L^2 boundedness for the solutions to the 2D semilinear heat equations and the 2D Navier-Stokes equations

Takayuki KOBAYASHI

Saga Univ.

November, 2012

joint work with Prof. Masashi Misawa (Kumamoto Univ.)

$\Omega = \mathbb{R}^2$ or an exterior domain in $\mathbb{R}^2$

$L^2((0,\infty)\times\Omega)$ – boundedness for the solutions to the

- 1. Semilinear heat equations in 2D exterior domains.
- 2. Navier-Stokes equations in $\mathbb{R}^2$

1.Heat equations in $\mathbb{R}^2$

1.1. Linear heat equations

Heat equations in $\mathbb{R}^2$

$$\begin{cases} u_t - \Delta u = 0 \\ u(0) = u_0 \end{cases} \qquad \text{(H)}$$

If $u_0 \in L^1(\mathbb{R}^2)$, then

$$\lim_{t\to\infty} t\|u(t)\|_{L^2}^2 = \frac{1}{8\pi}\left|\int_{\mathbb{R}^2} u_0(x)\,dx\right|^2$$

$\Longrightarrow$

$L^2((0,\infty)\times\mathbb{R}^2)$ – boundedness does not hold in general

Definition (Hardy space $\mathcal{H}^1(\mathbb{R}^n)$)

The Hardy space consists of functions f in $L^1(\mathbb{R}^n)$ such that

$$\|f\|_{\mathcal{H}^1(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \sup_{r>0} |\phi_r * f(x)| dx$$

is finite, where $\phi_r(x) = r^{-n}\phi(r^{-1}x)$ for $r>0$ and ϕ is a smooth function on $\mathbb{R}^n$ with compact support in an unit ball with center of the origin $B_1(0) = \{x \in \mathbb{R}^n; |x|<1\}$.

Remark.

- $f \in \mathcal{H}^1(\Omega)$ if and only if the zero extension $\tilde{f} \in \mathcal{H}^1(\mathbb{R}^n)$
- $\mathcal{H}^1 \subset L^1$
- $f \in \mathcal{H}^1(\mathbb{R}^n) \Longrightarrow \int_{\mathbb{R}^n} f(x)dx = 0$

Known Results.

- T. Miyakawa 1996

$$\|\nabla^{\alpha} u(t)\|_{\mathcal{H}^1(\mathbb{R}^2)} \leq C t^{-\frac{|\alpha|}{2}} \|u_0\|_{\mathcal{H}^1(\mathbb{R}^2)}$$

- T. Ogawa and S. Shimizu 2008

$$\int_0^t \|\nabla u(t)\|^2_{\mathcal{H}^1(\mathbb{R}^2)} ds \leq C \|u_0\|^2_{\mathcal{H}^1(\mathbb{R}^2)}$$

Proposition 1(L^2-boundedness)

$$\int_0^t \|u(t)\|_{L^2(\mathbb{R}^2)}^2 ds \le C\|u_0\|_{\mathcal{H}^1(\mathbb{R}^2)}^2$$

Proof.

$$\begin{aligned} \int_0^t \|u(t)\|_{L^2}^2 ds &\le C \int_0^t \|\nabla u(t)\|_{L^1}^2 ds && (\because W^{1,1} \subset L^2) \\ &\le C \int_0^t \|\nabla u(t)\|_{\mathcal{H}^1}^2 ds && (\because \mathcal{H}^1 \subset L^1) \\ &\le C\|u_0\|_{\mathcal{H}^1}^2 && (\because \text{T. Ogawa and S. Shimizu}) \end{aligned}$$

□

Our method:

Definition ($BMO(\mathbb{R}^n)$)

Let $f \in L^1_{loc}(\mathbb{R}^n)$. We say that f is of bounded mean oscillation (abbreviated as BMO) if

$$\|f\|_{BMO(\mathbb{R}^n)} = \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|}\int_B |f-(f)_B|dx<\infty,$$

where the supremum ranges over all finite ball $B \subset \mathbb{R}^n$, $|B|$ is the n-dimensional Lebesgue measure of B, and $(f)_B$ denotes the integral mean of f over B, namely $(f)_B = \frac{1}{|B|}\int_B f(x)dx$.

The class of functions of BMO, modulo constants, is a Banach space with the norm $\|\cdot\|_{BMO}$ defined above.

Lemma 1 (Poincaré inequality in $\mathbb{R}^2$)

$$\|f\|_{BMO(\mathbb{R}^2)} \le C\|\nabla f\|_{L^2(\mathbb{R}^2)}$$

Lemma 2 (Fefferman-Stein inequality)

$$\left|\int_{\mathbb{R}^n} fg\,dx\right| \le C\|f\|_{\mathcal{H}^1(\mathbb{R}^2)}\|g\|_{BMO(\mathbb{R}^2)}$$

$$\le C\|f\|_{\mathcal{H}^1(\mathbb{R}^2)}\|\nabla g\|_{L^2(\mathbb{R}^2)} \qquad (\because Lemma 1)$$

Lemma 3 (Hardy type inequality)

Let Ω be an exterior domain with smooth boundary in $\mathbb{R}^2$. Then, for $f \in C_0^1(\Omega)$

$$\left\| \frac{f}{W(\cdot)} \right\|_{L^2(\Omega)} \le C \|\nabla f\|_{L^2(\Omega)}$$

where

$$W(x) = |x| \log(B|x|).$$

1.2. Nonlinear heat equations:

Let Ω be an exterior domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. We consider

the semilinear 2D Heat equations

$$\begin{cases} u_t - \Delta u = \dfrac{|u|^{p-1}u}{W(x)} \\ u|_{\partial\Omega} = 0, \quad u(0) = u_0 \end{cases} \tag{NH}$$

where

$$W(x) = |x| \log(B|x|).$$

Theorem 1 (T.K. and M.M.)

Assume that $p > \frac{3}{2}$ and the initial data $u_0 \in \mathcal{H}^1(\Omega) \cap H^1(\Omega)$ is sufficiently small. Then the solutions of (NH) satisfy

$$\int_0^t \|u(s)\|_{L^2}^2 ds \leq C$$

where C is independent of t.

Remark.

- $|u|^{p-1}u \cdots p > 2 = 1 + \dfrac{2}{n}$: Fujita exponent
- $\dfrac{|u|^{p-1}u}{W} \cdots p > 3/2$ (cf. R.G. Pinsky 1997, 2009 etc)

  $\Longrightarrow p = 1 + \dfrac{2}{n+2}$

Sketch of Proof.)

Setting;

$$\|u(t)\|_2 \le K\, I_0\, (1+t)^{-1/2},$$
$$\|\nabla u(t)\|_2 \le K\, I_0\, (1+t)^{-1}$$

$$I_0 = \|u_0\|_{H^1(\Omega)} + \|u_0\|_{\mathcal{H}^1(\Omega)}$$

$$F(u(t)) = |u(t)|^{p-1} u(t) / W(x)$$

(i-1) Energy methods;

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_2^2 + \|\nabla u(t)\|_2^2 = (F(u(t)), u(t)) \tag{1}$$

$$\|u_t(t)\|_2^2 + \frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|_2^2 = (F(u(t)), u_t(t)) \tag{2}$$

$(1) \Longrightarrow$

$$\frac{1}{2}\|u(t)\|_2^2 + \int_0^t \|\nabla u(s)\|_2^2 ds = \frac{\|u_0\|_2^2}{2} + \int_0^t (F(u(s)), u(s))ds \tag{3}$$

$$\frac{(1+t)}{2}\|u(t)\|_2^2 + \int_0^t (1+s)\|\nabla u(s)\|_2^2 ds - \frac{1}{2}\int_0^t \|u(s)\|_2^2 ds$$
$$= \frac{\|u_0\|_2^2}{2} + \int_0^t (1+s)(F(u(s)), u(s))ds \tag{4}$$

$(2) \Longrightarrow$

$$\int_0^t (1+s)^2 \|u_s(s)\|_2^2 ds + \frac{(1+t)^2}{2} \|\nabla u(s)\|_2^2 - \int_0^t (1+s) \|\nabla u(s)\|_2^2 ds$$
$$= \frac{\|\nabla u_0\|_2^2}{2} + \int_0^t (1+s)^2 (F(u(s)), u_s(s)) ds. \quad (5)$$

(i-2) Morawetz's methods;

$$w(t) = w(t,x) = \int_0^t u(s,x)ds.$$

$\Longrightarrow$

$$w_t(t,x) - \Delta w(t,x) = u_0(x) + \int_0^t F(u(s))ds,$$
$$w(t,x)|_{\partial\Omega} = 0, \quad w(0,x) = 0,$$

$\Longrightarrow$

$$\|w_t(t)\|_2^2 + \frac{1}{2}\frac{d}{dt}\|\nabla w(t)\|_2^2 = (u_0, w_t(t)) + (\int_0^t F(u(s))ds, w_t(t)).$$

$\Longrightarrow$ $(w_t = u)$

$$\int_0^t \|u(s)\|_2^2 ds + \frac{1}{2}\|\nabla w(t)\|_2^2 = (u_0, w(t)) + \int_0^t (\int_0^s (F(u(\tau))d\tau, w_s(s))ds. \quad (6)$$

(ii) Estimates; $(u_0, w(t))$

· Fefferman-Stein inequality

$\Longrightarrow$

$$|(u_0, w(t))| = |(\tilde{u}_0, \tilde{w}(t))| \leq C\|\tilde{u}_0\|_{\mathcal{H}^1(\mathbb{R}^2)}\|\tilde{w}(t)\|_{BMO(\mathbb{R}^2)}$$

·Poincaré inequality

$\Longrightarrow$

$$\|\tilde{w}(t)\|_{BMO(\mathbb{R}^2)} \leq C\|\nabla w(t)\|_2$$

Then

$$|(u_0, w(t))| \leq C\|u_0\|_{\mathcal{H}^1(\Omega)}\|\nabla w(t)\|_2. \quad (7)$$

(ii-2) Estimates for the nonlinear terms $F(u(s))$;

· Gagliardo-Nirenberg inq., Hardy type inq.

$\Longrightarrow$

$$\begin{aligned}
\|u(t)/W^{\frac{1}{p+1}}\|_{p+1}^{p+1} &= \int_\Omega |u|^p \cdot |u|/W(x)dx \\
&\leq \||u|^p\|_2 \|u/W\|_2 \\
&= \|u\|_{2p}^p \|u/W\|_2 \\
&\leq C_s^p C_h \|\nabla u(t)\|_2^p \|u(t)\|_2 \\
&\leq C_s^p C_h K^{p+1} I_0^{p+1} (1+t)^{-p-1/2}.
\end{aligned}$$

$$\cdot \left| \int_0^t (1+s)^\alpha (F(u(s)), u(s)) ds \right| = \int_0^t (1+s)^\alpha \|u(s)/W^{\frac{1}{p+1}}\|_{p+1}^{p+1} ds \quad (8)$$
$$\leq C_s^p C_h K^{p+1} I_0^{p+1} \int_0^t (1+s)^{\alpha-p-1/2} ds.$$

$$
\begin{aligned}
&\cdot \int_0^t (1+s)^2 (F(u(s)), u_s(s)) ds \qquad (9)\\
= \;& \frac{1}{p+1} \int_0^t (1+s)^2 \frac{d}{ds} \| u(s)/W^{\frac{1}{p+1}} \|_{p+1}^{p+1} ds \\
= \;& \frac{1}{p+1} \{ (1+t)^2 \| u(t)/W^{\frac{1}{p+1}} \|_{p+1}^{p+1} - \| u_0/W^{\frac{1}{p+1}} \|_{p+1}^{p+1} \} \\
& - \frac{2}{p+1} \int_0^t (1+s) \| u(s)/W^{\frac{1}{p+1}} \|_{p+1}^{p+1} ds. \\
\leq \;& C_s^p C_h K^{p+1} I_0^{p+1} \{ \int_0^t (1+s)^{1/2-p} ds + (1+t)^{3/2-p} \} \\
& + \frac{1}{p+1} \| u_0/W^{\frac{1}{p+1}} \|_{p+1}^{p+1}.
\end{aligned}
$$

$$
\begin{aligned}
&\cdot \int_0^t \Big(\int_0^s F(u(\tau))d\tau, w_s(s)\Big) ds \\
&= \Big(\int_0^t F(u(s))ds, w(t)\Big) - \int_0^t (F(u(s)), w(s))ds = I_1 + I_2
\end{aligned}
$$

$$
\begin{aligned}
|I_1| &= \Big| \int_\Omega \int_0^t |u(s)|^{p-1} u(s) ds \cdot \frac{w(t)}{W(x)} dx \Big| \\
&\leq \int_0^t \|u(s)\|_{2p}^p ds \cdot \Big\| \frac{w(t)}{W} \Big\|_2 \\
&\leq C_s^p C_h K^p I_0^p \int_0^t (1+s)^{1/2-p} ds \|\nabla w(t)\|_2
\end{aligned}
\tag{10}
$$

$$
|I_2| \leq C_s^p C_h K^p I_0^p \int_0^t (1+s)^{1/2-p} ds \sup_{0<s<t} \|\nabla w(s)\|_2 \tag{11}
$$

By (3) – (6), (7) – (11), if $p > 3/2$, then

$$X(t) \leq \ell_p(I_0) + C_p X(t)^q, \quad (q = q(p) > 1)$$

where $\ell_p(I_0) \to 0$ as $I_0 \to 0$ and

$$X(t) = \sup_{0<s<t} \{(1+s)\|u(s)\|_2^2 + (1+s)^2\|\nabla u(s)\|_2^2 + \|\nabla \int_0^s u(\tau)d\tau\|_2^2\}$$
$$+ \int_0^t (1+s)\|\nabla u(s)\|_2^2 ds + \int_0^t (1+s)^2\|u_s(s)\|_2^2 ds + \int_0^t \|u(s)\|_2^2 ds.$$

□

2.Navier Stokes equations in $\mathbb{R}^2$

Navier Stokes equation

$$\begin{cases} \partial_t u - \Delta u + (u \cdot \nabla)u + \nabla \pi = 0, & \text{in } (0,\infty) \times \mathbb{R}^2, \\ \nabla \cdot u = 0 & \text{in } (0,\infty) \times \mathbb{R}^2, \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^2 \end{cases} \qquad \text{(NS)}$$

where

$u(x,t)$: Velocity field (unknown),
$\pi(x,t)$: Scalar pressure (unknown),
$u_0(x)$: a given vector function.

Theorem 4 (T.K. and M.M.)

Assume that $u_0 \in L^2_\sigma \cap \mathcal{H}^1$, then the solutions of (NS) satisfy

$$\int_0^t \|u(s)\|_{L^2}^2 ds \leq C$$

where C is independent of t.

Known Results:

- Leray, Serrin, Kato, Masuda, Borchers, Miyakawa etc
- M. Wiegner 1987 $\cdots$ $\|u(t)\|_{L^2} \leq C(1+t)^{-1/2}$ for weak solutions
- H. Kozono and T. Ogawa 1993 $\cdots$ Large data global
- T. Miyakawa 1996 $\cdots$ $\|\nabla^\alpha u(t)\|_{\mathcal{H}^1} \leq Ct^{-\frac{|\alpha|}{2}}\|u_0\|_{\mathcal{H}^1}$

Key Lemma:

Lemma 5 (R.Coifman, P.L.Lions, Y. Meyer and S.Semmes 1993)

$$\|(u\cdot\nabla)u\|_{\mathcal{H}^1} \leq C\|u\|_{L^2}\|\nabla u\|_{L^2}$$