Dirac Concentrations in Lotka-Volterra parabolic PDE

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Stating the problem

Main questions

Main result

References

- Benoit Perthame and Guy Barles, *Dirac Concentrations in Lotka-Volterra parabolic PDE*, Indiana University Mathematics Journal, vol. 57, No. 7, 2008
- Benoit Perthame, Guy Barles and Sepideh Mirrahimi, *Concentrations in Lotka-Volterra parabolic or integral equations: a general convergence result*, Method. Appl. Ana., Volume 16, No. 3, 2009

Introduction

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- Darwin's Law

"In the struggle for survival, the fittest win out at the expense of their rivals because they succeed in adapting themselves best to their environment."

Introduction

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- ## Darwin's Law

  "In the struggle for survival, the fittest win out at the expense of their rivals because they succeed in adapting themselves best to their environment."

- ## Question

  Is it possible to support Darwin's Law mathematically?

Stating the problem

$$\begin{cases} \frac{\partial}{\partial t} n_\varepsilon(t,x) - \varepsilon \Delta n_\varepsilon(t,x) = \frac{n_\varepsilon(t,x)}{\varepsilon} R(x, I_\varepsilon(t)), & in \quad \mathbb{R}_+ \times \mathbb{R}^d \\ n_\varepsilon(0,x) = n_\varepsilon^0(x),\ n_\varepsilon^0(x) \geq 0, & on \quad \{t=0\} \times \mathbb{R}^d, \end{cases}$$

where $I_\varepsilon(t) = \int\limits_{\mathbb{R}^d} n_\varepsilon(t,x)\psi(x)\mathrm{d}x$.

$n_\varepsilon(t,x)$: density of species x at time t

$R(x, I_\varepsilon(t))$: growth rate of species x

$I_\varepsilon(t)$: enviromental state (nutrition available at time t)

$\psi(x)$: predation of species x

ε : scaling such that mutation is small

Assumptions

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$$\psi \in W^{2,\infty}(\mathbb{R}^d),\ 0 < \psi_m \leq \psi \leq \psi_M < \infty$$

$$\exists I_m, I_M :\ 0 < I_m \leq I_M < \infty \text{ and } \min_{x\in\mathbb{R}^d} R(x, I_m) = 0,\ \max_{x\in\mathbb{R}^d} R(x, I_M) = 0$$

$$\exists K > 0 \forall x \in \mathbb{R}^d, I \in \mathbb{R} :\ -K \leq \frac{\partial R}{\partial I} \leq -\frac{1}{K} < 0,\ \sup_{\frac{I_m}{2} \leq I \leq 2I_M} \|R(\cdot, I)\|_{W^{2,\infty}} \leq K$$

$$n_\varepsilon^0 \in L^\infty(\mathbb{R}^d),\ \nabla n_\varepsilon^0 \in L^1(\mathbb{R}^d) \text{ and } I_m \leq \int_{\mathbb{R}^d} \psi(x) n_\varepsilon^0(x) \mathrm{d}x \leq I_M$$

Existence result

Theorem

Assume additionally $I_m - C\varepsilon^2 \leq I_\varepsilon(0) \leq I_M + C\varepsilon^2$*, then there exists a unique weak solution* $n_\varepsilon \in C(\mathbb{R}_+, L^1(\mathbb{R}^d))$ *to the equation above, which satisfies* $I_m - C\varepsilon^2 \leq I_\varepsilon(t) \leq I_M + C\varepsilon^2$*.*

Proof.

Banach's fixed point theorem + iteration in time □

Main questions

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1. What happens if mutation vanishes, i.e. $\varepsilon \to 0$?

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3. Does this new equation have a solution?

Main questions

1. What happens if mutation vanishes, i.e. $\varepsilon \to 0$?
2. What equation do we obtain?
3. Does this new equation have a solution?
4. What structure does this solution have?

Ansatz

We make the following ansatz. We set

$$\varphi_\varepsilon(t, x) = \varepsilon \log(n_\varepsilon(t, x)),$$

which is equivalent to

$$n_\varepsilon(t, x) = e^{\frac{\varphi_\varepsilon(t,x)}{\varepsilon}}.$$

Heuristic approach

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Suppose

$$\varphi_\varepsilon(t,x) \leq 0 \ \forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d,$$

then we can obtain the following limit

$$\lim_{\varepsilon \to 0} n_\varepsilon(t,x) = \lim_{\varepsilon \to 0} e^{\frac{\varphi_\varepsilon(t,x)}{\varepsilon}} = \begin{cases} 1, & \varphi_\varepsilon(t,x) = 0 \\ 0, & else \end{cases} = \sum_i \delta\left(x - x_i(t)\right)$$

Transformed equation

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Inserting $n_\varepsilon(t,x) = e^{\frac{\varphi_\varepsilon(t,x)}{\varepsilon}}$ into the equation yields

$$\begin{cases} \frac{\partial}{\partial t}\varphi_\varepsilon(t,x) = \left|\nabla\varphi_\varepsilon(t,x)\right|^2 + R(x, I_\varepsilon(t)) + \varepsilon\Delta\varphi_\varepsilon(t,x), & \textit{in} \quad \mathbb{R}_+ \times \mathbb{R}^d, \\ \varphi_\varepsilon(0,x) = \varphi_\varepsilon^0(x) = \varepsilon\log(n_\varepsilon^0(x)), & \textit{on} \quad \{t=0\} \times \mathbb{R}^d. \end{cases}$$

Viscosity solutions

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Definition

Assume u is bounded and uniformly continuous on $\mathbb{R}^d \times [0, T]$, for each $T > 0$.
We say u is a viscosity solution of the initial value problem

$$\begin{cases} u_t + H(\nabla u, x) = 0, & \text{in} \quad \mathbb{R}^d \times (0, \infty) \\ u = g, & \text{on} \quad \mathbb{R}^d \times \{t = 0\} \end{cases}$$

if

i) $u = g$ on $\mathbb{R}^d \times \{t = 0\}$

ii) for each $v \in C^\infty\left(\mathbb{R}^d \times (0, \infty)\right)$, if $u - v$ has a local maximum at a point $(t_0, x_0) \in \mathbb{R}^d \times (0, \infty)$, then

Viscosity solution

$$v_t(t_0, x_0) + H(\nabla v(t_0, x_0), x_0) \leq 0.$$

and if $u - v$ has a local minimum at a point $(t_0, x_0) \in (0, \infty) \times \mathbb{R}^d$, then

$$v_t(t_0, x_0) + H(\nabla v(t_0, x_0), x_0) \geq 0.$$

Main result

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Theorem

Assume additionally $n_\varepsilon^0(x) \leq e^{\frac{-A|x|+B}{\varepsilon}}$. *Let* $\varphi_\varepsilon(t,x) = \varepsilon \log(n_\varepsilon(t,x))$, *then after extraction of a subsequence,* φ_ε *converges locally uniformly to a Lipschitz-continuous viscosity solution* $\varphi \in C((0,\infty) \times \mathbb{R}^d)$ *of*

$$\begin{cases} \frac{\partial}{\partial t}\varphi(t,x) = \left|\nabla\varphi(t,x)\right|^2 + R(x, I(t)), & \textit{in} \quad \mathbb{R}_+ \times \mathbb{R}^d, \\ \max\limits_{x\in\mathbb{R}^d} \varphi(t,x) = 0, & \\ \varphi(0,x) = \varphi^0(x), & \textit{on} \quad \{t=0\} \times \mathbb{R}^d, \end{cases}$$

where $I_\varepsilon \xrightarrow{\varepsilon\to 0} I$ *a.e. In particular supp* $\{n\} \subset \{\varphi = 0\}$.

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Thank you for your attention!