Concentration-Diffusion Phenomena for the Boussinesq System

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Boussinesq Equations

$$
\begin{aligned}
u_t - \Delta u + u \cdot \nabla u + \nabla p &= g\theta \quad \text{in } \mathbb{R}^n \times (0,T) \\
\theta_t - \Delta\theta + u \cdot \nabla\theta &= 0 \quad \text{in } \mathbb{R}^n \times (0,T) \\
\operatorname{div} u &= 0 \quad \text{in } \mathbb{R}^n \times (0,T) \\
\theta(0) = \theta_0,\ u(0) &= u_0 \quad \text{at } t = 0
\end{aligned}
$$

where the gravitational field $g(x)$ decays as $\frac{1}{|x|^{n-1}}$

Aims

- Existence and uniqueness of mild solutions (u, θ):

$$u(t) = e^{t\Delta}u_0 - \int_0^t e^{(t-\tau)\Delta} P(u \cdot \nabla u)(\tau)\, d\tau + \int_0^t e^{(t-\tau)\Delta} P(g\theta)(\tau)\, d\tau$$

$$\theta(t) = e^{t\Delta}\theta_0 - \int_0^t e^{(t-\tau)\Delta} (u \cdot \nabla \theta)(\tau)\, d\tau$$

- Strong solutions
- Spatial asymptotic behavior (for fixed t)
- Concentration-diffusion phenomena

Definition (Weighted L^∞-spaces)

$$L^\infty_\mu = \{u \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n) : \|u\|_{L^\infty_\mu} = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} (1+|x|)^\mu |u(x)| < \infty\}$$

Theorem (Mild Solutions)

Let $u_0 \in L^\infty_{\mu,\sigma}$, $\theta_0 \in L^\infty_\nu$ *where* $\mu \in (0,n]$, $\nu > \max(0, \mu - n + 1)$ *and* $g \in L^\infty_{n-1}$. *Then*

$$\exists!\ (u,\theta) \in C_{\omega^*}([0,T]; L^\infty_\mu) \times C_{\omega^*}([0,T]; L^\infty_\nu)$$

where

$$C\big(\sqrt{T}+T^{1+\nu/2}\big)\Big(\big(\|u_0\|_{L^\infty_\mu} + \|\theta_0\|_{L^\infty_\mu}\big)^{1/2} + \big(\|u_0\|_{L^\infty_\mu} + \|\theta_0\|_{L^\infty_\mu} + \|g\|_{L^\infty_{n-1}}\big)\Big) < 1$$

Ideas of proof

- semigroup $e^{t\Delta}, t \geq 0$ is only weakly-* continuous on L^∞_μ
- analyze the bilinear integral operator

$$\mathcal{B}(u,v) = -\int_0^t e^{(t-\tau)\Delta} P(u \cdot \nabla v)(\tau)\, d\tau$$

on $C_{\omega^*}([0,T]; L^\infty_\mu) \times C_{\omega^*}([0,T]; L^\infty_\mu) \to C_{\omega^*}([0,T]; L^\infty_\mu)$

- By analogy, $\mathcal{D}(u,\theta) = -\int_0^t e^{(t-\tau)\Delta}(u \cdot \nabla \theta)\, d\tau$
- Analyze on $C_{\omega^*}([0,T]; L^\infty_\nu) \to C_{\omega^*}([0,T]; L^\infty_\mu)$

$$\mathcal{C}(\theta) = \int_0^t e^{(t-\tau)\Delta} P(g\theta)(\tau)\, d\tau$$

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- Technical tool: estimate of convolution integrals

$$\int_0^t \int_{\mathbb{R}^n} \frac{dy\, ds}{(|x-y| + \sqrt{t-s})^{n+1}(1+|y|)^\mu}$$

- Banach's fixed point theorem for small data T or (u_0, θ_0)

Theorem (Strong Solutions)

Let $u_0 \in L^\infty_{\mu,\sigma}$, $\theta_0 \in L^\infty_\nu$ *where* $\mu \in (0, n]$, $\nu > \max(0, \mu - n + 1)$ *and* $g \in W^{1,\infty}_{n-1}$. *Then additionally*

$$(u, \theta) \in C^1((0, T); \mathrm{BUC}) \cap C((0, T); W^{2,\infty})$$

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Ideas of proof

Analysis of the analytic semigroup $e^{t\Delta}$, $t > 0$ on BUC and boot strapping argument in homogeneous Besov spaces $\dot{B}^0_{\infty,1}$ and $\dot{B}^s_{\infty,1}$, $s > 0$, with norm $\sum_{j\in\mathbb{Z}} 2^{js} \|\varphi_j * u\|_\infty$

Theorem (Spatial asymptotic behavior)

Let $\mu > \frac{n+2}{2}$, $\nu > 3$, $u_0 \in L^\infty_{\mu,\sigma}$, $\theta_0 \in L^\infty_\nu$ *and* $g \in W^{1,\infty}_{n-1}$. *Then the strong solution* (u, θ) *satisfies for* $|x| \gg \sqrt{t}$

$$u(x,t) = e^{t\Delta}u_0 - \nabla\left(\nabla\Big(\frac{1}{\omega_n |x|^{n-2}}\Big) \cdot \int_0^t \int_{\mathbb{R}^n} (g\theta)\, dy\, ds\right)$$
$$- \nabla\left(\nabla^2\Big(\frac{1}{\omega_n |x|^{n-2}}\Big) : \int_0^t \int_{\mathbb{R}^n} (u \otimes u + y \otimes g\theta)\, dy\, ds\right)$$
$$+ O_t\big(|x|^{-n-2+\varepsilon}\big)$$

$$\theta(x,t) = e^{t\Delta}\theta_0 + O_t(|x|^{-\mu-\nu})$$

where $\varepsilon > 0$, $\omega_n = |\partial B_1|$ *for* $B_1 \subset \mathbb{R}^n$.

Comparison with Navier-Stokes

- For Navier-Stokes (cf. Brandolese-Vigneron JMPA 2007)

$$u(x,t) = e^{t\Delta}u_0 - \nabla\left(\nabla^2\Big(\frac{1}{\omega_n|x|^{n-2}}\Big) : \int_0^t \int_{\mathbb{R}^n} u \otimes u\right) + \dots,$$

i.e., $u(t,x) \sim e^{t\Delta}u_0 + O\big(|x|^{-n-1}\big)$.

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- In our case,

$$u(x,t) = e^{t\Delta}u_0 - \nabla\left(\nabla\Big(\frac{1}{\omega_n|x|^{n-2}}\Big) \cdot \int_0^t \int_{\mathbb{R}^n} (g\theta)\,dy\,ds\right)$$
$$- \nabla\left(\nabla^2\Big(\frac{1}{\omega_n|x|^{n-2}}\Big) : \int_0^t \int_{\mathbb{R}^n} (u \otimes u + y \otimes g\theta)\,dy\,ds\right) + \dots$$

  i.e., $u(t,x) \sim e^{t\Delta}u_0 + O\big(|x|^{-n}\big)$

- Is there a chance to let vanish the leading terms for all times or from time to time?

Ideas of proof

- The Stokes semigroup $e^{t\Delta}P$ has due to the Helmholtz correction term $\partial_i\partial_j(-\Delta)^{-1}e^{t\Delta}$ the kernel

$$E(x,t) = \nabla^2\Big(\frac{1}{\omega_n|x|^{n-2}}\Big) + \ldots \text{for } \; |x| \gg \sqrt{t}$$

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- Taylor's formula for convolutions (for smooth decaying functions)

$$\begin{aligned} f * h(x) &= \int h(y)\Big(\sum_{|\alpha|\le m}\frac{(-y)^\alpha}{\alpha!}\partial^\alpha f(x) + \ldots\Big)dy \\ &= \sum_{|\alpha|\le m}\frac{(-1)^{|\alpha|}}{\alpha!}\Big(\int y^\alpha h(y)\,dy\Big)\partial^\alpha f(x) + \ldots \end{aligned}$$

applied to $f(x) = E(x,t)$, $f(x) = \nabla E(x,t)$ and $h(x) = g\theta$, $h(x) = u\otimes u$

Theorem (Concentration-Diffusion)

Let $0 \neq g \in W^{2,\infty}_{n-1}$, $\Delta g \in L^1 \cap L^2$ *and* $u_0 \in L^\infty_{n+2}$ *satisfy certain parity/symmetry assumptions. Choose* N *arbitrary epochs*

$$0 < t_1 < \dots t_N < T.$$

Then there exists a temperature $\theta_0 \in \mathcal{S}(\mathbb{R}^n)$ *and epochs* t_i', t_i'' *arbitrarily close to* t_i *such that (more or less)*

$$|u(x, t_i')| \leq C|x|^{-n-1}, \quad |u(x, t_i'')| \geq c|x|^{-n}.$$

Here (u, θ) *is the solution of the Boussinesq system with initial data* $(\eta u_0, \eta\theta_0)$ *with a sufficiently small* η*.*

Parity: g *is in* x *odd (or even (unphysical)),*
$\tilde{}$*-symmetry:* $g(\tilde{x}) = \tilde{g}(x)$, $u_0(\tilde{x}) = \tilde{u}_0(x)$ *where*
$\tilde{x} = (x_2, x_3, \dots, x_n, x_1)$
(Choose θ_0 *odd (or even) in* x *and* $\tilde{}$*-symmetric)*

Ideas of proof (I)

- Find θ_0 $\tilde{}$-symmetric such that $\int_{\mathbb{R}^n} g_j \theta_0 \, dy \neq 0$ is j-independent, but

$$\int_0^t \int_{\mathbb{R}^n} g_1 \cdot (e^{s\Delta}\theta_0) \, dy \, ds \quad \text{has a simple zero at each } t_i$$

- Choose $\eta > 0$ so small that for the solution (u, θ) with initial value $(\eta u_0, \eta \theta_0)$

$$\int_0^t \int_{\mathbb{R}^n} g_1 \, \theta \, dy \, ds \sim \int_0^t \int_{\mathbb{R}^n} g_1 \, (e^{s\Delta} \, \eta \, \theta_0) \, dy \, ds$$

changes sign nearby each t_i

Ideas of proof (II)

- Plancherel's Theorem $\Rightarrow$

$$\int_0^t \int_{\mathbb{R}^n} g_1 \cdot (e^{s\Delta}\theta_0)\, dy\, ds$$

$$= F(t) := \int_{\mathbb{R}^n} \left(1 - e^{-t|\xi|^2}\right) \frac{\widehat{\Delta g_1}(\xi)}{|\xi|^4} \cdot \hat{\theta}_0(\xi)\, d\xi$$

- Choose $\hat{\theta}_0$ as a linear combination of "almost" δ-distributions: $\hat{\theta}_0(\xi) \sim \sum_{j=0}^{N} \lambda_j \delta_{\alpha_j}(\xi)$ where $\mathbb{R}^n \ni \alpha_j = \sqrt{1+\delta j}\, \alpha_0 \neq 0$
- Find $\tilde{\lambda}_j$ such that

$$F(t) = \sum_{j=0}^{N} \lambda_j \left(1 - e^{-t|\alpha_j|^2}\right) \frac{\widehat{\Delta g_1}(\alpha_j)}{|\alpha_j|^4} = \sum_{j=0}^{N} \tilde{\lambda}_j (1 - T^{1+\delta j})$$

$(T(t) := e^{-|\alpha_0|^2 t})$ changes sign at t_i:

$$F(t_i) = 0 \text{ but } F'(t_1) \neq 0$$

Ideas of proof (III)

- Find $\tilde{\lambda} \in \mathbb{R}^{N+1}$ solving the linear system $M(\delta)\tilde{\lambda} = e_{N+1}$:

$$M(\delta) = \begin{pmatrix} 1 - T_1^1 & 1 - T_1^{1+\delta} & \dots & 1 - T_1^{1+\delta N} \\ \vdots & \vdots & \ddots & \vdots \\ 1 - T_N^1 & 1 - T_N^{1+\delta} & \dots & 1 - T_N^{1+\delta N} \\ 1 \cdot T_1^1 & (1+\delta) T_1^{1+\delta} & \dots & (1+\delta N) T_1^{1+\delta N} \end{pmatrix}$$

Ideas of proof (III)

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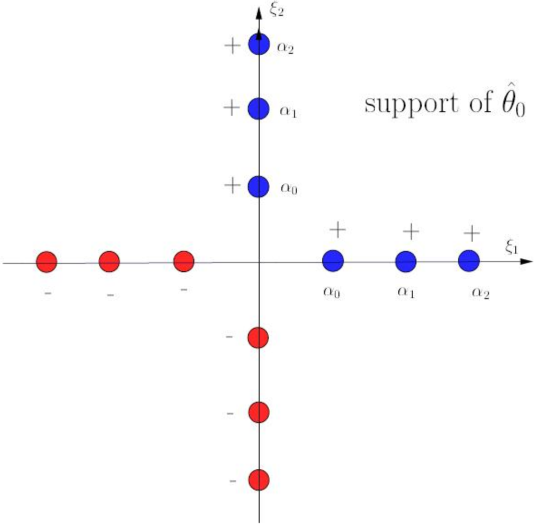
$$M(\delta) = \begin{pmatrix} 1 - T_1^1 & 1 - T_1^{1+\delta} & \dots & 1 - T_1^{1+\delta N} \\ \vdots & \vdots & \ddots & \vdots \\ 1 - T_N^1 & 1 - T_N^{1+\delta} & \dots & 1 - T_N^{1+\delta N} \\ 1 \cdot T_1^1 & (1+\delta)T_1^{1+\delta} & \dots & (1+\delta N)T_1^{1+\delta N} \end{pmatrix}$$

- Actually, $\det M(1) \neq 0$, see Brandolese (Indiana UMJ 2009). Since $\det M(\delta)$ is analytic, we find small $\delta > 0$ such that $\det M(\delta) \neq 0$
- We find $\tilde{\lambda} \in \mathbb{R}^{N+1}$ solving $F(t_i) = 0$ for all i, but $F'(t_1) = 1$. Changing t_1 with t_i, we also get $F'(t_i) \neq 0$.



ξ_2
ξ_1
support of $\hat{\theta}_0$
α_2
α_1
α_0
α_0
α_1
α_2
+
-