Stochastic Fractal Burgers type equations with conservation laws

BIN XIE

International Young Researchers Empowerment Center,
Shinshu University, Japan

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Outline

Consider the stochastic fractal Burgers type equation

$$\begin{aligned}&\partial_t u(t,x) + div(f(u))(t,x) \\ =&\gamma\mathfrak{D}(x,D)u(t,x) + \sigma(x,u(t,x))\dot{w}(t,x), x \in \mathbb{R}^d,\end{aligned} \tag{1.1}$$

where $\gamma \geq 0$, $f : \mathbb{R} \to \mathbb{R}^d$, $\dot{w}(t,x)$ denotes some kind of noises, $\mathfrak{D}(x,D)$ in (1.1) denotes a pseudo-differential operator (a generator of stable-like process) including the non-local operator $-(-\Delta)^{\alpha/2}$ as its example.
It is applied to fluid dynamics, hydrodynamics, statistical mechanics, physiology, molecular biology, traffic jams, cellular detonations in gases(P. Clavin et al. (2001, 2002)).
Many authors studied such equations, such as, Albeverio, Bertini, Da Prato, Debussche, Flandoli, Gyongy, Nualart, Roeckner, Sinai, Brzezniak, Jacob, Truman, . . .

- KPZ equations(1986)(growing interface);
- Scalar conservation law ($\gamma = 0$);
- 1-D Navier-Stokes equations.

For each test function $\psi \in \mathcal{S}(\mathbb{R}^d)$, the operator $\mathfrak{D}(x, D)$ is defined by the following

$$\begin{aligned}&\mathfrak{D}(x,D)\psi(x)\\ = &\int_0^\infty \int_{S^{d-1}} \left(\psi(x+y) - \psi(y) - \frac{\langle y, \nabla\psi(x)\rangle}{1+|y|^2}\right) \frac{d|y|}{|y|^{1+\alpha(x)}} \tilde{\mu}(x, ds),\end{aligned}$$

where $\alpha : \mathbb{R}^d \to (0, 2)$ is a continuous function, S^{d-1} is the unit sphere of $\mathbb{R}^d$, $\langle \cdot, \cdot \rangle$ stands for the inner product on $\mathbb{R}^d$, y is represented by its magnitude $|y|$ and the unit vector $s := \frac{y}{|y|} \in S^{d-1}$ in the direction y, and $\tilde{\mu}$ is symmetric and is called spectral measure which is a finite measure on $\mathcal{B}(S^{d-1})$ depending smoothly on $x \in \mathbb{R}^d$.
If $\tilde{\mu}$ is rotation invariant, then $\mathfrak{D}(x, D) = -(-\Delta)^{\alpha(x)/2}$.
In particular, $\mathfrak{D}(x, D) = -(-\Delta)^{\alpha/2}$ is a fractional Laplacian operator whenever $\tilde{\mu}$ and α are indenpent of x.

The symbol of $\mathfrak{D}(x, D)$ has the form

$$\begin{aligned} \mathfrak{D}(x,\xi) &= \int_0^\infty \int_{S^{d-1}} \left(e^{i\langle \xi, y\rangle} - 1 - \frac{i\langle \xi, y\rangle}{1+|y|^2} \right) \frac{d|y|}{|y|^{1+\alpha(x)}} \tilde{\mu}(x, ds) \\ &= -\int_{S^{d-1}} |\langle \xi, y\rangle|^{\alpha(x)} \mu(x, ds), \end{aligned} \tag{1.2}$$

with

$$\mu(x, ds) := \frac{1}{\alpha(x)} \Gamma(|1-\alpha(x)|) \cos(\frac{1}{2}\pi\alpha(x)) \tilde{\mu}(x, ds) \text{ for } \alpha(x) \neq 1$$

$$\mu(x, ds) := \frac{1}{2}\pi \tilde{\mu}(x, ds) \text{ for } \alpha(x) = 1.$$

Such operator is studied by many authors, Bass, Komatsu, Kikuchi, Negoro, Jacob, Kolokoltsov, Tsuchiya, . . .

Let $(E, \mathcal{E})$ be a measurable space and let ν be a σ-finite measure on $(E, \mathcal{E})$. Given a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. Let us denote by π a Poisson random measure on $[0, \infty) \times \mathbb{R}^d \times E$ with intensity measure $dtdx\nu(d\xi)$, that is,

- $\pi([0, T] \times A \times B)$ has the Poisson distribution with parameter $T|A|\nu(B)$.
- For any disjoint subsets $\Gamma_1, \cdots, \Gamma_n$, the random variables $\pi(\Gamma_1), \cdots, \pi(\Gamma_n)$ are independent.

The compensated Poisson random measure is given by

$$\hat{\pi}(dt, dx, d\xi) := \pi(dt, dx, d\xi) - dtdx\nu(d\xi).$$

Proposition

The stochastic integral

$$\int_0^t \int_{\mathbb{R}^d} \phi(s,x,\xi)\hat{\pi}(ds,dx,d\xi)$$

is well defined in $L^p(\Omega,\mathcal{F},\mathbb{P})$ *if* $\phi:[0,\infty)\times\mathbb{R}^d\times E\times\Omega\to\mathbb{R}$ *satisfying*

$$\int_0^t \int_{\mathbb{R}^d} \int_E \mathbb{E}[|\phi(s,x,\xi)|^p] ds dx \nu(d\xi) < \infty.$$

Furthermore, we have

$$\mathbb{E}\left[\left|\int_0^t \int_{\mathbb{R}^d} \int_E \phi(s,x,\xi)\hat{\pi}(ds,dx,d\xi)\right|^p\right] \leq C \int_0^t \int_{\mathbb{R}^d} \int_E \mathbb{E}[|\phi(s,x,\xi)|^p] ds dx \nu(d\xi).$$

Let $\mathfrak{p}(t,x,y)$ denote the solution of the following initial value problem

$$\frac{\partial \mathfrak{p}}{\partial t}(t,x,y) = \mathfrak{D}(x,D)\mathfrak{p}(t,x,y), \qquad \lim_{t\downarrow 0} \mathfrak{p}(t,x,y) = \delta_x(y).$$

Definition

An $\mathcal{F}_t$-adapted $L^p(\mathbb{R}^d)$-valued stochastic process $u(t,x,\omega)$ is a mild solution of (1.1) *if the following integral equation is satisfied:*

$$\begin{aligned} u(t,x) &= \int_{\mathbb{R}^d} \mathfrak{p}(t,x,y)u_0(y)dy \\ &+ \sum_{j=1}^{d} \int_0^t \int_{\mathbb{R}^d} \frac{\partial \mathfrak{p}}{\partial y_j}(t-s,x,y) f_j(y,u(s,y))dyds \\ &+ \int_0^t \int_{\mathbb{R}^d} \int_E \mathfrak{p}(t-s,x,y)\sigma(y,u(s,y);\xi)\hat{\pi}(ds,dy,d\xi) \quad a.s. \end{aligned} \qquad (2.1)$$

$(A1)$ f satisfies the following polynomial growth and and Lipschitz conditions: there exist positive functions $a_1 \in L^1(\mathbb{R}^d), a_2 \in L^p(\mathbb{R}^d)$ and $a_3 \in L^q(\mathbb{R}^d)$ such that

$$\sum_{j=1}^{d} |f_j(x,z)| \le C(a_1(x) + a_2(x)^{p-1}|z| + |z|^p)$$

$$\sum_{j=0}^{d} |f_j(x,z) - f_j(x,z')| \le C(a_3(x) + |z|^{p-1} + |z'|^{p-1})|z - z'|$$

$(A2)$ The coefficient σ fulfills the following linear growth and Lipschitz conditions: there exists a positive function $a_4 \in L^1(\mathbb{R}^d)$ such that

$$\int_E |\sigma(x,z;\xi)|^p \nu(d\xi) \le Ca_4(x),$$

$$\int_E |\sigma(x,z;\xi) - \sigma(x,z';\xi)|^p \nu(d\xi) \le C|z - z'|^p.$$

Theorem (Main Theorem 1)

Assume $\alpha_m = \inf_{x\in\mathbb{R}^d} \alpha(x) > 3/2$.
(1) *(Wu-Xie(2012)) If* $d \geq 1$ *and* $\dot{w}(t,x)$ *is a pure jump space time noise, then there exists a unique mild solution in* $L^p(\mathbb{R}^d)$ *with*

$$d(p-1) + p \leq \alpha_m p.$$

Furthermore, there exists a predictable modification for the solution $u(t)$ *of* (1.1).
(2) *(Truman-Wu (2006) Jacob et al. (2010)) If* $d = 1$ *and* $\dot{w}(t,x)$ *is a space time noise with Gaussian part, then there exists a unique mild solution in* $L^2(\mathbb{R})$.

Lemma

(1) *The Chapman-Kolmogorov equation holds:*

$$\mathfrak{p}(t+s,x,y)=\int_{\mathbb{R}^d}\mathfrak{p}(t,x,z)\mathfrak{p}(s,z,y)dz,\quad s,t\in[0,\infty),\ x,y\in\mathbb{R}^d.$$

(2) *Let* $\beta\in(0,1/(d+\alpha))$ *and* $\gamma\in(0,1-\beta(d+\alpha))$.

$$\mathfrak{p}(t,x,y)=\mathfrak{p}_\alpha(t,x-y)(1+O(t^\beta))+\frac{O(t^\gamma)}{1+|x-y|^{\alpha+d}}.$$

$$\frac{\partial\mathfrak{p}}{\partial t}(t,x,y)=\frac{\partial\mathfrak{p}_\alpha}{\partial t}(t,x-y)+O(t^{\beta-1})\mathfrak{p}_\alpha(t,x-y)+\frac{O(t^{\gamma-1})}{1+|x-y|^{\alpha+d}}.$$

$$\frac{\partial\mathfrak{p}}{\partial x_j}(t,x,y)=\frac{\partial\mathfrak{p}_\alpha}{\partial x_j}(t,x-y)+O(t^{\beta-1/\alpha})\mathfrak{p}_\alpha(t,x-y)+\frac{O(t^{\gamma-1/\alpha})}{1+|x-y|^{\alpha+d}}.$$

Lemma

For each $p \in (1,2]$, *we have that*

$$(3) \sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} |\mathfrak{p}(t,x,y)|^p dx \le C t^{-\frac{d(p-1)}{\alpha}}.$$

$$(4) \left| \int_0^t \int_{\mathbb{R}^d} \mathfrak{p}(t-s,\cdot,y) u(s,y) ds dy \right|_{L^p} \le C \int_0^t |u(s)|_{L^p} ds$$

$$(5) \left| \int_0^t \int_{\mathbb{R}^d} \frac{\partial \mathfrak{p}}{\partial y_j}(t-s,\cdot,y) u(s,y) ds dy \right|_{L^p} \le C \int_0^t (t-s)^{-\frac{d(p-1)+p}{\alpha p}} |u(s)|_{L^1} ds$$

From now, we will consider a special case of Equation (1.1)

$$\begin{aligned}&\partial_t u(t,x) + div(f(u))(t,x)\\ =&-(-\Delta)^{\alpha/2}u(t,x) + \sigma(x,u(t,x))\dot{w}(t), x \in \mathbb{R}^d,\end{aligned} \tag{3.1}$$

If $\gamma = 0, \sigma = 0, f(u) = u^2/2$, (1.1) is the classical inviscid Burgers equation.

According to α, this equation is called to be

- supercritical if $\alpha \in (0,1)$,
- critical if $\alpha = 1$,
- subcritical if $\alpha \in (1,2)$.

Hope-Cole transform can not be applied to this case.

Remark (Some results for Fractal PDEs)

- *Biler, Funaki and Woyczynski(1998) established the existence and uniqueness of weak solution for $\alpha \in (3/2, 2)$ and $d = 1$. However, for small α, the well-posed question is open.*
- *Droniou, Gallouet and Vovelle(2003) studied regularity of solution for subcritical case in L^∞ space.*
- *Alibaud (2007) introduced definition of entropy solution for supercritical in L^∞ space.*
- *Kiselev et al(2008) proved the existence and uniqueness of weak solution for critical. However, for supercritical, the uniqueness of weak solution for supercritical fails, see Alibaud and Andreianov (2010).*
- ...

Lemma

For each $\alpha \in (0,2)$, there exists a positive constat $\kappa = \kappa(\alpha)$ such that for any $\psi \in \mathcal{S}(R^d)$ and for any $r > 0$,

$$\begin{aligned} -(-\Delta)^{\alpha/2}(\psi)(x) =& \kappa \int_{|z|\geq r} \frac{\psi(x+z)-\psi(x)}{|z|^{d+\alpha}} dz \\ &+ \kappa \int_{|z|<r} \frac{\psi(x+z)-\psi(x)-\langle \nabla\psi(x), z\rangle}{|z|^{d+\alpha}} dz \\ :=& \mathcal{D}_\alpha^r \psi(x) + \mathcal{D}_{\alpha,r}\psi(x) \end{aligned}$$

We also consider the entropy function $\eta \in C^2(\mathbb{R})$ is convex and for each η and the flux f, we denote by $\Psi : \mathbb{R} \to \mathbb{R}^d$ a primitive function of $\eta' f'$, where $f' = (f_1', \cdots, f_d')$.
The functions η and Ψ will be called entropy and entropy flux.

Definition

(SE) (**Stochastic entropy solution**) *An* $L^2(\mathbb{R}^d)$*-valued* $\mathcal{F}_t$*-adapted process* $u(t,\cdot)$ *is called a stochastic entropy solution of* (3.1), *if*
(1) *For each* $T>0$, $\mathbb{E}[\int_0^T \|u(t)\|^p_{L^p(\mathbb{R}^d)}dt] < \infty$ *for any* $p \in \mathbb{N}$;
(2) *For any* $0 \le \psi \in C_c^\infty(\mathbb{R}^d)$ *and* (η, Ψ) *of entropy and entropy-flux such that for any* $0 \le s < t \le T$

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \eta(u(t,x))\psi(x)dx \\
&\le \int_{\mathbb{R}^d} \eta(u(s,x))\psi(x)dx + \int_s^t \int_{\mathbb{R}^d} \langle \Psi(u(\tau,x)), \nabla\psi(x)\rangle dxd\tau \\
&+ \int_s^t \int_{\mathbb{R}^d} \eta'(u(\tau,x))\psi(x)\mathcal{D}_\alpha^r u(\tau,x)dxd\tau + \int_s^t \int_{\mathbb{R}^d} \eta(u(\tau,x))\mathcal{D}_{\alpha,r}\psi(x)dxd\tau \\
&+\frac{1}{2}\int_s^t \int_{\mathbb{R}^d} \eta''(u(\tau,x))\sigma^2(u(\tau,x))\psi(x)dxd\tau \\
&+ \int_s^t \int_{\mathbb{R}^d} \eta'(u(\tau,x))\sigma(u(\tau,x))\psi(x)dxdw(\tau) \ a.s.,
\end{aligned}
$$

Definition

(SSE) (**Strong stochastic entropy solution**) *We say $u(t,x)$ is a strong stochastic entropy solution of* (3.1) *if it is a stochastic entropy solution and furthermore the following holds:*
For any $L^2(\mathbb{R}^d)$-valued $\mathcal{F}_t$-adapted process $v(t)$ satisfying the properties in (SE)-(1),

$$\mathbb{E}\Big[\int_s^t \int_{\mathbb{R}^d} \varrho(\tau, u(t,y), y) dy dw(\tau)\Big]$$
$$\leq \mathbb{E}\Big[\int_s^t \int_{\mathbb{R}^d} \partial_u \varrho(\tau, u(\tau,y), y)\sigma(y, u(\tau,y)) dy d\tau\Big] + e(s,t),$$

where $\varrho(\tau,u,y) = \int_{\mathbb{R}^d} \eta'(v(\tau,x)-u)\sigma(v(\tau,x))\phi(x,y)dx$ with $0 \leq \phi \in C_c(\mathbb{R}^d \times \mathbb{R}^d)$ and $\lim_{\Delta_n \to 0} \sum_{i=0}^{n-1} e(t_i, t_{i+1}) = 0$ for each partition $0 = t_0 < t_1 < t_2 < \cdots < t_n = t$ of $[0,t]$ with mesh Δ_n.

Remark

(1) *The stochastic entropy solution* $u(t,x)$ *can be roughly explained as to a solution of the inequality*

$$\begin{aligned}&\partial_t \eta(u(t,x)) + div_x \Psi(u)(t,x)\\ \leq&\mathcal{D}_\alpha u(t,x) + \frac{1}{2}\eta''(u(t,x))\sigma^2(u(t,x)) + \eta'(u(t,x))\sigma(u(t,x))\dot{w}(t).\end{aligned}$$

in distribution sense for any η *and* Ψ*.*
(2) *Stochastic entropy solution* $u(t,x)$ *of* (3.1) *is a weak one, i.e., for any test function* $\psi \in C_c^\infty(\mathbb{R}^d)$,

$$\begin{aligned}&\int_{\mathbb{R}^d} u(t,x)\psi(x)dx = \int_{\mathbb{R}^d} u_0(x)\psi(x)dx + \int_0^t \int_{\mathbb{R}^d} u(s,x)\mathcal{D}_\alpha(\psi(x))dxds\\ &+ \int_0^t \int_{\mathbb{R}^d} \langle f(u(s,x)), \nabla\psi(x)\rangle dxds + \int_0^t \int_{\mathbb{R}^d} \sigma(u(s,x))\psi(x)dxdw(s)\ a.s.\end{aligned}$$

Assumption:

We assume the following holds:
(i) The flux $f \in C^2(\mathbb{R};\mathbb{R}^d)$ and there exist $C > 0$ and $p \in \mathbb{N}$ such that

$$|f(u)| \le C(1 + |u|^p),\ u \in \mathbb{R}.$$

(ii) There exists a non-negative increasing function h with $h(0) = 0$ such that

$$|\sigma(u) - \sigma(v)| \le |u - v|^{1/2} h(|u - v|).$$

Theorem (Main theorem 2)

Assume u and v are stochastic entropy solution with initial data u_0 and respectively stochastic strong entropy solution with initial data v_0. Then, the L^1-contractive property holds, that is

$$\mathbb{E}[\|u(t,\cdot)-v(t,\cdot)\|_{L^1(\mathbb{R}^d)}] \leq \mathbb{E}[\|u_0(\cdot)-v_0(\cdot)\|_{L^1(\mathbb{R}^d)}].$$

In particular, we have that if $u_0 = v_0$ $a.e.$ in x and $a.s.$, then $u(t,x) = v(t,x)$ $a.e.$ in x and $a.s.$ for each $t > 0$.

Remark

- *It is proved by approximation and the doubling variables technique of Kruzkhov.*
- $\mathbb{E}[\|\big(u(t,\cdot)-v(t,\cdot)\big)^+\|_{L^1(\mathbb{R}^d)}] \leq \mathbb{E}[\|\big(u_0(\cdot)-v_0(\cdot)\big)^+\|_{L^1(\mathbb{R}^d)}]$.
  This implies the comparison principle holds: if $u_0 \leq v_0$, then $u(t,x) \leq v(t,x)$.

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Thank you for your attention!