Time global stability of the Lax-Friedrichs scheme

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Introduction

Let \( c, h(c) \in \mathbb{R} \) be given constants and consider

\[
\begin{align*}
(\text{CL}) & \quad \begin{cases} 
  u_t + H(x, t, c + u)_x = 0 \text{ in } \mathbb{T} \times (0, T], \\
  u(x, 0) = u^0(x) \in L^\infty(\mathbb{T}) \text{ on } \mathbb{T} := \mathbb{R}/\mathbb{Z}, \\
  \int_{\mathbb{T}} u^0 dx = 0, \quad ||u^0||_{L^\infty} \leq r.
\end{cases} \\
(\text{HJ}) & \quad \begin{cases} 
  v_t + H(x, t, c + v_x) = h(c) \text{ in } \mathbb{T} \times (0, T], \\
  v(x, 0) = v^0(x) \in \text{Lip}(\mathbb{T}) \text{ on } \mathbb{T}, \quad ||v^0||_{L^\infty} \leq r.
\end{cases}
\end{align*}
\]

If \( u^0 = v^0_x \), then

\[ \exists u \in C^0((0, T]; L^\infty): \text{entropy sol.}, \]

\[ \exists v \in \text{Lip}(\mathbb{T} \times [0, T]): \text{viscosity sol. s.t. } v_x = u. \]

(\text{CL}) is well-studied with the variational structures of (\text{HJ}).
Consider discretization of (CL) and (HJ) by the L-F scheme:

$$(CL)_\Delta \quad \begin{cases} 
D_t u_{m+1}^{k+1} + D_x H(x_{m+2}, t_k, c + u_{m+2}^k) = 0 \\
u_{m\pm2N}^k = v_m^k, \quad u_{m}^0 = u_\Delta^0(x_m). 
\end{cases}$$

$$(HJ)_\Delta \quad \begin{cases} 
D_t v_{m}^{k+1} + H(x_m, t_k, c + D_x v_{m+1}^k) = h(c) \\
v_{m+1\pm2N}^k = v_{m+1}^k, \quad v_{m+1}^0 = v_\Delta^0(x_{m+1}). 
\end{cases}$$

\[
D_t v_{m}^{k+1} := \frac{v_{m+1}^{k+1} - v_{m-1}^{k+1} + v_{m+1}^k}{2\Delta t}, \quad D_x v_{m+1}^k := \frac{v_{m+1}^k - v_{m-1}^k}{2\Delta x} \tag{Lax-Friedrichs scheme}
\]

$$(CL)_\Delta$$ and $$(HJ)_\Delta$$ are equivalent: $u_{m}^k = D_x v_{m+1}^k$, if $u_{0}^0 = v_{x}^0$.

We want to show under hyperbolic scaling $0 < \lambda_0 \leq \lambda = \Delta t/\Delta x < \lambda_1$

- Stability of $$(CL)_\Delta$$ for $k \to \infty$ with fixed $\Delta = (\Delta x, \Delta t)$,
- Error estimate between $u_{m}^k$ and $u$ for $\Delta \to 0$,
- Asymptotic behavior of $$(CL)_\Delta$$ for $k \to \infty$ with fixed $\Delta$. 
Known Results on \((\text{CL})_{\Delta}\)

(i) The case of \(H(x,t,p) = f(p)\)

Huge literature on
- time global stability,
- \(L^1\)-convergence,
- \(L^1\)-error estimates,
- long time behavior.

(ii) The case of \(H(x,t,p) = f(p) + F(x,t)\)

[T] Takeno ('01): stability within \([0, 1]\) for existence of periodic sol.
[NS] Nishida-S ('12): time global stability, \(L^1\)-convergence, long time behavior.

(iii) The case of \(H(x,t,p)\)

[O] Oleinik ('57): stability within a restricted time interval, \(L^1\)-convergence.

- All, except [S], are based on \(L^1\)-framework.
- [S] is based on stochastic and variational methods with the theory of viscosity sol. of Hamilton-Jacobi eq.
Main Results
Suppose that the $C^2$-flux function $H(x, t, p) : T^2 \times \mathbb{R} \to \mathbb{R}$ satisfies
$$H_{pp} > 0, \quad \lim_{|p| \to \infty} \frac{H(x, t, p)}{|p|} = \infty, \quad |L_x| \leq \alpha(1 + |L|) \ (L := H^*).$$

$u_\Delta$: the step function given by $u^k_m$,
$v_\Delta$: the linear interpolation of $v^k_{m+1}$.

**Thm.** $\exists \lambda_1 > 0$ such that if $0 < \lambda_0 \leq \lambda := \Delta t / \Delta x < \lambda_1$, the following hold:

1. The L-F scheme $(CL)_\Delta$ is globally stable: For all $k \geq 0$ and $m$
   $$|H_p(x_m, t_k, c + u^k_m)| \leq \lambda_1^{-1} < \lambda^{-1} \ (\text{CFL-condition}).$$

2. For each $t > 0$, $\exists \alpha(t) > 0$ independent of $\Delta x, \Delta t$ and initial datas s.t. $\| u_\Delta(\cdot, t) - u(\cdot, t) \|_{L^1(T)} \leq \alpha(t) \Delta x^4$.

3. For each $c$, $\exists! \bar{u}^c_\Delta$: periodic difference entropy sol., which absorbs any other difference entropy sol. exponentially as $t \to \infty$. 
Idea of Proof.

1. Use variational techniques, not usual $L^1$-framework,
2. Convert the equation into the Hamilton-Jacobi type with the theory of viscosity solutions,
3. Use Variational characterization of (CL), (HJ), (CL)$_\Delta$, (HJ)$_\Delta$: Let $L_c := L(x, t, \xi) - c\xi$.
   \[
   (1) \quad v(x, t) = \inf_{\gamma \in AC, \gamma(t) = x} \left[ \int_0^t L_c(\gamma(s), s, \gamma'(s))ds + v^0(\gamma(0)) \right] + h(c)t,
   \]
   (2) If $(x, t)$: regular point of $v$ (i.e. $\exists v_x(x, t)$) and $\gamma^*$: minimizer,
   \[
   u(x, t) = \int_0^t L_c(\gamma^*(s), s, \gamma^*(s))ds + u^0(\gamma^*(0)),
   \]
   (3) Stochastic variational problems for $v_\Delta, u_\Delta ([S])$.
4. Necessary boundedness is derived from calculus of variations,
5. Due to periodic setting, iteration of time-1 analysis yields time-global properties.
Stability

- Due to calculus of variations, velocity of every minimizing curve $\gamma^*$ for $v(\cdot, t)$ is uniformly bounded: For each $t > 0$
  \[ |\gamma^*(t)| \leq \beta_1(t). \quad (\beta_1(t) \text{ is indep. of } r, \| v_x^0 \|_{L^\infty} \leq r) \]
- Since $u(x, t) = v_x(x, t) = L^c_{\xi}(x, t, \gamma^*(t))$ a.e. $x \in \mathbb{T}$, $u$ is uniformly bounded: For each $t > 0$
  \[ \| u(\cdot, t) \|_{L^\infty} = \| v_x(\cdot, t) \|_{L^\infty} \leq \beta_2(t). \]
- $u_{\Delta}(\cdot, 1) \to u(\cdot, 1) (\Delta \to 0)$ in $L^1(\mathbb{T})$ is uniform w.r.t. initial datas bounded by $r$ ($\| u^0 \|_{L^\infty} \leq r$).
- $\exists E(t) > 0$ indep. of initial datas bounded by $r$ s.t.
  \[ \frac{u_{\Delta}(x, t) - u_{\Delta}(y, t)}{x - y} \leq E(t). \quad \text{(entropy condition)} \]
- By [S], stability holds in $[0, 1]$ for all initial datas bounded by $r$.

Therefore, $\exists \delta > 0$ s.t. if $\Delta x, \Delta t \leq \delta$
\[ \| u_{\Delta}(\cdot, 1) \|_{L^\infty} \leq \beta_2(1) + 1. \]
Otherwise the entropy condition is violated. Take $r \geq \beta_2(1) + 1$. 

**Error Estimate**

- Due to stochastic and variational approach \([S]\)
  \[
  \| v_{\Delta}(\cdot, t) - v(\cdot, t) \|_{C^0} \leq \alpha_1(t) \sqrt{\Delta x}.
  \]

- Due to the entropy condition, \( u_{\Delta}(\cdot, t), u(\cdot, t) \) is B.V. In particular for \( w(\cdot) := u_{\Delta}(\cdot, t) - u(\cdot, t) \)
  \[
  \sum_j |w(x_j) - w(x_{j-1})| \leq 4E(t).
  \]

- Let \( 0 = x_0 < x_1 < x_2 < \cdots < x_n = 1 \) be s.t. \( w(\cdot) \geq 0 \) on \( I_{2j} = [x_{2j}, x_{2j+1}] \) and \( < 0 \) on \( I_{2j+1} \). Define

  \[
  J := \{ j \mid \max\{|I_{2j}|, |I_{2j+1}|\} \leq \Delta x^{\frac{1}{4}} \}, \quad \tilde{J} := \{ j \} \setminus J. \quad (\#\tilde{J} \leq \Delta x^{-\frac{1}{4}})
  \]

Then

\[
\| u_{\Delta}(\cdot, t) - u(\cdot, t) \|_{L^1(T)} \leq \sum_{j \in J} \max_{x \in I_{2j}} w(x)|I_{2j}| - \min_{x \in I_{2j+1}} w(x)|I_{2j+1}|
\]
\[
+ \sum_{j \in \tilde{J}} \int_{I_{2j}} w(x)dx - \int_{I_{2j+1}} w(x)dx
\]
\[
\leq 4E(t)\Delta x^{\frac{1}{4}} + \#\tilde{J} \cdot 4\alpha_1(t)\sqrt{\Delta x} \leq \alpha(t)\Delta x^{\frac{1}{4}}.
\]
Long Time Behavior

- Initial datas belong to $L^\infty_r(\mathbb{T}) := \{ u^0 \in L^\infty(\mathbb{T}) \mid \| u^0 \|_{L^\infty} \leq r \}$.

- Take $r \geq \beta_2(t) + 1$. Then the time-1 map of $(\mathcal{C}\mathcal{L})_\Delta$ is 
  \[ \phi^1_\Delta(\cdot; c) : L^\infty_r(\mathbb{T}) \ni u^0_\Delta \mapsto u_\Delta(\cdot, 1) \in L^\infty_r(\mathbb{T}). \]

- $\phi^1_\Delta$ is essentially a map from $B_r(0) \subset \mathbb{R}^N$ into itself and a fixed point $\bar{u}^0_\Delta$ exists.

- $\bar{u}^c_\Delta(\cdot, t) := \phi^t_\Delta(\bar{u}^0_\Delta; c)$ is a periodic difference entropy sol. 

- $\phi^t_\Delta(\cdot; c)$ is strictly contractive.

Therefore $\bar{u}^c_\Delta$ is unique w.r.t. $c$ and absorbs any other sol.