Weak Neumann implies $\mathcal{H}^\infty$ calculus for the Stokes operator

Matthias Geissert

TU Darmstadt/TU Dresden
Germany

The 7th Japanese-German International Workshop on Mathematical Fluid Dynamics
Waseda
Nov. 2012

Outline

The Helmholtz projection

- Let $1 < q < \infty$, $\Omega \subset \mathbb{R}^n$ be a domain.
- We say that the *Helmholtz decomposition* exists if

$$L^q(\Omega)^n = L^q_\sigma(\Omega) \oplus G_q(\Omega),$$

where

$$G_q(\Omega) := \{g \in L^q(\Omega)^n : \exists h \in \widehat{W}^{1,q}(\Omega) \text{ such that } g = \nabla h\},$$
$$L^q_\sigma(\Omega) := \{f \in L^q(\Omega)^n : \int_\Omega f \nabla \varphi = 0,\ \varphi \in \widehat{W}^{1,q'}(\Omega)\}.$$

In this case there exists the *Helmholtz projection*

$$P_q : L^q(\Omega)^n \to L^q_\sigma(\Omega).$$

The weak Neumann problem

Consider

$$\begin{cases} \Delta v &= \nabla \cdot g \quad \text{in } \Omega, \\ n \cdot \nabla v &= n \cdot g \quad \text{on } \partial\Omega. \end{cases} \qquad (\mathrm{WNP_q})$$

Does there exists a unique weak solution $v \in \widehat{W}^{1,q}(\Omega)$ of $(\mathrm{WNP_q})$, i.e.

$$\int_\Omega \nabla v \nabla \varphi = \int_\Omega g \nabla \varphi, \quad \varphi \in \widehat{W}^{1,q'}(\Omega),$$

for $g \in L^q(\Omega)^n$ satisfying $\|v\|_{\widehat{W}^{1,q}(\Omega)} \leq C\|g\|_{L^q(\Omega)^n}$.

Proposition

(WNP_q) is uniquely solvable $\Leftrightarrow$ P_q exists.

The Stokes operator

Let $1 < q < \infty$ and $\Omega \subset \mathbb{R}^n$ be a domain such that the Helmholtz projection exists. Set

$$D(A_q) = W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) \cap L^q_\sigma(\Omega)$$

and define the *Stokes operator*

$$A_q : \begin{cases} D(A_q) & \rightarrow & L^q_\sigma(\Omega), \\ u & \mapsto & P_q \Delta u. \end{cases}$$

$\mathcal{H}^\infty$-calculus

Let

- $A : D(A) \to X$ be sectorial,
- $\theta \in (0, \pi - \Phi_A)$.

For $\psi \in (\theta, \pi - \Phi_A)$ we define

$$f(A)g := \frac{1}{2\pi i} \int_{\partial\Sigma_\psi} f(z)(z - A)^{-1} g \, \mathrm{d}z, \quad g \in X.$$

We say A admits a *bounded $\mathcal{H}^\infty$-calculus* if for $\theta \in (0, \pi - \Phi_A)$

$$\|f(A)\|_{\mathcal{L}(X)} \le C_\theta \|f\|_{L^\infty(\overline{\Sigma_\theta}^c)}, \quad f \in \mathcal{H}_0^\infty(\overline{\Sigma_\theta}^c).$$

$\mathcal{H}^\infty$-calculus

Theorem (P. Kunstmann, M. G.)

Assume that

- $\Omega \subset \mathbb{R}^n$ *has uniform* C^3*-boundary,*
- (WNP_q) *is uniquely solvable for some* $q_0 \in (1, \infty)$.

Then the Stokes operator $\lambda_0 - A_q$ *has a bounded* $\mathcal{H}^\infty$*-calculus for some* $\lambda_0 > 0$ *and* $q \in [\min\{q_0, q_0'\}, \max\{q_0, q_0'\}]$.

Related results

- Noll/Saal '03: C^3-boundary, bounded or exterior domain
- Abels '05: $C^{1,1}$-boundary, bounded or exterior domain, layer, layer-like, aperture domains
- Kalton/Kunstmann/Weis '06: $C^{2+\alpha}$-boundary, bounded domain
- Farwig/Ri '07: unbounded cylinders with several exists to inifinity
- Kunstmann '08: uniform $C^{2+\alpha}$-boundary, $L^q_\sigma(\Omega) \cap L^2_\sigma(\Omega)$, $2 < q < \infty$, or $L^q_\sigma(\Omega) + L^2_\sigma(\Omega)$, $1 < q < 2$
- Abels/Terasawa '09, Abels '10: uniform $W^{2-1/r,r}$-boundary, Helmholtz exists +(suitable decomposition of pressure)

Idea of Proof: Key tool

Proposition (N.J. Kalton, P. Kunstmann, L. Weis '06)

Assume that

- (X_0, X_1) *interpolation couple of reflexive and B-convex spaces,*
- $P_j : X_j \to Y_j$ *compatible surjections with compatible right inverses* $J_j : Y_j \to X_j$, $j = 0, 1$,
- L_j *has an* $\mathcal{H}^\infty$*-calculus in* X_j, A_j $\mathcal{R}$*-sectorial on* Y_j, *for* $\alpha > 0 > \beta$

$$P_0((X_0)^{\cdot}_{\alpha,L_0}) = (Y_0)^{\cdot}_{\alpha,A_0}, \quad P_1((X_1)^{\cdot}_{\beta,L_1}) = (Y_1)^{\cdot}_{\beta,A_1},$$
$$J_0((Y_0)^{\cdot}_{\alpha,A_0}) = (X_0)^{\cdot}_{\alpha,L_0}, \quad J_1((Y_1)^{\cdot}_{\beta,A_1}) = (X_1)^{\cdot}_{\beta,L_1},$$

Then, A_θ *has* $\mathcal{H}^\infty$*-calculus on* $Y_\theta = [Y_0, Y_1]_\theta$, $\theta \in (0, 1)$.

Idea of Proof

Apply the previous proposition to

- $$X_0 := L^2(\Omega), Y_0 := L^2_\sigma(\Omega),$$
  $$X_1 := L^q(\Omega), Y_1 := L^q_\sigma(\Omega)$$

- $$X_{\alpha,L_0} := \begin{pmatrix} D((-L_0)^\alpha) & ,\alpha \geq 0, \\ (X, \|(1+L_0)^\alpha \cdot \|)^\sim & ,\alpha < 0 \end{pmatrix},$$

  L_0/L_1 shifted Dirichlet-Laplacian, A_0/A_1 shifted Stokes-Operator.

- P_0, P_1 Helmholtz projection, $J_i := L_i A_i^{-1}$, $i = 0, 1$.

Idea of Proof: P_i and J_i, $i = 0, 1$

Then,

- $P_1 \in \mathcal{L}(X_{-1,L_1}, Y_{-1,A_1})$
  Idea: $|\langle A_1^{-1} Px, y\rangle| = |\langle x, (A_1^{-1})' P' y\rangle| = |\langle L_1 L_1^{-1} x, (A_1^{-1})' P' y\rangle|$
  $= |\langle L_1^{-1} x, L_1' (A_1^{-1})' P' y\rangle| \le \|x\|_{-1,L_1} \|y\|_{q'}$
- $J_1 \in \mathcal{L}(Y_{-1,A_1}, X_{-1,L_1})$
  Idea: $Y_{-1,A_1} \xrightarrow{A_1^{-1}} Y_{0,A_1} \xrightarrow{L_1} X_{-1,L_1}$.

Idea of Proof: P_i and J_i, $i = 0, 1$

Then,

- $P_1 \in \mathcal{L}(X_{-1,L_1}, Y_{-1,A_1})$
  Idea: $|\langle A_1^{-1}Px, y\rangle| = |\langle x, (A_1^{-1})'P'y\rangle| = |\langle L_1 L_1^{-1}x, (A_1^{-1})'P'y\rangle|$
  $= |\langle L_1^{-1}x, L_1'(A_1^{-1})'P'y\rangle| \leq \|x\|_{-1,L_1}\|y\|_{q'}$
- $J_1 \in \mathcal{L}(Y_{-1,A_1}, X_{-1,L_1})$
  Idea: $Y_{-1,A_1} \xrightarrow{A_1^{-1}} Y_{0,A_1} \xrightarrow{L_1} X_{-1,L_1}$.
- $P_0 \in \mathcal{L}(X_{\alpha,L_0}, Y_{\alpha,A_0}) = \mathcal{L}(D((-\Delta)_2^{\alpha}), D((-A_2)^{\alpha}), \quad \alpha \in (0, \frac{1}{4}),$
- $J_0 \in \mathcal{L}(Y_{\alpha,A_0}, X_{\alpha,L_0})$
  Idea: $Y_{\alpha,A_0} \xrightarrow{A_0^{-1}} Y_{1+\alpha,A_0} \xrightarrow{L_0} X_{\alpha,L_0}$

Idea of Proof: $\mathcal{R}$-sectorial

Theorem (H. Heck, M. Hieber, O. Sawada, M. G.)

Assume that

- $\Omega \subset \mathbb{R}^n$ *has uniform* C^3*-boundary,*
- (WNP_q) *is uniquely solvable for some* $q \in (1, \infty)$.

Then the (shifted) Stokes operator A_q *is* $\mathcal{R}$*-sectorial in* $L^q(\Omega)$.

Weak Neumann implies Stokes

- Abels/Terasawa ’09, Abels ’10: uniform $W^{2-1/r,r}$-boundary, Helmholtz exists + suitable decomposition of pressure
- Geissert/Heck/Hieber/Sawada ’12: uniform C^3-boundary, Helmholtz exists

Weak Neumann implies Stokes

- Abels/Terasawa '09, Abels '10: uniform $W^{2-1/r,r}$-boundary, Helmholtz exists + suitable decomposition of pressure
- Geissert/Heck/Hieber/Sawada '12: uniform C^3-boundary, Helmholtz exists
- Shibata '12: uniform $C^{1,1}$-boundary, Helmholtz exists ($W^{1,q}(\Omega)$ dense in $\widehat{W}^{1,q}(\Omega)$)

Maximal L^p-Regularity (MR)

Assume that

- $\Omega \subset \mathbb{R}^n$, uniform $C^{1,1}$-boundary,
- the Helmholtz decomposition exists on $L^q(\Omega)$ for some $q \in (1, \infty)$

For $f \in L^p(J; L^q(\Omega)^n)$ and $g \in L^p(J; W^{1,q}(\Omega)) \cap W^{1,q}(J; \widehat{W}_0^{-1,q}(\Omega))$ consider

$$\begin{array}{rcll} \partial_t u - \Delta u + \nabla \pi & = & f & \text{in } J \times \Omega, \\ div\, u & = & g & \text{in } J \times \Omega, \\ u & = & 0 & \text{on } J \times \partial\Omega, \\ u(0, \cdot) & = & 0 & \text{in } \Omega. \end{array}$$

Then,

- $\Omega = \mathbb{R}^n$, $\Omega = \mathbb{R}^n_+$,

Maximal L^p-Regularity (MR)

Assume that

- $\Omega \subset \mathbb{R}^n$, uniform $C^{1,1}$-boundary,
- the Helmholtz decomposition exists on $L^q(\Omega)$ for some $q \in (1, \infty)$

For $f \in L^p(J; L^q(\Omega)^n)$ and $g \in L^p(J; W^{1,q}(\Omega)) \cap W^{1,q}(J; \widehat{W}_0^{-1,q}(\Omega))$ consider

$$
\begin{array}{rcll}
\partial_t u - \Delta u + \nabla \pi & = & f & \text{in } J \times \Omega, \\
div\, u & = & g & \text{in } J \times \Omega, \\
u & = & 0 & \text{on } J \times \partial\Omega, \\
u(0, \cdot) & = & 0 & \text{in } \Omega.
\end{array}
$$

Then,

- $\Omega = \mathbb{R}^n$, $\Omega = \mathbb{R}^n_+$,Ω bent halfspace ✓

Maximal L^p-Regularity (MR)

Assume that

- $\Omega \subset \mathbb{R}^n$, uniform $C^{1,1}$-boundary,
- the Helmholtz decomposition exists on $L^q(\Omega)$ for some $q \in (1, \infty)$

For $f \in L^p(J; L^q(\Omega)^n)$ and $g \in L^p(J; W^{1,q}(\Omega)) \cap W^{1,q}(J; \widehat{W}_0^{-1,q}(\Omega))$ consider

$$\begin{array}{rcll} \partial_t u - \Delta u + \nabla \pi & = & f & \text{in } J \times \Omega, \\ div\, u & = & g & \text{in } J \times \Omega, \\ u & = & 0 & \text{on } J \times \partial\Omega, \\ u(0, \cdot) & = & 0 & \text{in } \Omega. \end{array}$$

Then,

- $\Omega = \mathbb{R}^n$, $\Omega = \mathbb{R}^n_+$, Ω bent halfspace ✓
- general Ω ???

The reduced Stokes problem (RMR)

Problem (MR) is equivalent to

$$
\begin{aligned}
\partial_t u - \Delta u + \nabla T(u) &= h, && \text{in } \mathbb{R}_+ \times \Omega,\\
u &= 0, && \text{on } \mathbb{R}_+ \times \partial\Omega,\\
u(0,\cdot) &= 0, && \text{in } \Omega.
\end{aligned}
$$

The reduced Stokes problem (RMR)

Problem (MR) is equivalent to

$$\begin{aligned} \partial_t u - \Delta u + \nabla T(u) &= h, && \text{in } \mathbb{R}_+ \times \Omega, \\ u &= 0, && \text{on } \mathbb{R}_+ \times \partial\Omega, \\ u(0,\cdot) &= 0, && \text{in } \Omega. \end{aligned}$$

Here:

$$\nabla T(u) = (Id - P)[\Delta - \nabla\, div]u$$

i.e.

$$\langle \nabla T(u), \nabla\Phi \rangle = \langle [\Delta - \nabla\, div]u, \nabla\Phi \rangle, \quad \Phi \in \widehat{W}^{1,q'}(\Omega)$$

Localization

Set

$$u := \sum \varphi_j u_j,$$

where u_j solves the corresponding problem in a bent halfspace. Then,

$$\partial_t u - \Delta u + \nabla T(u) = h - \sum(\Delta\varphi_j u + 2\nabla\varphi_j \nabla u_j) + \nabla T(u) - \sum \varphi_j \nabla T_j(u_j)$$

Localization: Estimate on Error terms

$$
\begin{aligned}
&\langle \nabla T(u) - \sum \varphi_j \nabla T_j(u_j), \psi_\sigma + \nabla \psi \rangle \\
&= \langle \nabla T(u), \nabla \psi \rangle - \langle \sum \varphi_j \nabla T_j(u_j), \psi_\sigma + \nabla \psi \rangle \\
&= \langle \nabla T(u), \nabla \psi \rangle - \sum \langle \varphi_j \nabla T_j(u_j), \nabla \psi_{c_j} \rangle + l.o \\
&= \langle \nabla T(u), \nabla \psi \rangle - \sum \langle \nabla T_j(u_j), \nabla(\varphi_j \psi_{c_j}) - \nabla \varphi_j \psi_{c_j} \rangle + l.o. \\
&= \langle [\Delta - \nabla\, div] u, \nabla \psi \rangle - \sum \langle [\Delta - \nabla\, div] u_j, \nabla(\varphi_j \psi_{c_j}) \rangle + l.o. \\
&= \sum \langle [\Delta - \nabla\, div] \varphi_j u_j, \nabla \psi \rangle - \sum \langle \varphi_j [\Delta - \nabla\, div] u_j, \nabla \psi_{c_j} \rangle + l.o. \\
&= l.o.
\end{aligned}
$$

Key estimate:

$$
\langle \nabla T_j(u_j), \nabla \varphi_j \psi_{c_j} \rangle \leq \|T_j(u_j)\|_{H^{1-\varepsilon,q}(supp\, \varphi_j)} \|\nabla \psi\|_{q'} = l.o.
$$

Flux condition on aperture domains

Note that

$$\widehat{W}^{1,q}(\Omega) = \overline{C_c^\infty(\overline{\Omega})}^{\widehat{W}^{1,q}(\Omega} \oplus span\{\chi\}, \quad q < n.$$

Flux condition on aperture domains

Note that

$$\widehat{W}^{1,q}(\Omega) = \overline{C_c^\infty(\overline{\Omega})}^{\widehat{W}^{1,q}(\Omega} \oplus span\{\chi\}, \quad q < n.$$

For $q > n'$ consider

$$g \in L^p(J; W^{1,q}(\Omega)), \partial_t g \in L^p(J; \widehat{W}_0^{-1,q}(\Omega))$$

Flux condition on aperture domains

Note that

$$\widehat{W}^{1,q}(\Omega) = \overline{C_c^\infty(\overline{\Omega})}^{\widehat{W}^{1,q}(\Omega} \oplus span\{\chi\}, \quad q < n.$$

For $q > n'$ consider

$$g \in L^p(J; W^{1,q}(\Omega)), \partial_t g \in L^p(J; \widehat{W}_0^{-1,q}(\Omega))$$

Then,

$$\langle \partial_t u, \nabla \chi \rangle = \langle \partial_t g, \chi \rangle$$

is the *flux condition*.