k+1}]}(u), \quad u \in [0, \pi].$$

Let $\theta_p^{(l)}$ be defined as $\xi_p^{(l)}$ but with the cosine terms replaced by sine terms.

Spectral approximate method

Then

$$Y_p^{(l)} := \int_0^\pi \xi_p^{(l)}(u)dW_1(u) - \int_0^\pi \theta_p^{(l)}(u)dW_2(u)$$
$$\stackrel{d}{=} \sum_{k=0}^{l-1} \sqrt{\frac{f_{\Delta t}(s_{k+1})}{l}} \{\cos(ps_{k+1})Z_k^{(0)} - \sin(pS_{k+1})Z_k^{(1)}\} =: X_p^{(l)}, \tag{5}$$

where $Z_k^{(0)}, Z_k^{(1)}$, $k = 0, 1, \ldots, l-1$, are iid standard normal random variables.

Advantage: RHS of (5) can be evaluated efficiently using **FFT** (with $n := N - 1$ and $l := N/2$ where $N = 2^m$ for some $m \in \mathbb{N}$).

Fact: $Y_p^{(l)}$ converges in $(L)^r$, $r \geq 1$, to Y_p and therefore $\mathbf{X}^{(l)}(n) := (X_0^{(l)}, X_1^{(l)}, \ldots, X_n^{(l)})$ converges in distribution to $\mathbf{X}(n)$ for $l \to \infty$.

We can prove: $\mathbb{E}(|Y_p^{(l)} - Y_p|^2)^{1/2} \geq Cl^{H-1}$ for some $C > 0$.

Drawback: The rate of convergence is quite slow and for $H > 1/2$ we approximate a long range dependent process by a short range dependent

Thank you for your attention!