Hadamard variational formula for the Green function for the velocity and pressure of the Stokes equations of the perturbations of domains

Erika Ushikoshi
Mathematical Institute, Tohoku University

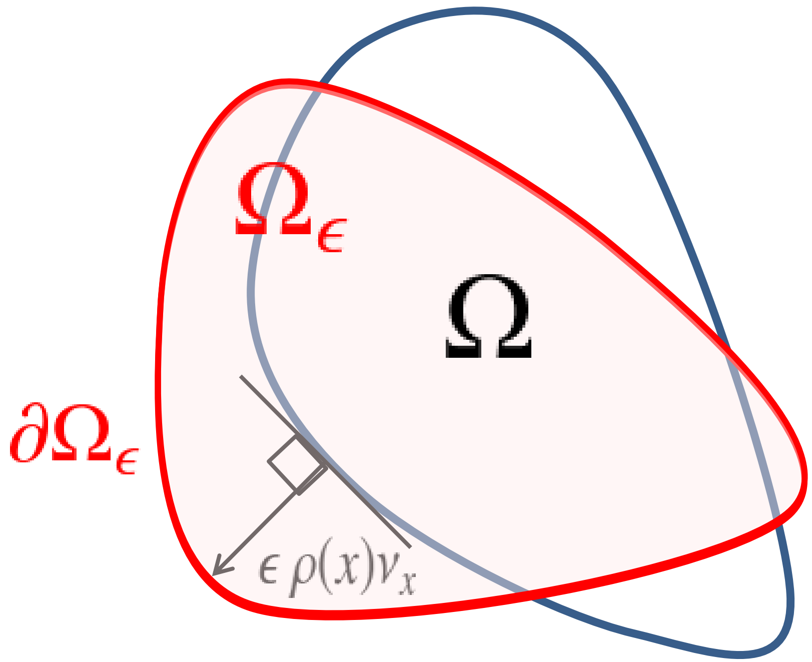
The 5th Japanese-German International Workshop on Mathematical Fluid Dynamics
2011. 6. 13. Waseda University

1. Introduction

$\Omega \subset \mathbb{R}^d (d \geq 2)$; bdd domain with smooth $\partial\Omega$,

$\forall \epsilon \geq 0, \ \rho \in C^\infty(\partial\Omega)$,

$$\partial\Omega_\epsilon := \{\tilde{x} = x + \epsilon\rho(x)\nu_x \ ; \ x \in \partial\Omega\}$$



2. Known Results(1)

· **P.R.Garabedian – M.Schiffer('52-53)** : Laplace eq.

$$\begin{cases} \Delta G_\epsilon = \delta_z & \text{in } \Omega_\epsilon, \\ G_\epsilon = 0, & \text{on } \partial\Omega_\epsilon. \end{cases}$$

Theorem A *Let G_ε be the Green function for the Laplace equation. Then there exists*

$$\delta G(y,z) := \lim_{\varepsilon \to 0} \frac{G_\varepsilon(y,z) - G(y,z)}{\varepsilon}$$

$(G_0 = G)$

for all $y, z \in \Omega$ with an expression

$$\delta G(y,z) = \int_{\partial\Omega} \frac{\partial G}{\partial \nu_x}(x,y) \frac{\partial G}{\partial \nu_x}(x,z) \rho(x)\, d\sigma_x,$$

where $\nu_x = (\nu_x^1, \cdots, \nu_x^d)$ is the unit outer normal to $\partial\Omega$ at $x \in \partial\Omega$ and σ_x denotes the surface element of $\partial\Omega$.

· **D.Fujiwara-S.Ozawa ('78)** : m-th order elliptic eq.

3. Known Results(2)

Laplace eq. → **General(single) m-th order elliptic eq.**

P.R.Garabedian – M.Schiffer

D.Fujiwara – S.Ozawa

↓

Elliptic system

H.Kozono –E.U.(2011 submitted)

Stokes equations

$$
\begin{cases}
\Delta_x \boldsymbol{G}_{\varepsilon,m}(x,z) - \nabla_x R_{\varepsilon,m}(x,z) = e_m \delta(x-z), & \text{in } \Omega_\varepsilon, \\
\operatorname{div} \boldsymbol{G}_{\varepsilon,m}(x,z) = 0, & \text{in } \Omega_\varepsilon, \\
\boldsymbol{G}_{\varepsilon,m}(x,z) = 0, & \text{on } \partial\Omega_\varepsilon
\end{cases}
$$

for $m = 1, \cdots, d$, where $\{\boldsymbol{G}_{\varepsilon,m}\}_{m=1,\cdots,d}$: *velocity*, $\{R_{\varepsilon,m}\}_{m=1,\cdots,d}$: *pressure*,

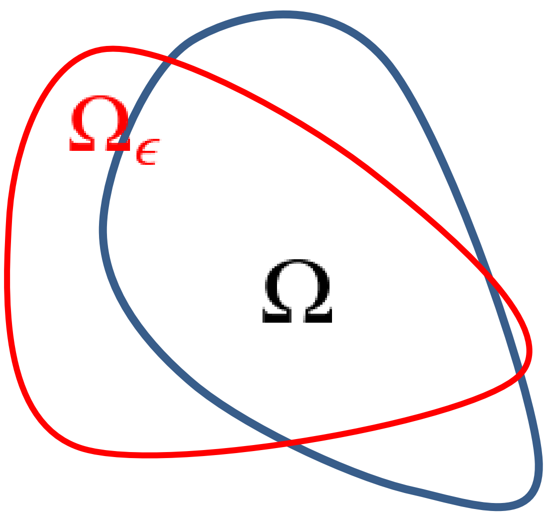
$\{e^m\}_{m=1,\cdots,d}$ is a canonical basis of $\mathbb{R}^d$.

4.Perturbation

(a) Laplace eq.

$$\forall \epsilon \geq 0, \quad \rho \in C^{\infty}(\partial\Omega),$$

$$\partial\Omega_{\epsilon} := \{\tilde{x} = x + \epsilon\rho(x)\nu_x \ ; \ x \in \partial\Omega\}$$



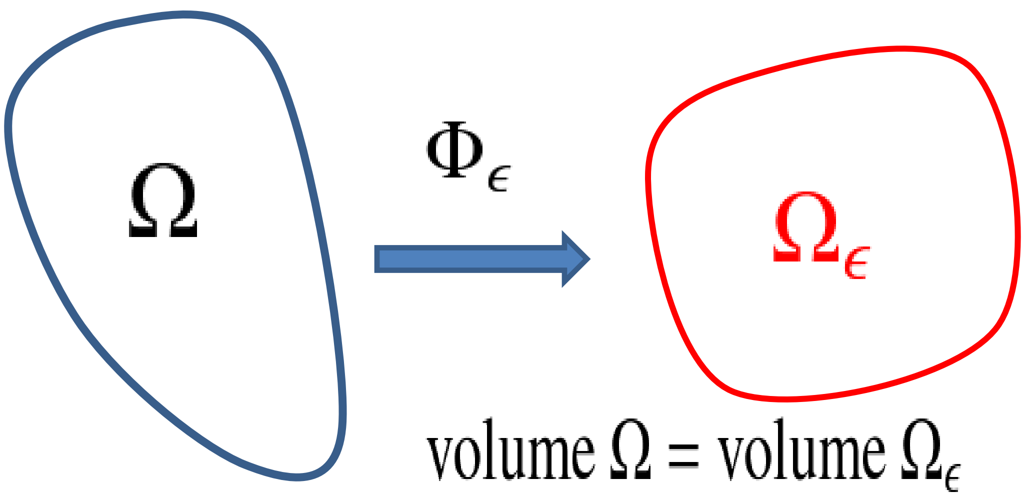
(b) Stokes eq.

$$\Omega_{\epsilon} := \Phi_{\epsilon}(\Omega),$$

(1) $\Phi_{\epsilon}(\cdot) \in C^{\infty}(\overline{\Omega}), \ \Phi_0(x) = x,$

(2) $\Phi_{\epsilon}(x) = x + \epsilon S(x) + O(\epsilon^2),$
$(S \in C^{\infty}(\overline{\Omega})^d), \quad \text{as } \epsilon \to 0 \text{ uniformly in } \overline{\Omega},$

(3) $\det\left(\frac{\partial \Phi_{\epsilon,i}}{\partial x_j}\right)_{1 \leq i,j \leq d} = 1 \ \text{ for } \forall x \in \Omega, \ \forall \epsilon \geq 0.$



volume Ω = volume Ω_{ϵ}

Stokes eq.(H.Kozono- E.U.)

Theorem B *Let* $\{G^n_{\varepsilon,m}\}_{m,n=1,\cdots,d}$ *be the Green matrix of the boundary value problem of the Stokes equations. Then there exists*

$$\delta G^n_m(y,z) := \lim_{\varepsilon\to 0} \frac{G^n_{\varepsilon,m}(y,z) - G^n_m(y,z)}{\varepsilon}, \quad m,n = 1,\cdots,d.$$

for all $y, z \in \Omega$ *with an expression*

$$\delta G^n_m(y,z) = \int_{\partial\Omega} \sum_{i=1}^{d} \frac{\partial G^i_n}{\partial \nu_x}(x,y) \frac{\partial G^i_m}{\partial \nu_x}(x,z) S(x)\cdot \nu_x \, d\sigma_x, \quad m,n = 1,\cdots,d,$$

where $\nu_x = (\nu^1_x, \cdots, \nu^d_x)$ *is the unit outer normal to* $\partial\Omega$ *at* $x \in \partial\Omega$ *and* σ_x *denotes the surface element of* $\partial\Omega$.

$$\Phi_\epsilon(x) = x + \epsilon S(x) + O(\epsilon^2) \text{ as } \epsilon \to 0.$$

$(S \in C^\infty(\overline{\Omega}))$

Laplace eq.(P.R.Garabedian-M.Schiffer)

Theorem A *Let G_ε be the Green function for the Laplace equation. Then there exists*

$$\delta G(y,z) := \lim_{\varepsilon\to 0}\frac{G_\varepsilon(y,z)-G(y,z)}{\varepsilon}$$

for all $y, z \in \Omega$ with an expression

$$\delta G(y,z) = \int_{\partial\Omega}\frac{\partial G}{\partial \nu_x}(x,y)\frac{\partial G}{\partial \nu_x}(x,z)\rho(x)\,d\sigma_x,$$

where $\nu_x = (\nu_x^1, \cdots, \nu_x^d)$ is the unit outer normal to $\partial\Omega$ at $x \in \partial\Omega$ and σ_x denotes the surface element of $\partial\Omega$.

Stokes eq.(H.Kozono – E.U.)

Theorem B *Let $\{G^n_{\varepsilon,m}\}_{m,n=1,\cdots,d}$ be the Green matrix of the boundary value problem of the Stokes equations. Then there exists*

$$\delta G^n_m(y,z) := \lim_{\varepsilon\to 0}\frac{G^n_{\varepsilon,m}(y,z)-G^n_m(y,z)}{\varepsilon}, \quad m,n = 1,\cdots,d.$$

for all $y, z \in \Omega$ with an expression

$$\delta G^n_m(y,z) = \int_{\partial\Omega}\sum_{i=1}^{d}\frac{\partial G^i_n}{\partial \nu_x}(x,y)\frac{\partial G^i_m}{\partial \nu_x}(x,z)S(x)\cdot\nu_x\,d\sigma_x, \quad m,n = 1,\cdots,d,$$

where $\nu_x = (\nu_x^1, \cdots, \nu_x^d)$ is the unit outer normal to $\partial\Omega$ at $x \in \partial\Omega$ and σ_x denotes the surface element of $\partial\Omega$.

<u>Aim</u> To refine the proof of Theorem B

5. Background

(1) explicit representation of $\delta G \; := \left(\dfrac{dG_\varepsilon}{d\varepsilon}\right)\Big|_{\varepsilon=0}$

(2) existence of δG

① Green Integral formula

$$\int_\Omega \{\mathcal{L}v(x)w(x) - \mathcal{L}^* w(x)v(x)\}\, dx = \int_{\partial\Omega} \{Bv(x)w(x) - B^* w(x)v(x)\}\, d\sigma_x,$$

where $\mathcal{L}$: elliptic op. (ex. $\mathcal{L} = \Delta$)

(E) $$\begin{cases} \mathcal{L}u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Green function

$$\begin{cases} \mathcal{L}G = \delta_z, & \text{in } \Omega, \\ G = 0, & \text{on } \partial\Omega. \end{cases}$$

Sol. of (E) $u(y) = (G * f)(y) = \int_\Omega G(y,z) f(y)\, dz.$

② Parametrix (P : approximation of the Green function G)

ex. $\deg \mathcal{L} = 2$

$$\begin{cases} (1)\mathcal{L}(G-P) = O\left(\frac{1}{|x-y|^{d-1}}\right), & \text{as } x \to y, \\ (2) P = 0, & \text{on } \partial\Omega. \end{cases}$$

$\mathcal{L} = \Delta$ **(P.R.Garabedian)**

$$P(x,y) = \alpha(x,y)\Gamma(x,y)$$

where $\Gamma(x,y) = \dfrac{-1}{d(2-d)\omega_d}|x-y|^{2-d}$, $\alpha(\cdot,y) \in C_0^\infty(\Omega)$ with $\alpha(y,y) = 1$

③ A priori estimate $\|v\|_{C^{2+\theta}(\overline{\Omega})} \leq C\left(\|\mathcal{L}v\|_{C^{\theta}(\overline{\Omega})} + \|v\|_{C^{2+\theta}(\partial\Omega)}\right)$

$\exists\, C > 0$ s.t. $\left\|q_\varepsilon(\cdot,y)\right\|_{C^{2+\theta}(\overline{\Omega})} \leq C$ for $0 \leq \varepsilon < 1$.

where $G_\varepsilon(x,y) := \Gamma(x,y) - q_\varepsilon(x,y)$.

There exists $\delta G(y,z) := \lim\limits_{\varepsilon \to 0} \dfrac{G_\varepsilon(y,z) - G(y,z)}{\varepsilon}$?

Stokes eq. (H.Kozono-E.U.)

① Green Integral formula

② Parametrix

Stokes op.

$$(1)\ \mathcal{L}(G-P) = O\left(\frac{1}{|x-y|^{d-1}}\right), \quad \text{as } x \to y.$$

$$(2)\ \operatorname{div} P = 0, \quad \text{in } \Omega,$$

$$(3)\ P = 0, \quad \text{on } \partial\Omega.$$

$\mathcal{L}$ is Stokes op. **(H.Kozono-U.)**

$$P(x,y) = \alpha(x,y)\Gamma(x,y) \Rightarrow \operatorname{div}_x P(x,y) = \nabla_x \alpha(x,y) \cdot \Gamma(x,y) \neq 0,$$

where $\Gamma^i_m(x,y) = \frac{-1}{d(2-d)\omega_d}\{\delta^{im}|x-y|^{2-d} + (d-2)(x^i-y^i)(x^m-y^m)|x-y|^{-d}\}$, $\alpha(\cdot,y) \in C_0^\infty(\Omega)$ with $\alpha(y,y)=1$

Bogovskii formula

$$\begin{cases} \operatorname{div}_x Q(x,y) = \nabla_x \alpha(x,y) \cdot \Gamma(x,y), & \text{in } \Omega, \\ Q(x,y) = 0, & \text{on } \partial\Omega. \end{cases}$$

$$\tilde{P}(x,y) := P(x,y) - Q(x,y) \Rightarrow \operatorname{div} \tilde{P} = 0 \text{ and } (1)\ (3)$$

③ a priori estimate $\|\boldsymbol{v}\|_{C^{2+\theta}(\overline{\Omega})} + \|\nabla\pi\|_{C^\theta(\overline{\Omega})} \leq C\left(\|\mathcal{L}(\boldsymbol{v},\pi)\|_{C^\theta(\overline{\Omega})} + \|\boldsymbol{v}\|_{C^{2+\theta}(\partial\Omega)}\right)$

$\exists C > 0$ s.t. $\|\boldsymbol{q}_{\varepsilon,m}(\cdot,y)\|_{C^{2+\theta}(\overline{\Omega})} \leq C$ for $0 \leq \varepsilon < 1$.

where $\boldsymbol{G}_{\varepsilon,m} := \Gamma - \boldsymbol{q}_{\varepsilon.m}$.

$$\text{There exists } \delta G_m^n(y,z) := \lim_{\varepsilon\to 0} \frac{G_{\varepsilon,m}^n(y,z) - G_m^n(y,z)}{\varepsilon}.$$

Laplace eq. and **Stokes eq.**

P.R.Garabedian – M.Schiffer　　H.Kozono –E.U.

(1) explicit representation of δG
(2) existence of δG

① Green Integral formula
② Parametrix
③ A priori estimate

} Hadamard variational formula

Aim To refine the proof of Theorem B

·**D.Fujiwara-S.Ozawa ('78)** : m-th order elliptic eq.

6. Key Lemma

Lemma 0.1. *For any fixed z in Ω, there exist $\{\delta G_m(y,z)\}_{m=1,\dots,d}$ and $\{\delta R_m(y,z)\}_{m=1,\cdots,d}$ for all $y \in \overline{\Omega}$ with $y \neq z$.*

$$\delta G_m(y,z) := \lim_{\varepsilon\to 0} \frac{G_{\varepsilon,m}(y,z) - G_m(y,z)}{\varepsilon} \qquad \delta R_m(y,z) := \lim_{\varepsilon\to 0} \frac{R_{\varepsilon,m}(y,z) - R_m(y,z)}{\varepsilon}$$

7. Results(1)

Theorem 0.1. *Let* $\{G^n_{\varepsilon,m}\}_{m,n=1,\cdots,d}$ *be the Green matrix of the boundary value problem of the Stokes equations. Then there exists*

$$\delta G^n_m(y,z) := \lim_{\varepsilon\to 0}\frac{G^n_{\varepsilon,m}(y,z) - G^n_m(y,z)}{\varepsilon}, \quad m,n = 1,\cdots,d.$$

for all $y,z \in \Omega$ *with an expression*

$$\delta G^n_m(y,z) = \int_{\partial\Omega}\sum_{i=1}^{d}\frac{\partial G^i_n}{\partial \nu_x}(x,y)\frac{\partial G^i_m}{\partial \nu_x}(x,z)S(x)\cdot\nu_x\,d\sigma_x, \quad m,n = 1,\cdots,d,$$

where $\nu_x = (\nu^1_x,\cdots,\nu^d_x)$ *is the unit outer normal to* $\partial\Omega$ *at* $x \in \partial\Omega$ *and* σ_x *denotes the surface element of* $\partial\Omega$.

8. Results(2)

Theorem 0.1. *Let* $\{R_{\varepsilon,m}\}_{m,n=1,\cdots,d}$ *be the Green function of the boundary value problem for the Stokes equations. Then there exists*

$$\delta R_m(y,z) := \lim_{\varepsilon\to 0}\frac{R_{\varepsilon,m}(y,z)-R_m(y,z)}{\varepsilon}, \quad m=1,\cdots,d.$$

for all $y,z\in\Omega$ *with an expression*

$$\begin{aligned}
&\delta R_m(y,z)\\
&=\int_{\partial\Omega}\sum_{i,j,k=1}^{d}\Big\{\left(-S^k(x)\frac{\partial R_i(y,x)}{\partial x^k}\nu_x^j+\frac{\partial S^i}{\partial x^k}(x)R_j(y,x)\nu_x^k\right)T^{ij}(\boldsymbol{G}_m,R_m)(x,z)\\
&\quad+\frac{\partial S^i}{\partial x^j}(x)\left(-R_i(y,x)\frac{\partial G_m^j}{\partial x^k}(x,z)\nu_x^k-R_k(y,x)\frac{\partial G_m^i}{\partial x^j}(x,z)\nu_x^k+R_k(y,x)\frac{\partial G_m^k}{\partial x^i}(x,z)\nu_x^j\right)\Big\}\,d\sigma_x
\end{aligned}$$

for $m=1,\cdots,d$, *where* $\{T^{ij}\}_{i,j=1,\cdots,d}$ *is the stress tensor for the velocity* $\boldsymbol{v}$ *and the pressure* π *defined by*

$$T^{ij}(\boldsymbol{v},\pi)(x) := -\delta^{ij}\pi(x)+\left(\frac{\partial v^i}{\partial x^j}(x)+\frac{\partial v^j}{\partial x^i}(x)\right), \qquad i,j=1,\cdots,d,$$

$\nu_x=(\nu_x^1,\cdots,\nu_x^d)$ *is the unit outer normal to* $\partial\Omega$ *at* $x\in\partial\Omega$ *and* σ_x *denotes the surface element of* $\partial\Omega$.

9.Remark

(a) Laplace eq.

$$\delta G(y,z) := \lim_{\varepsilon\to 0}\frac{G_\varepsilon(y,z)-G(y,z)}{\varepsilon}$$
$$= \int_{\partial\Omega}\frac{\partial G}{\partial \nu_x}(x,y)\frac{\partial G}{\partial \nu_x}(x,z)\rho(x)\,d\sigma_x.$$

- $\delta G(y,z) = \delta G(z,y)$
- $\rho \in C^\infty(\partial\Omega)$

(b) Stokes eq.

$$\delta G^n_m(y,z) := \lim_{\varepsilon\to 0}\frac{G^n_{\varepsilon,m}(y,z)-G^n_m(y,z)}{\varepsilon}$$
$$= \int_{\partial\Omega}\sum_{i=1}^{d}\frac{\partial G^i_n}{\partial \nu_x}(x,y)\frac{\partial G^i_m}{\partial \nu_x}(x,z)S(x)\cdot\nu_x\,d\sigma_x.$$

$$\delta R_m(y,z)$$
$$= \int_{\partial\Omega}\sum_{i,j,k=1}^{d}\Bigg\{\left(-S^k(x)\frac{\partial R_i(y,x)}{\partial x^k}\nu^j_x+\frac{\partial S^i}{\partial x^k}(x)R_j(y,x)\nu^k_x\right)T^{ij}(\mathbf{G}_m,R_m)(x,z)$$
$$+\frac{\partial S^i}{\partial x^j}(x)\left(-R_i(y,x)\frac{\partial G^j_m}{\partial x^k}(x,z)\nu^k_x-R_k(y,x)\frac{\partial G^i_m}{\partial x^j}(x,z)\nu^k_x+R_k(y,x)\frac{\partial G^k_m}{\partial x^i}(x,z)\nu^j_x\right)\Bigg\}d\sigma_x$$

- $\delta G^m_n(y,z) = \delta G^n_m(z,y)$
- $\Phi_\epsilon(x) = x + \epsilon S(x) + O(\epsilon^2)$

$(S \in C^\infty(\overline{\Omega}))$ as $\epsilon \to 0.$

$\Rightarrow$ $\operatorname{div} S = 0$ in Ω

10.How to Construct Hadamard variational formula

(H.Kozono – E.U.)

(1) explicit representation of δG
(2) existence of δG

① Green Integral formula
② Parametrix
③ A priori estimate

(D.Fujiwara – S.Ozawa)

(1) explicit representation of δG

① Green Integral formula
② Whitney extension theorem

(2) existence of δG

① A priori estimate

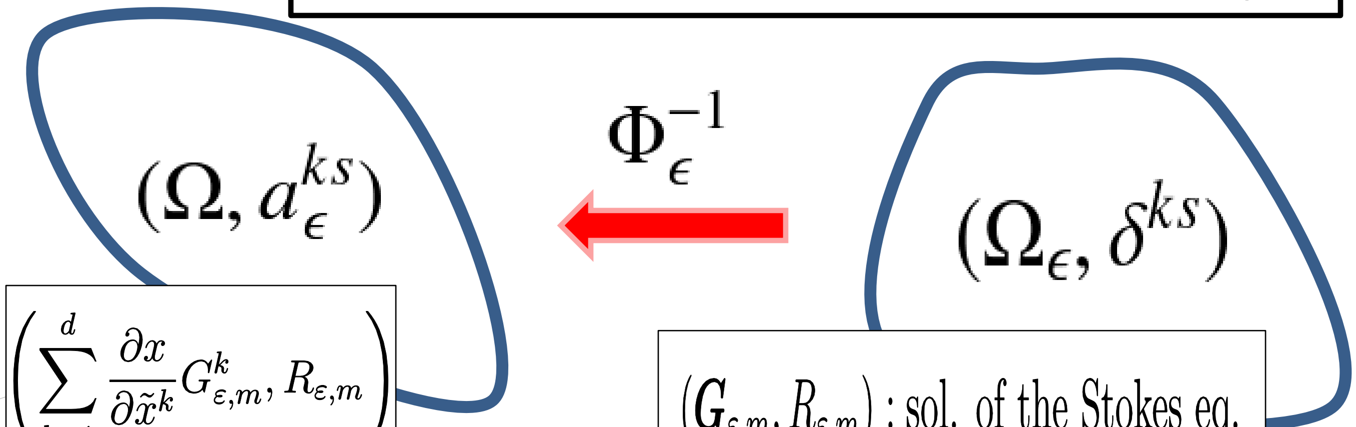
Lemma 0.1. *For any fixed z in Ω, there exist $\{\delta G_m(y,z)\}_{m=1,\ldots,d}$ and $\{\delta R_m(y,z)\}_{m=1,\cdots,d}$ for all $y \in \overline{\Omega}$ with $y \neq z$.*

11.Idea of the Proof

Inoue – Wakimoto('77)
(Moving boundary problem for the Navier-Stokes eq.)

$$a_\varepsilon^{ks} = \sum_{j=1}^{d} \frac{\partial x^k}{\partial \tilde{x}^j} \frac{\partial x^s}{\partial \tilde{x}^j}$$

$$\tilde{x} = \Phi_\epsilon(x)$$



$$\left(\sum_{k=1}^{d} \frac{\partial x}{\partial \tilde{x}^k} G_{\varepsilon,m}^k, R_{\varepsilon,m} \right)$$

$(G_{\varepsilon,m}, R_{\varepsilon,m})$: sol. of the Stokes eq.

$$\mathcal{L}_\varepsilon^j(u,\pi) := \sum_{i,k,s,p}^{d} \frac{\partial}{\partial x^k} \left(a_\varepsilon^{ks} \frac{\partial}{\partial x^k} \tilde{u}^i \right) \frac{\partial \tilde{x}^i}{\partial x^p} a_\varepsilon^{pj} - a_\varepsilon^{ji} \frac{\partial \pi}{\partial x^i}$$

$(\mathcal{L}_0(v,\pi) = \Delta v - \nabla \pi)$ $\qquad (j = 1, \cdots, d)$

$$\tilde{u}_i := \sum_{l=1}^{d} \frac{\partial \tilde{x}_i}{\partial x_l} u_l$$

$$\mathcal{L}_\varepsilon \left(\sum_{k=1}^{d} \frac{\partial x}{\partial \tilde{x}^k} G_{\varepsilon,m}^k, R_{\varepsilon,m} \right)(x,z) = 0 \quad \text{for } x \in \Omega$$

$$:= g_{\varepsilon,m}$$

$$\mathrm{div} \left(\sum_{k=1}^{d} \frac{\partial x}{\partial \tilde{x}^k} G_{\varepsilon,m}^k \right)(x,z) = 0 \ \text{for } x \in \Omega$$

10. Outline of the Proof

$$\left(\frac{d}{d\varepsilon}\mathcal{L}_\varepsilon\right)(\boldsymbol{g}_{\varepsilon,m}, R_{\varepsilon,m}) + \mathcal{L}_\varepsilon\left(\frac{d}{d\varepsilon}\boldsymbol{g}_{\varepsilon,m}, \frac{d}{d\varepsilon}R_{\varepsilon,m}\right) = 0$$

$$\left(\frac{d}{d\varepsilon}\mathcal{L}_\varepsilon\right)\Big|_{\varepsilon=0}(\boldsymbol{G}_m, R_m) + \mathcal{L}_0(\delta\boldsymbol{G}_m, \delta R_m) = 0$$

$$(\mathrm{SE}_\varepsilon)\begin{cases} \mathcal{L}_\varepsilon(\boldsymbol{g}_{\varepsilon,m}, R_{\varepsilon,m})(x,z) = 0, & x\in\Omega,\\ \operatorname{div}\boldsymbol{g}_{\varepsilon,m}(x,z) = 0, & x\in\Omega,\\ \boldsymbol{g}_{\varepsilon,m}(x,z) = 0, & x\in\partial\Omega, \quad m=1,\cdots,d. \end{cases}$$

$$\begin{cases} \frac{d}{d\varepsilon}\mathcal{L}_\varepsilon(\boldsymbol{g}_{\varepsilon,m}, R_{\varepsilon,m})(x,z)\Big|_{\varepsilon=0} = 0, & x\in\Omega,\\ \frac{d}{d\varepsilon}\operatorname{div}\boldsymbol{g}_{\varepsilon,m}(y,z)\Big|_{\varepsilon=0} = 0, & x\in\Omega,\\ \frac{d}{d\varepsilon}\boldsymbol{g}_{\varepsilon,m}(x,z)\Big|_{\varepsilon=0} = 0, & x\in\partial\Omega, \quad m=1,\cdots,d. \end{cases}$$

where $\mathcal{L}^j_\varepsilon(u,\pi) := \sum_{i,k,l,s,p}^{d} \frac{\partial}{\partial x^k}\left\{a^{ks}_\varepsilon \frac{\partial}{\partial x^k}\left(\frac{\partial\tilde{x}^i}{\partial x^l}u^l\right)\right\}\frac{\partial\tilde{x}^i}{\partial x^p}a^{pj}_\varepsilon - a^{ji}_\varepsilon\frac{\partial\pi}{\partial x^i}$, $\quad \boldsymbol{g}_m := \sum_{k=1}^{d}\frac{\partial x}{\partial\tilde{x}^k}G^k_{\varepsilon,m}$,

$$a^{ks}_\varepsilon := \sum_{j=1}^{d}\frac{\partial x^k}{\partial\tilde{x}^j}\frac{\partial x^s}{\partial\tilde{x}^j}, \qquad \tilde{x} = \Phi_\varepsilon(x) = x + \varepsilon S(x) + O(\varepsilon^2), \quad \text{as } \varepsilon\to 0.\ (S\in C^\infty(\overline{\Omega}))$$

$\exists\delta G$ and $\exists\delta R$ **(By key Lemma)**

$$(\mathrm{ISE})\begin{cases} \Delta\delta\boldsymbol{G}_m(x,z) - \nabla\delta R_m(x,z) = \boldsymbol{f}_m(x,z), & x\in\Omega,\\ \operatorname{div}\delta\boldsymbol{G}_m(x,z) = 0, & x\in\Omega,\\ \delta\boldsymbol{G}_m(x,z) = 0, & x\in\partial\Omega, \quad m=1,\cdots,d. \end{cases}$$

where $f^i_m(x,z) := -\sum_{k,s=1}^{d}\frac{\partial}{\partial x^k}\left(\delta a^{ks}\frac{\partial G^i_m}{\partial x^s}(x,z)\right) - \sum_{j=1}^{d}\Delta\left(\frac{\partial S^i}{\partial x^j}G^j_m\right)(x,z) - \sum_{j=1}^{d}\frac{\partial S^j}{\partial x^i}\Delta G^j_m(x,z)$, $\quad \delta a^{ks} := \frac{d a^{ks}_\varepsilon}{d\varepsilon}\Big|_{\varepsilon=0}$

$$\text{(ISE)}\begin{cases} \Delta\delta \boldsymbol{G}_m(x,z) - \nabla\delta R_m(x,z) = \boldsymbol{f}_m(x,z), & x\in\Omega, \\ \operatorname{div}\delta\boldsymbol{G}_m(x,z) = 0, & x\in\Omega, \\ \delta\boldsymbol{G}_m(x,z) = 0, & x\in\partial\Omega, \qquad m=1,\cdots,d, \end{cases}$$

where $f^i_m(x,z) := -\sum_{k,s=1}^{d}\frac{\partial}{\partial x^k}\left(\delta a^{ks}\frac{\partial G^i_m}{\partial x^s}(x,z)\right) - \sum_{j=1}^{d}\Delta\left(\frac{\partial S^i}{\partial x^j}G^j_m\right)(x,z) - \sum_{j=1}^{d}\frac{\partial S^j}{\partial x^i}\Delta G^j_m(x,z), \quad \delta a^{ks} := \left.\frac{da^{ks}_\varepsilon}{d\varepsilon}\right|_{\varepsilon=0}.$

Green function

$$\begin{cases} \Delta\boldsymbol{G}_n(x,y) - \nabla R_n(x,y) = e_n\delta(x-y), & x\in\Omega, \\ \operatorname{div}\boldsymbol{G}_n(x,y) = 0, & x\in\Omega, \\ \boldsymbol{G}_n(x,y) = 0, & x\in\partial\Omega, \qquad n=1,\cdots,d. \end{cases}$$

Green integral formula

$$T^{ij}(\boldsymbol{v},\pi)(x) := -\delta^{ij}\pi(x) + \left(\frac{\partial v^i}{\partial x^j}(x) + \frac{\partial v^j}{\partial x^i}(x)\right), \quad i,j=1,\cdots,d.$$

$$\int_\Omega \sum_{i=1}^{d}\left\{\mathcal{L}^i(\boldsymbol{v},\pi)(x)w^i(x) - \mathcal{L}^i(\boldsymbol{w},-\tilde{\pi})(x)v^i(x)\right\}dx = \int_{\partial\Omega}\sum_{i,j=1}^{d}\left\{T^{ij}(\boldsymbol{v},\pi)(x)w^i(x) - T^{ij}(\boldsymbol{w},-\tilde{\pi})(x)v^i(x)\right\}\nu^j_x\, d\sigma_x,$$

Sol. of (ISE)

$$\begin{cases} \delta G^n_m(y,z) = \int_\Omega \sum_{i=1}^{d} G^i_n(x,y) f^i_m(x,z)\,dx, \\ \delta R_m(y,z) = \int_\Omega \sum_{i=1}^{d} R_i(x,y) f^i_m(x,z)\,dx \end{cases}$$

Theorem 0.1. *Let* $\{G^n_{\varepsilon,m}\}_{m,n=1,\cdots,d}$ *be the Green matrix of the boundary value problem of the Stokes equations. Then there exists*

$$\delta G^n_m(y,z) := \lim_{\varepsilon\to 0} \frac{G^n_{\varepsilon,m}(y,z) - G^n_m(y,z)}{\varepsilon}, \quad m,n = 1,\cdots,d.$$

for all $y, z \in \Omega$ *with an expression*

$$\delta G^n_m(y,z) = \int_{\partial\Omega} \sum_{i=1}^{d} \frac{\partial G^i_n}{\partial \nu_x}(x,y) \frac{\partial G^i_m}{\partial \nu_x}(x,z) S(x) \cdot \nu_x \, d\sigma_x, \quad m,n = 1,\cdots,d,$$

where $\nu_x = (\nu^1_x, \cdots, \nu^d_x)$ *is the unit outer normal to* $\partial\Omega$ *at* $x \in \partial\Omega$ *and* σ_x *denotes the surface element of* $\partial\Omega$.

Theorem 0.1. *Let $\{R_{\varepsilon,m}\}_{m,n=1,\cdots,d}$ be the Green function of the boundary value problem for the Stokes equations. Then there exists*

$$\delta R_m(y,z) := \lim_{\varepsilon\to 0} \frac{R_{\varepsilon,m}(y,z) - R_m(y,z)}{\varepsilon}, \quad m = 1,\cdots,d.$$

for all $y, z \in \Omega$ with an expression

$$\begin{aligned}
&\delta R_m(y,z) \\
&= \int_{\partial\Omega} \sum_{i,j,k=1}^{d} \Bigg\{ \left(-S^k(x) \frac{\partial R_i(y,x)}{\partial x^k} \nu_x^j + \frac{\partial S^i}{\partial x^k}(x) R_j(y,x) \nu_x^k \right) T^{ij}(G_m, R_m)(x,z) \\
&\qquad + \frac{\partial S^i}{\partial x^j}(x) \left(-R_i(y,x) \frac{\partial G_m^j}{\partial x^k}(x,z)\nu_x^k - R_k(y,x)\frac{\partial G_m^i}{\partial x^j}(x,z)\nu_x^k + R_k(y,x)\frac{\partial G_m^k}{\partial x^i}(x,z)\nu_x^j \right) \Bigg\} d\sigma_x
\end{aligned}$$

for $m = 1,\cdots,d$, where $\{T^{ij}\}_{i,j=1,\cdots,d}$ is the stress tensor for the velocity v and the pressure π defined by

$$T^{ij}(v,\pi)(x) := -\delta^{ij}\pi(x) + \left(\frac{\partial v^i}{\partial x^j}(x) + \frac{\partial v^j}{\partial x^i}(x) \right), \qquad i,j = 1,\cdots,d,$$

$\nu_x = (\nu_x^1,\cdots,\nu_x^d)$ is the unit outer normal to $\partial\Omega$ at $x \in \partial\Omega$ and σ_x denotes the surface element of $\partial\Omega$.

6. Outline of Proof

$\forall u, w \in C^\infty(\overline{\Omega}),\ \operatorname{div} u = \operatorname{div} w = 0,$

$\forall\, \pi, \tilde{\pi} \in C^\infty(\overline{\Omega})$

$$\mathcal{L}_\epsilon^j(u,\pi) := \sum_{i,k,s,p=1}^{d} \frac{\partial}{\partial x_k}\left(a_\epsilon^{ks}\frac{\partial}{\partial x_s}\tilde{u}_i\right)\frac{\partial \tilde{x}_i}{\partial x_p}a_\epsilon^{pj} - a_\epsilon^{ji}\frac{\partial \pi}{\partial x_i}$$

$$\int_\Omega \sum_{k=1}^{d}\left\{\mathcal{L}_\epsilon^k(u,\pi)(x)w_k(x) - \mathcal{L}_\epsilon^k(w,-\tilde{\pi})(x)u_k(x)\right\}dx$$

$$= \int_{\partial\Omega}\sum_{k,l=1}^{d}\left\{T_{\epsilon,l}^k(u,\pi)(x)w_l(x)\nu_k - T_{\epsilon,l}^k(w,-\tilde{\pi})(x)u_l(x)\nu_k\right\}d\sigma_x.$$

$$u_k(x) = \sum_{i=1}^{d}\frac{\partial x_k}{\partial \tilde{x}_i}G_{\epsilon,i}^m(\tilde{x},\tilde{z}),$$

$$\pi(x) = R_\epsilon^m(\tilde{x},\tilde{z}),$$

$$w_k(x) = G_k^n(x,y) - P_{0,k}^n(x,y) + P_{\epsilon,k}^n(x,y),$$

$$\tilde{\pi}(x) = R^n(x,y) - r_0^n(x,y) + r_\epsilon^n(x,y),$$

$m, n = 1, \cdots, d.$

5. Main Theorem

(1) $\Phi_\epsilon(\cdot) \in C^\infty(\overline{\Omega}),\ \Phi_0(x) = x,$

(2) $\Phi_\epsilon(x) = x + \epsilon S(x) + O(\epsilon^2),$
$(S \in C^\infty(\overline{\Omega})^d),$ as $\epsilon \to 0$ uniformly in $\overline{\Omega}$,

(3) $\det\left(\frac{\partial \Phi_{\epsilon,i}}{\partial x_j}\right)_{1\le i,j\le d} = 1$ for $\forall x \in \Omega,\ \forall \epsilon \ge 0.$

$\Longrightarrow$

$$\delta G_n^m(y,z) := \lim_{\epsilon\to 0} \frac{G_{\epsilon,n}^m(y,z) - G_n^m(y,z)}{\epsilon}$$

$$= \int_{\partial\Omega} \sum_{i=1}^{d} \left\{ \frac{\partial G_i^n(x,y)}{\partial \nu_x} \frac{\partial G_i^m(x,z)}{\partial \nu_x} - \left(R^n(x,y) \frac{\partial G_i^m(x,z)}{\partial \nu_x} + R^m(x,z) \frac{\partial G_i^n(x,y)}{\partial \nu_x} \right) \nu_i \right\} S(x) \cdot \nu_x \, d\sigma_x,$$

$m, n = 1, \cdots, d.$

Application

Asymptotic formula of the eigenvalue of Laplacian (S.Ozawa '78)

$0 > \mu_1 \geq \mu_2 \geq \cdots;$ eigenvalue of Δ,

$$T_r(t,\epsilon) := \sum_{j=1}^{\infty} e^{\mu_j(\epsilon)t},$$

$$\delta T_r(t) := \lim_{\epsilon \to 0} \frac{T_r(t,\epsilon) - T_r(t,0)}{\epsilon},$$

⇒

$$\delta T_r(t) = t \sum_{j=1}^{\infty} e^{\mu_j t} \int_{\partial\Omega} \left(\frac{\partial \psi_j}{\partial \nu_x}(x) \right)^2 \rho(x)\, d\sigma_x.$$

where

ψ_j : function of eigenvalue μ_j