Time periodic solutions to the Navier-Stokes equations in the rotational framework

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The 5th Japanese-German International Workshop on Mathematical Fluid Dynamics

Waseda University, Tokyo, JAPAN

June 13, 2012

The Navier-Stokes equations with the Coriolis force

$$\text{(NSC)} \begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + \Omega e_3 \times u + (u \cdot \nabla) u + \nabla p = f & \text{in } \mathbb{R}^3_x \times \mathbb{R}_t, \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^3_x \times \mathbb{R}_t, \end{cases}$$

where

$u = (u^1(x,t), u^2(x,t), u^3(x,t))$: velocity filed

$p = p(x,t)$: pressure

$f = (f^1(x,t), f^2(x,t), f^3(x,t))$: time periodic external force

$\Omega \in \mathbb{R}$: the Coriolis parameter

$e_3 := (0, 0, 1)$

Known Results for $\Omega = 0$

The Navier-Stokes equations in unbounded domain

$$\text{(NS)} \begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + (u \cdot \nabla) u + \nabla p = f & \text{in } D \times \mathbb{R}, \\ \operatorname{div} u = 0 & \text{in } D \times \mathbb{R}. \end{cases}$$

Existence of time periodic solutions

- Maremonti ('91) : $D = \mathbb{R}^3, \mathbb{R}^3_+$
- Kozono-Nakao ('96) :

  $D = \mathbb{R}^n, \mathbb{R}^n_+$ $(n \geqslant 3)$ or $D \subset \mathbb{R}^n$: exterior domain $(n \geqslant 4)$
- Yamazaki ('00) :

  $D = \mathbb{R}^n, \mathbb{R}^n_+$ $(n \geqslant 3)$ or $D \subset \mathbb{R}^n$: exterior domain $(n \geqslant 3)$
- Kubo ('05) :

  $D \subset \mathbb{R}^n$: perturbed $\mathbb{R}^n_+$, aperture domain $(n \geqslant 3)$

Integral equations of (NS)

$$u(t) = \int_{-\infty}^{t} e^{(t-s)\Delta}\mathbb{P} f(s)ds - \int_{-\infty}^{t} e^{(t-s)\Delta}\mathbb{P}[(u(s)\cdot\nabla)u(s)]ds$$

Difficulty : **Convergence of time integral on $(-\infty, t)$**

($\Longleftarrow$ The external force f must satisfy decay property)

- The case $\mathbb{R}^n$ $(n \geqslant 4)$:

  $f \in BC(\mathbb{R}; L^p(\mathbb{R}^3) \cap L^l(\mathbb{R}^3))$ *for some* $1 < p, l < \infty$

- The case $\mathbb{R}^3$:

  $f \in BC(\mathbb{R}; \dot{W}^{-1,p}(\mathbb{R}^3) \cap L^l(\mathbb{R}^3))$ *for some* $1 < p, l < \infty$

  (i.e. $f = \operatorname{div} F$ *for some* $F = \{F_{ij}\}_{1\leqslant i,j\leqslant 3} \in L^p(\mathbb{R}^3)$)

$$\left\| e^{(t-s)\Delta}\mathbb{P} \operatorname{div} F(s) \right\|_{L^q} \leqslant C(t-s)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{2}} \|F(s)\|_{L^p}$$

Integral equations for (NSC)

$$u(t) = \int_{-\infty}^{t} T_\Omega(t-s)\mathbb{P} f(s)ds - \int_{-\infty}^{t} T_\Omega(t-s)\mathbb{P}\left[(u(s)\cdot\nabla)u(s)\right]ds$$

where

$$Lu := -\Delta u + \mathbb{P}\Omega e_3 \times u, \quad \mathbb{P} := (\delta_{ij} + R_i R_j)_{1\leqslant i,j\leqslant 3}$$

$$T_\Omega(t)f := e^{-tL} f$$
$$= \mathscr{F}^{-1}\left[\cos\left(\Omega\frac{\xi_3}{|\xi|}t\right)e^{-|\xi|^2 t}\widehat{f}(\xi) + \sin\left(\Omega\frac{\xi_3}{|\xi|}t\right)e^{-|\xi|^2 t}R(\xi)\widehat{f}(\xi)\right]$$

$$R(\xi) := \frac{1}{|\xi|}\begin{pmatrix} 0 & \xi_3 & -\xi_2 \\ -\xi_3 & 0 & \xi_1 \\ \xi_2 & -\xi_1 & 0 \end{pmatrix} \quad \xi \in \mathbb{R}^3 \setminus \{0\}$$

Main Result

Theorem (Main Theorem)

$$\frac{5}{2} < r < 3, \quad 2 < q \leqslant \frac{12}{5}, \quad \frac{1}{3} + \frac{5}{3r} < \frac{1}{p} < 1, \quad \frac{5}{3q} < \frac{1}{l} < \frac{1}{q} + \frac{1}{3}$$

$\Rightarrow \exists C = C(r,q,p,l) > 0$ & $\exists K = K(r,q,p,l) > 0$ *s.t.*

$\forall \Omega \in \mathbb{R} \setminus \{0\}$ & $\forall f \in BC(\mathbb{R}; L^p(\mathbb{R}^3) \cap L^l(\mathbb{R}^3))$ *with*

$$(1) \quad \exists T > 0 \ \ \text{s.t.} \ \ f(t) = f(t+T) \quad \forall t \in \mathbb{R}$$

$$(2) \quad \sup_{t \in \mathbb{R}} \|f(t)\|_{L^p \cap L^l} \leqslant C \min \left\{ |\Omega|^{1+\frac{3}{2r}-\frac{3}{2p}}, |\Omega|^{\frac{1}{2}+\frac{3}{2q}-\frac{3}{2l}} \right\}$$

$\exists! u \in X_K$: *time periodic mild sol. to* (NSC) *with*

$$u(t) = u(t+T) \quad \forall t \in \mathbb{R}$$

where

$$X_K := \left\{ u \in BC(\mathbb{R}; L^r_\sigma(\mathbb{R}^3) \cap \dot{W}^{1,q}(\mathbb{R}^3)) \ \Big| \ \sup_{t \in \mathbb{R}} \|u(t)\|_{L^r \cap \dot{W}^{1,q}} \leqslant K \right\}$$

Main Result

Remark 1

In the case $\Omega = 0$

(Maremonti ('91), Kozono-Nakao ('96), Yamazaki ('00))

- In $\mathbb{R}^n$ $(n \geqslant 4)$:

  $f \in BC(\mathbb{R}; L^p(\mathbb{R}^3) \cap L^l(\mathbb{R}^3))$ *for some* $1 < p, l < \infty$

- In $\mathbb{R}^3$:

  $f \in BC(\mathbb{R}; \dot{W}^{-1,p}(\mathbb{R}^3) \cap L^l(\mathbb{R}^3))$ *for some* $1 < p, l < \infty$

  (i.e. $f = \operatorname{div} F$ *for some* $F = \{F_{ij}\}_{1 \leqslant i,j \leqslant 3} \in L^p(\mathbb{R}^3)$)

In the case $\Omega \in \mathbb{R} \setminus \{0\}$ (Today's talk)

- In $\mathbb{R}^3$:

  $f \in BC(\mathbb{R}; L^p(\mathbb{R}^3) \cap L^l(\mathbb{R}^3))$ *for some* $1 < p, l < \infty$

Main Result

Remark 2

Size condition on the external forece f

$$\sup_{t\in\mathbb{R}} \|f(t)\|_{L^p\cap L^l} \leqslant C \min\left\{|\Omega|^{1+\frac{3}{2r}-\frac{3}{2p}}, |\Omega|^{\frac{1}{2}+\frac{3}{2q}-\frac{3}{2l}}\right\}$$

It follows from our assumptions on (r, q, p, l) that

$$1+\frac{3}{2r}-\frac{3}{2p} > 0 \quad \& \quad \frac{1}{2}+\frac{3}{2q}-\frac{3}{2l} > 0$$

Hence our theorem means that

$\forall f \in BC(\mathbb{R}; L^p(\mathbb{R}^3) \cap L^l(\mathbb{R}^3)), \ \exists\Omega_f > 0$ *s.t.*

$|\Omega| \geqslant \Omega_f \implies \exists! u \in X_K$: *time periodic sol. to* (NSC)

(**Large data global existence**)

Linear Estimates

The operator $\mathscr{G}_{\pm}(\tau)$ ($\tau \in \mathbb{R}$) of oscillatory integral type

$$\mathscr{G}_{\pm}(\tau)[f](x) := \mathscr{F}^{-1}\left[e^{\pm i\tau\frac{\xi_3}{|\xi|}}\mathscr{F}[f]\right](x) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{\pm i\tau\frac{\xi_3}{|\xi|}} e^{ix\cdot\xi}\widehat{f}(\xi)d\xi$$

Decomposition of the semigroup $T_{\Omega}(t)$

$$\begin{aligned} T_{\Omega}(t)f &= \mathscr{F}^{-1}\left[\cos\left(\Omega\frac{\xi_3}{|\xi|}t\right)e^{-|\xi|^2 t}\widehat{f}(\xi) + \sin\left(\Omega\frac{\xi_3}{|\xi|}t\right)e^{-|\xi|^2 t}R(\xi)\widehat{f}(\xi)\right] \\ &= \frac{1}{2}\mathscr{G}_{+}(\Omega t)\left[e^{t\Delta}(I+\mathscr{R})f\right] + \frac{1}{2}\mathscr{G}_{-}(\Omega t)\left[e^{t\Delta}(I-\mathscr{R})f\right] \end{aligned}$$

where

$$\mathscr{R} := \begin{pmatrix} 0 & R_3 & -R_2 \\ -R_3 & 0 & R_1 \\ R_2 & -R_1 & 0 \end{pmatrix}, \quad R_j : \text{the Riesz transform}$$

Linear Estimates

Dispersive estimate for $\mathscr{G}_{\pm}(\tau)$

Lemma (Dispersive estimate)

$2 \leqslant \forall p \leqslant \infty,\ \exists C = C(p) > 0$ *s.t.*

$$\left\| \mathscr{G}_{\pm}(\tau)[f] \right\|_{\dot{B}^{s}_{p,q}} \leqslant C \left\{ \frac{\log(e + |\tau|)}{1 + |\tau|} \right\}^{\frac{1}{2}\left(1 - \frac{2}{p}\right)} \| f \|_{\dot{B}^{s+3(1-\frac{2}{p})}_{p',q}}$$

for $\forall \tau \in \mathbb{R}, \forall s \in \mathbb{R}, 1 \leqslant \forall q \leqslant \infty, \forall f \in \dot{B}^{s+3(1-\frac{2}{p})}_{p',q}(\mathbb{R}^3)$.

[Idea of the proof]

1. $L^1 - L^\infty$ estimates $\Longleftarrow$ stationary phase method
2. $L^2 - L^2$ estimates $\Longleftarrow$ the Plancherel theorem
3. the Riesz-Thorin real interpolation theorem

Linear Estimates

Lemma (Estimate for the external force)

$$\frac{5}{2} < r < 6, \quad \max\left\{1 - \frac{1}{r}, \frac{1}{3} + \frac{5}{3r}\right\} < \frac{1}{p} < \min\left\{1, \frac{2}{3} + \frac{1}{r}\right\}$$

$\Rightarrow \ \exists C = C(r,p) > 0$ *s.t.*

$$\sup_{t\in\mathbb{R}} \left\| \int_{-\infty}^{t} T_{\Omega}(t-s)\mathbb{P}f(s)ds \right\|_{L^r} \leqslant \frac{C}{|\Omega|^{1+\frac{3}{2r}-\frac{3}{2p}}} \sup_{t\in\mathbb{R}} \|f(t)\|_{L^p}$$

for $\forall \Omega \in \mathbb{R}\setminus\{0\}$ & $\forall f \in BC(\mathbb{R}; L^p(\mathbb{R}^3))$

[*Proof*]
It suffices to prove that

$$\sup_{t\in\mathbb{R}} \left\| \int_{-\infty}^{t} \mathcal{G}_{\pm}(\Omega(t-s))e^{(t-s)\Delta}\mathbb{P}f(s)ds \right\|_{L^r} \leqslant \frac{C}{|\Omega|^{1+\frac{3}{2r}-\frac{3}{2p}}} \sup_{t\in\mathbb{R}} \|f(t)\|_{L^p}.$$

Linear Estimates

By $\dot{B}^0_{r,2}(\mathbb{R}^3) \hookrightarrow L^r(\mathbb{R}^3)$ and the dispersive estimate

$$\begin{aligned}
&\left\| \int_{-\infty}^{t} \mathcal{G}_\pm(\Omega(t-s)) e^{(t-s)\Delta} \mathbb{P} f(s) ds \right\|_{L^r} \\
&\leqslant C \int_{-\infty}^{t} \left\| \mathcal{G}_\pm(\Omega(t-s)) e^{(t-s)\Delta} \mathbb{P} f(s) \right\|_{\dot{B}^0_{r,2}} ds \\
&\leqslant C \int_{-\infty}^{t} \left\{ \frac{\log(e + |\Omega|(t-s))}{1 + |\Omega|(t-s)} \right\}^{\frac{1}{2}(1-\frac{2}{r})} \left\| e^{(t-s)\Delta} \mathbb{P} f(s) \right\|_{\dot{B}^{3(1-\frac{2}{r})}_{r',2}} ds
\end{aligned}$$

By the smoothing effect of $e^{t\Delta}$ and $L^{r'}(\mathbb{R}^3) \hookrightarrow \dot{B}^0_{r',2}(\mathbb{R}^3)$

$$\begin{aligned}
\left\| e^{(t-s)\Delta} \mathbb{P} f(s) \right\|_{\dot{B}^{3(1-\frac{2}{r})}_{r',2}} &\leqslant C(t-s)^{-\frac{3}{2}(1-\frac{2}{r})} \left\| e^{\frac{t-s}{2}\Delta} \mathbb{P} f(s) \right\|_{\dot{B}^0_{r',2}} \\
&\leqslant C(t-s)^{-\frac{3}{2}(\frac{1}{r'}-\frac{1}{r})} \left\| e^{\frac{t-s}{2}\Delta} \mathbb{P} f(s) \right\|_{L^{r'}} \\
&\leqslant C(t-s)^{-\frac{3}{2}(\frac{1}{r'}-\frac{1}{r})-\frac{3}{2}(\frac{1}{p}-\frac{1}{r'})} \|\mathbb{P} f(s)\|_{L^p}
\end{aligned}$$

Linear Estimates

Hence we have that

$$\left\| \int_{-\infty}^{t} \mathscr{G}_{\pm}(\Omega(t-s)) e^{(t-s)\Delta} \mathbb{P} f(s) ds \right\|_{L^r}$$

$$\leqslant C \int_{-\infty}^{t} \left\{ \frac{\log(e + |\Omega|(t-s))}{1 + |\Omega|(t-s)} \right\}^{\frac{1}{2}(1-\frac{2}{r})} (t-s)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{r})} \|\mathbb{P} f(s)\|_{L^p} ds$$

$$\leqslant C \int_{0}^{\infty} \left\{ \frac{\log(e + |\Omega|\tau)}{1 + |\Omega|\tau} \right\}^{\frac{1}{2}(1-\frac{2}{r})} \tau^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{r})} d\tau \times \sup_{t\in\mathbb{R}} \|f(t)\|_{L^p}$$

Our assumption : $1/3 + 5/(3r) < 1/p < 2/3 + 1/r$ yields that

$$\int_{0}^{\infty} \left\{ \frac{\log(e + |\Omega|\tau)}{1 + |\Omega|\tau} \right\}^{\frac{1}{2}(1-\frac{2}{r})} \tau^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{r})} d\tau$$

$$= \frac{1}{|\Omega|^{1+\frac{3}{2r}-\frac{3}{2p}}} \int_{0}^{\infty} \left\{ \frac{\log(e + s)}{1 + s} \right\}^{\frac{1}{2}(1-\frac{2}{r})} s^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{r})} d\tau < \infty$$