Spatial Decay Estimates for Elliptic Integro-Differential Equations

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Tokyo 2012

Elliptic Integro-Differential Operator

Let $n \in \mathbb{N}$, $n \geq 2$, $b \in \mathbb{R}^n$ and ν a Borel measure defined on $\mathcal{B}(\mathbb{R}^n \backslash \{0\})$

$$\nu(\mathbb{R}^n \backslash B_1) < \infty, \int_{B_1 \backslash \{0\}} |y|^2 \nu(dy) < \infty.$$

We consider the Elliptic Integro-Differential Operator

$$(Lu)(x) = b \cdot \nabla u - \int_{\mathbb{R}^n \backslash \{0\}} \big(u(x+y) - u - \chi_{|y| \leq 1} y \cdot \nabla u\big) \nu(dy).$$

Example

If $b = 0$ and $\nu(dy) = \frac{1}{|y|^{n+2s}} dy$ then

$$(Lu)(x) = -\int_{\mathbb{R}^n \backslash \{0\}} \frac{u(x+y) - u(x)}{|y|^{n+2s}} dy.$$

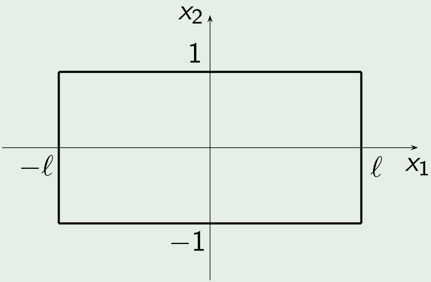
Cylinder

Let $1 \leq p < n$, $\omega \subset \mathbb{R}^{n-p}$ open with bounded diameter and for $\ell > 0$

$$\Omega_\ell = (-\ell, \ell)^p \times \omega.$$

Example

For $n = 2$, $p = 1$ and $\omega = (-1, 1)$



Dirichlet Problem in the Cylinder

$f_\ell \in L^2(\Omega_\ell)$

$$\begin{cases} Lu_\ell = f_\ell \text{ in } \Omega_\ell, \\ u_\ell = 0 \text{ in } \mathbb{R}^n \backslash \Omega_\ell. \end{cases}$$

Spatial Decay Estimate

In the case of usual Dirichlet problem we have for some $\alpha = \alpha(\omega) > 0$

$$\int_{\Omega_\ell} |\nabla u_\ell|^2 e^{-\alpha|x|} dx \leq C \int_{\Omega_\ell} |f_\ell|^2 e^{-\alpha|x|} dx.$$

In the case of Integro-Differential equations our aim is for suitable $\rho(x) = \rho_\omega(x) > 0$

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n \backslash \{0\}} (u_\ell(x+y) - u_\ell(x))^2 \nu_s(dy) \rho dx \leq C \int_{\Omega_\ell} f_\ell^2 \rho dx.$$

Levy Processes

Let $X = X(t, \omega)$, $t \geq 0$ be a stochastic process defined on a probability space $(A, \mathcal{F}, P)$.
X is a Levy Process if

1. $X(0) = 0$ a.s.,
2. X has independent and stationary increments,
3. X is stochastically continuous, i.e. $X(t) \to X(s)$ in measure as $t \to s$.

The Semigroup Associated with X

Let us define the operators $T_t : C_0(\mathbb{R}^n) \to C_0(\mathbb{R}^n)$ as follows

$$(T_t u)(x) = E[u(x + X(t))] = \int_{\mathbb{R}^n} u(x + y) p_{X(t)}(dy).$$

then T_t satisfies the contraction semigroup conditions

1. $T_{s+t} = T_s T_t$,
2. $T_0 = I$,
3. $\|T_t\| \leq 1$,
4. $\lim_{t \to +0} \|T_t u - u\| = 0$, for all $u \in C_0(\mathbb{R}^n)$.

Generator of T_t

The generator of T_t, that is

$$Gu = \lim_{t \to +0} \frac{T_t u - u}{t}$$

for $u \in C_c^\infty(\mathbb{R}^n)$ has the form

$$(Gu)(x) = \frac{1}{2}\operatorname{div}(Q\nabla u)(x) - b \cdot \nabla u(x) \\ + \int_{\mathbb{R}^n \setminus \{0\}} \{u(x+y) - u(x) - \chi_{|y|\le 1} y \cdot \nabla u(x)\} \nu(dy).$$

$H^\mu(\mathbb{R}^n)$

Let μ be a symmetric Borel measure defined on $\mathcal{B}(\mathbb{R}^n\backslash\{0\})$ such that

$$\int_{\mathbb{R}^n\backslash\{0\}} \min(1,|y|^2)\mu(dy) < \infty.$$

Let us denote by $H^\mu(\mathbb{R}^n)$ the set of functions in $L^2(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n\backslash\{0\}} (u(x+y)-u(x))^2\mu(dy)dx < \infty.$$

For short notation let us denote

$$(\delta u)(x,y) = u(x+y) - u(x).$$

Inner product in $H^\mu(\mathbb{R}^n)$

$H^\mu(\mathbb{R}^n)$ is a Hilbert space with the inner product

$$(u,v)_{H^\mu(\mathbb{R}^n)} = (u,v)_{L^2(\mathbb{R}^n)} + (\delta u, \delta v)_{L^2(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}), \lambda \otimes \mu)}.$$

$C_c^\infty(\mathbb{R}^n)$ is dense in $H^\mu(\mathbb{R}^n)$.

Example

If $\mu(dy) = \frac{1}{|y|^{n+2s}} dy$ then $H^\mu(\mathbb{R}^n) = H^s(\mathbb{R}^n)$.

$\tilde{H}_0^\mu(\Omega)$

We denote

$$\tilde{H}_0^\mu(\Omega) = \left\{ v \in H^\mu(\mathbb{R}^n) \;\middle|\; v = 0 \text{ in } \mathbb{R}^n \setminus \Omega \right\}.$$

Let ν_s and ν_a denote respectively the symmetric and antisymmetric parts of ν.

Let us define for $u, v \in C_c^\infty(\mathbb{R}^n)$, $B(u, v) = (Lu, v)$.

Lemma

Let there be $a_1, a_2 \in \mathbb{R}$ and $\zeta \in \mathbb{R}^n$ such that

$$\Omega \subset \left\{x \in \mathbb{R}^n \;\middle|\; a_1 < x \cdot \zeta < a_2\right\},$$

$$\nu_s\left(\left\{x \in \mathbb{R}^n \;\middle|\; x \cdot \zeta \neq 0\right\}\right) > 0$$

and for some $C > 0$

$$|B_a(u, v)| \leq C\|u\|_{H^{\nu_s(\mathbb{R}^n)}}\|v\|_{H^{\nu_s}(\mathbb{R}^n)}, \;\; \forall u, v \in C_c^\infty(\mathbb{R}^n)$$

then for $f \in (\tilde{H}_0^{\nu_s}(\Omega))^*$ and $g \in H^{\nu_s}(\mathbb{R}^n)$ there exists a unique $u \in H^{\nu_s}(\mathbb{R}^n)$ such that $u - g \in \tilde{H}_0^{\nu_s}(\Omega)$ and

$$B(u, v) = \langle f, v \rangle, \;\; \forall v \in \tilde{H}_0^{\nu_s}(\Omega).$$

Let ρ be a nonnegative, bounded and Lipschitz function on $\mathbb{R}^n$.

Let us define

$$(S(\rho))(x) = \int_{\mathbb{R}^n\setminus\{0\}} (\sqrt{\rho(x+y)} - \sqrt{\rho(x)})^2 \nu_s(dy),$$

$$\begin{aligned}(A\rho)(x) &= b\cdot\nabla\rho(x)\\ &\quad - \frac{1}{2}\int_{\mathbb{R}^n\setminus\{0\}} \{\rho(x+y) - \rho(x-y) - 2\chi_{|y|\leq 1} y\cdot\nabla\rho\}\nu_a(dy).\end{aligned}$$

Let $f_\ell \in L^2(\Omega_\ell)$ and u_ℓ the solution to

$$\begin{cases} Lu_\ell = f_\ell \text{ in } (\tilde{H}_0^{\nu_s}(\Omega_\ell))^*,\\ u_\ell \in \tilde{H}_0^{\nu_s}(\Omega_\ell).\end{cases}$$

Theorem

If $0 < \nu_s(\mathbb{R}^p \times (\mathbb{R}^{n-p} \backslash \{0_{n-p}\}))$ *then there exist positive constants* $C, \gamma > 0$ *such that if*

$$S(\rho) \leq \gamma\rho,\ A\rho \leq \gamma\rho$$

then

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n \backslash \{0\}} (u_\ell(x+y) - u_\ell(x))^2 \nu_s(dy)\rho dx \leq C \int_{\Omega_\ell} f_\ell^2 \rho dx.$$

Example

$n = 2$, $p = 1$, let $0 < \nu_s(\mathbb{R} \times (\mathbb{R}\backslash\{0\}))$ and there exists a $C > 0$ such that for $A \in \mathcal{B}(\mathbb{R})$ we have

$$\nu(A \times \mathbb{R}) \leq C \int_A \frac{dt}{t^2}.$$

For $\lambda > 0$ define

$$\rho_\lambda(t) = \frac{1}{1 + (\frac{t}{\lambda})^2}$$

then we have

$$S(\rho_\lambda) \leq \frac{C_1}{\lambda}\rho_\lambda,\ |A\rho_\lambda| \leq \frac{C_2}{\lambda}\rho_\lambda.$$

So for large enough λ the condition of the previous theorem are satisfied and we have

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2\backslash\{0\}} (u_\ell(x+y) - u_\ell(x))^2 \nu_s(dy)\rho_\lambda dx \leq \int_{\Omega_\ell} f_\ell^2 \rho_\lambda dx.$$

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Thank You For Your Attention