Time global stability of the Lax-Friedrichs scheme

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Introduction

Let $c, h(c) \in \mathbb{R}$ be given constants and consider

$$\text{(CL)} \quad \begin{cases} u_t + H(x,t,c+u)_x = 0 \text{ in } \mathbb{T} \times (0,T], \\ u(x,0) = u^0(x) \in L^\infty(\mathbb{T}) \text{ on } \mathbb{T} := \mathbb{R}/\mathbb{Z}, \\ \int_{\mathbb{T}} u^0 dx = 0, \ \| u^0 \|_{L^\infty} \le r. \end{cases}$$

$$\text{(HJ)} \quad \begin{cases} v_t + H(x,t,c+v_x) = h(c) \text{ in } \mathbb{T} \times (0,T], \\ v(x,0) = v^0(x) \in Lip(\mathbb{T}) \text{ on } \mathbb{T}, \ \| v^0_x \|_{L^\infty} \le r. \end{cases}$$

If $u^0 = v^0_x$, then

$\exists /\, u \in C^0((0,T]; L^\infty)$: entropy sol.,

$\exists /\, v \in Lip(\mathbb{T} \times [0,T])$: viscosity sol. s.t. $v_x = u$.

(CL) is well-studied with the variational structures of (HJ).

Consider discretization of (CL) and (HJ) by the L-F scheme:

$$(\mathrm{CL})_\Delta \quad \begin{cases} D_t u^{k+1}_{m+1} + D_x H(x_{m+2}, t_k, c + u^k_{m+2}) = 0 \\ u^k_{m\pm 2N} = v^k_m, \quad u^0_m = u^0_\Delta(x_m). \end{cases}$$

$$(\mathrm{HJ})_\Delta \quad \begin{cases} D_t v^{k+1}_m + H(x_m, t_k, c + D_x v^k_{m+1}) = h(c) \\ v^k_{m+1\pm 2N} = v^k_{m+1}, \quad v^0_{m+1} = v^0_\Delta(x_{m+1}). \end{cases}$$

$D_t v^{k+1}_m := \dfrac{v^{k+1}_m - \frac{v^k_{m-1} + v^k_{m+1}}{2}}{\Delta t}$, $D_x v^k_{m+1} := \dfrac{v^k_{m+1} - v^l_{m-1}}{2\Delta x}$ (Lax-Friedrichs scheme)

$(\mathrm{CL})_\Delta$ and $(\mathrm{HJ})_\Delta$ are equivalent: $u^k_m = D_x v^k_{m+1}$, if $u^0 = v^0_x$.

We want to show under hyperbolic scaling $0 < \lambda_0 \le \lambda = \Delta t/\Delta x < \lambda_1$

- Stability of $(\mathrm{CL})_\Delta$ for $k \to \infty$ with fixed $\Delta = (\Delta x, \Delta t)$,
- Error estimate between u^k_m and u for $\Delta \to 0$,
- Asymptotic behavior of $(\mathrm{CL})_\Delta$ for $k \to \infty$ with fixed Δ.

Known Results on $(\mathrm{CL})_\triangle$

(i) The case of $H(x,t,p)=f(p)$

Huge literature on

time global stability, L^1-convergence,
L^1-error estimates, long time behavior.

(ii) The case of $H(x,t,p)=f(p)+F(x,t)$

[T] Takeno ('01): stability within $[0,1]$ for existence of periodic sol.

[NS] Nishida-S ('12): time global stability, L^1-convergence, long time behavior.

(iii) The case of $H(x,t,p)$

[O] Oleinik ('57): stability within a restricted time interval, L^1-convergence.

[S] S (submitted): stability within arbitrary time intervals, pointwise convergence, approximation of characteristic curves.

- All, except [S], are based on L^1-framework.
- [S] is based on stochastic and variational methods with the theory of viscosity sol. of Hamilton-Jacobi eq.

Main Results

Suppose that the C^2-flux function $H(x,t,p) : \mathbb{T}^2 \times \mathbb{R} \to \mathbb{R}$ satisfies

$$H_{pp} > 0, \quad \lim_{|p|\to\infty} \frac{H(x,t,p)}{|p|} = \infty, \quad |L_x| \leq \alpha(1 + |L|) \ (L := H^*).$$

u_Δ: the step function given by u^k_m,
v_Δ: the linear interpolation of v^k_{m+1}.

Thm. $\exists\, \lambda_1 > 0$ such that if $0 < \lambda_0 \leq \lambda := \Delta t/\Delta x < \lambda_1$, the following hold:

1. The L-F scheme $(\mathrm{CL})_\Delta$ is globally stable: For all $k \geq 0$ and m

$$|H_p(x_m, t_k, c + u^k_m)| \leq \lambda_1^{-1} < \lambda^{-1} \text{ (CFL-condition).}$$

2. For each $t > 0$, $\exists\, \alpha(t) > 0$ independent of $\Delta x, \Delta t$ and initial datas s.t. $\| u_\Delta(\cdot,t) - u(\cdot,t) \|_{L^1(\mathbb{T})} \leq \alpha(t)\Delta x^{\frac{1}{4}}$.

3. For each c, $\exists!\, \bar{u}^c_\Delta$: periodic difference entropy sol., which absorbs any other difference entropy sol. exponentially as $t \to \infty$.

Idea of Proof.

1. Use variational techniques, not usual L^1-framework,
2. Convert the equation into the Hamilton-Jacobi type with the theory of viscosity solutions,
3. Use Variational characterization of (CL), (HJ), (CL)$_\triangle$, (HJ)$_\triangle$: Let $L^c := L(x,t,\xi) - c\xi$.

   (1) $v(x,t) = \inf_{\gamma \in AC, \gamma(t)=x} \Big[\int_0^t L^c(\gamma(s), s, \gamma'(s))ds + v^0(\gamma(0)) \Big] + h(c)t,$

   (2) If (x,t): regular point of v (i.e. $\exists\, v_x(x,t)$) and γ^*: minimizer,

$$u(x,t) = \int_0^t L^c_x(\gamma^*(s), s, \gamma^{*\prime}(s))ds + u^0(\gamma^*(0)),$$

   (3) Stochastic variational problems for $v_\triangle$, $u_\triangle$ ([S]).
4. Necessary boundedness is derived from calculus of variations,
5. Due to periodic setting, iteration of time-1 analysis yields time-global properties.

Stability

- Due to calculus of variations, velocity of every minimizing curve γ^* for $v(\cdot,t)$ is uniformly bounded: For each $t>0$

$$|\gamma^{*\prime}(t)| \le \beta_1(t). \quad (\beta_1(t) \text{ is indep. of } r,\ \| v_x^0 \|_{L^\infty} \le r)$$

- Since $u(x,t) = v_x(x,t) = L^c_\xi(x,t,\gamma^{*\prime}(t))$ a.e. $x \in \mathbb{T}$, u is uniformly bounded: For each $t>0$

$$\| u(\cdot,t) \|_{L^\infty} = \| v_x(\cdot,t) \|_{L^\infty} \le \beta_2(t).$$

- $u_\Delta(\cdot,1) \to u(\cdot,1)$ $(\Delta \to 0)$ in $L^1(\mathbb{T})$ is uniform w.r.t. initial datas bounded by r ($\| u^0 \|_{L^\infty} \le r$).

- $\exists\, E(t) > 0$ indep. of initial datas bounded by r s.t.

$$\frac{u_\Delta(x,t) - u_\Delta(y,t)}{x-y} \le E(t). \quad (\text{entropy condition})$$

- By [S], stability holds in $[0,1]$ for all initial datas bounded by r.

Therefore, $\exists\, \delta > 0$ s.t. if $\Delta x, \Delta t \le \delta$

$$\| u_\Delta(\cdot,1) \|_{L^\infty} \le \beta_2(1) + 1.$$

Otherwise the entropy condition is violated. Take $r \ge \beta_2(1) + 1$.

Error Estimate

- Due to stochastic and variational approach [S]

$$\| v_\Delta(\cdot,t) - v(\cdot,t) \|_{C^0} \le \alpha_1(t)\sqrt{\Delta x}.$$

- Due to the entropy condition, $u_\Delta(\cdot,t), u(\cdot,t)$ is B.V. In Particular for $w(\cdot) := u_\Delta(\cdot,t) - u(\cdot,t)$

$$\sum_j |w(x_j) - w(x_{j-1})| \le 4E(t).$$

- Let $0 = x_0 < x_1 < x_2 < \cdots < x_n = 1$ be s.t. $w(\cdot) \ge 0$ on $I_{2j} = [x_{2j}, x_{2j+1}]$ and < 0 on I_{2j+1}. Define

$$J := \{j \mid \max\{|I_{2j}|, |I_{2j+1}|\} \le \Delta x^{\frac{1}{4}}\},\ \tilde{J} := \{j\} \setminus J.\ (\sharp\tilde{J} \le \Delta x^{-\frac{1}{4}})$$

Then

$$\begin{aligned}
\| u_\Delta(\cdot,t) - u(\cdot,t) \|_{L^1(\mathbb{T})} &\le \sum_{j\in J} \max_{x\in I_{2j}} w(x)|I_{2j}| - \min_{x\in I_{2j+1}} w(x)|I_{2j+1}| \\
&\quad + \sum_{j\in\tilde{J}} \int_{I_{2j}} w(x)dx - \int_{I_{2j+1}} w(x)dx \\
&\le 4E(t)\Delta x^{\frac{1}{4}} + \sharp\tilde{J}\cdot 4\alpha_1(t)\sqrt{\Delta x} \le \alpha(t)\Delta x^{\frac{1}{4}}.
\end{aligned}$$

Long Time Behavior

- Initial datas belong to $L^\infty_r(\mathbb{T}) := \{u^0 \in L^\infty(\mathbb{T}) \mid \| u^0 \|_{L^\infty} \le r\}$.

- Take $r \ge \beta_2(t) + 1$. Then the time-1 map of $(\mathrm{CL})_\Delta$ is

$$\phi^1_\Delta(\cdot; c) : L^\infty_r(\mathbb{T}) \ni u^0_\Delta \mapsto u_\Delta(\cdot, 1) \in L^\infty_r(\mathbb{T}).$$

- ϕ^1_Δ is essentially a map from $B_r(0) \subset \mathbb{R}^N$ into itself and a fixed point $\bar{u}^0_\Delta$ exists.

- $\bar{u}^c_\Delta(\cdot, t) := \phi^t_\Delta(\bar{u}^0_\Delta; c)$ is a periodic difference entropy sol.

- $\phi^t_\Delta(\cdot; c)$ is strictly contractive.

Therefore $\bar{u}^c_\Delta$ is unique w.r.t. c and absorbs any other sol.