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Asymptotics of small exterior Navier-Stokes flows with non-decaying boundary data

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$0 \notin \Omega \subset \mathbb{R}^3$, $\partial\Omega \in C^\infty$

$I = \mathbb{R}$ or $\mathbb{R}_+$

$$\begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p = 0 & I \times \Omega \\ \operatorname{div} u = 0 \quad \text{in} & I \times \Omega \\ u|_{\partial\Omega} = u_* \quad \text{in} \quad I, & \lim_{x \to \infty} u = 0 \quad \text{in } I \end{cases}$$

$u_* = u_*(t, x) \not\to 0$, $t \to \infty$

Known results. (time-independent case)

Borchers–Miyakawa (1994)

$u_* \in C^2(\partial\Omega)$ small $\Rightarrow$ $\exists_1$ stable stati. sol. u

$|u(x)| \le C|x|^{-1}$

Korolev–Sverak (2011)

$|u(x)| \le \varepsilon |x|^{-1}$ ε: small $\Rightarrow$ $|u(x) - {}^{\exists}U(x)| \le C\varepsilon|x|^{-1-\delta}$

$\delta \in (0, 1)$

for $|x| \gg 1$

when $U^b(x) \sim |x|^{-1}$: sol of $-\Delta U^b + U^b \nabla U^b + \nabla p^b$
$= b \cdot \delta$ in $\mathbb{R}^3$

$(\ 0$ in $\Omega)$

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$\{U^b\}_{b\in\mathbb{R}}$: Landau sol. -1 homogeneous
axisymmetric

Results

(i) $I = \mathbb{R}_+$ $\quad u|_{t=0} = u_0 \in L^{3,\infty}(\Omega)$, $\quad u_* \in C^1(I; C^2(\partial\Omega))$
: small

$\Rightarrow$ $\exists_1\, u \in L^\infty(\mathbb{R}_+; L^{3,\infty}(\Omega))$ sol. of (NS)

(ii) $I = \mathbb{R}$, $u_* \in C^1(I; C^2(\partial\Omega))$ small, periodic in time

$\Rightarrow$ $\exists_1\, u$: time periodic solution $\quad |u(t,x)| \le \varepsilon |x|^{-1}$

$$|u(t,x) - {}^{\exists}U^b(x)| \le C\varepsilon |x|^{-1-\delta} \qquad |x| \gg 1.$$

(iii) u_p: periodic sol. given in (ii)

$I = \mathbb{R}_+$ $\quad \underbrace{u_0 - u_p(0)}_{= w_0} \in L^{3,\infty}$ small

$$|w_0(x) - \overline{W}_0(x)| \le \varepsilon\, |x|^{-\theta-1} \qquad |x| \gg 1$$

$\overline{W}_0$: -1 homo. div. free

$\Rightarrow$

$$\| u(t) - (u_p(t) + W(t)) \|_{L^{3,\infty}} \to 0 \quad \text{as } t \to \infty,$$

where

$$\begin{cases} \partial_t W - \Delta W + W\cdot\nabla(W + U^b) + U^b\cdot\nabla W + \nabla\pi = 0 & \text{on } I\times\mathbb{R}^3 \\ \operatorname{div} W = 0, \quad W|_{t=0} = U^b - \overline{W}_0 & \text{self-similar} \end{cases}$$

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Remark $u_*(t) \to 0$ as $t \to 0$, Solonnikov, Amann, Grubb, Farwig-Kozono-Sohr $u_* \in L^s(\mathbb{R}_+; W^{-1/q, q}(\partial\Omega))$.

(i) Yamazaki estimates in $L^{3,\infty}$ and div. free extension.

(ii) $\partial_t v - \Delta v + \nabla \pi = g$ in $\mathbb{R} \times \mathbb{R}^3$
$\mathrm{div}\, v = 0$

$$\|u\|_{L^\infty_\alpha} = \| |x|^\alpha u \|_{L^\infty}$$

$\alpha > 1$.

$$\int_0^T \int_{\mathbb{R}^3} g \, dx dt = 0$$

$\Rightarrow$

$$\|v\|_{L^\infty(\mathbb{R}; L^\infty_2)} \leq C_\alpha \|g\|_{L^\infty(\mathbb{R}; L^\infty_\alpha)}$$

$(T \to \infty \Rightarrow C_\alpha \to \infty)$

$$b = \frac{1}{T} \int_0^T \int_{|x|=\rho} T_{ij} \cdot N \, dS dt$$