# Boundary layer solution to the symmetric hyperbolic-parabolic system

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The 7th Japanese–German International Workshop on Mathematical Fluid Dynamics

November 5–8, 2012, Waseda University, Tokyo

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# §1. Hyperbolic–Parabolic System

System of viscous conservation laws over $\mathbb{R}_+ = (0, \infty)$,

$$f^0(u)_t + f(u)_x = (G(u)u_x)_x, \quad x \in \mathbb{R}_+, \ t > 0. \tag{HPS}$$

| | | |
|---|---|---|
| $\mathscr{O} \subset \mathbb{R}^m$ | : | state space (open & convex). |
| $u = u(t,x) \in \mathscr{O}$ | : | unknown $m$-vector function. |
| $f^0(u) \in \mathbb{R}^m$ | : | smooth, $\det D_u f^0(u) \neq 0$ for $u \in \mathscr{O}$. |
| $f(u) \in \mathbb{R}^m$ | : | flux function, smooth in $u \in \mathscr{O}$. |
| $G(u) \in \mathbb{R}^{mm}$ | : | viscosity matrix, smooth in $u \in \mathscr{O}$. |

Assume that

$$G(u) = \begin{array}{cc} \begin{array}{cc} \overset{m_1}{\frown} & \overset{m_2}{\frown} \end{array} & \\ \begin{bmatrix} 0 & 0 \\ 0 & G_2(u) \end{bmatrix} & \begin{array}{l} )\, m_1 \\ )\, m_2 \end{array} \end{array} \qquad (m_1 + m_2 = m),$$

$G_2(u)$ : $m_2 \times m_2$ matrix, $\det G_2(u) \neq 0$ for $u \in \mathscr{O}$.

System (HPS) $=$ $m_1$-hyperbolic eq. $\oplus$ $m_2$-parabolic eq.

## ◇ Motivation To study motion of compressible and viscous gases.

(I) Barotropic model :

$$\begin{aligned} &\rho_t + (\rho u)_x = 0, \\ &(\rho u)_t + \big(\rho u^2 + p(\rho)\big)_x = (\mu u_x)_x. \end{aligned} \tag{BM}$$

(II) Heat-conductive model :

$$\begin{aligned} &\rho_t + (\rho u)_x = 0, \\ &(\rho u)_t + \big(\rho u^2 + p(\rho,\theta)\big)_x = (\mu u_x)_x, \\ &\left\{\rho\left(c_{\mathrm{v}}\theta + \frac{u^2}{2}\right)\right\}_t + \left\{\rho u\left(c_{\mathrm{v}}\theta + \frac{u^2}{2}\right) + p(\rho,\theta)u\right\}_x = (\mu u u_x + \kappa\theta_x)_x. \end{aligned} \tag{HCM}$$

$\rho$ : Density of fluid $\qquad u$ : Fluid velocity $\qquad \theta$ : Absolute temperature

$p$ : Pressure $\qquad \mu, \kappa, c_{\mathrm{v}}$ : Physical constants

† These models are governed by hyperbolic-parabolic system.

## ♢ Known results (1) ∼ Full space $\mathbb{R}^n$ ∼

- **Matsumura–Nishida ('80) :**
  Global existence & asymptotic stability of constant state (stationary sol.) for (HCM) in $\mathbb{R}^3$ by **energy method**.

- **Matsumura ('81) :**
  Optimal convergence rate for (HCM) in $\mathbb{R}^3$ by **weighted energy method**.

- **Umeda–Kawashima–Shizuta ('84) :**
  Stability & convergence rate for (HPS) under **Condition (K)**.

- **Shizuta–Kawashima ('85) :**
  Condition (K) $\Longleftrightarrow$ **Stability condition**.
  † **Stability condition** is easy to check.

## ◇ Known results (2) ~ Half space $\mathbb{R}^n_+$ ~

Barotropic model (BM)

- **Kawashima–Nishibata–Zhu ('03)** : Outflow problem $(u < 0)$ in $\mathbb{R}_+$
  Existence & stability of boundary layer solution.
- **Matsumura–Nishihara ('01) :** Inflow problem $(u > 0)$ in $\mathbb{R}_+$
  Existence & stability of boundary layer solution (in Lagrangian coordinate).
- **Kagei–Kawashima ('06) :** Outflow problem $(u < 0)$ in $\mathbb{R}^n_+$
  Asymptotic stability of planar boundary layer solution.

Heat-conductive model (HCM)

- **Matsumura–Nishida ('83) :** In $\mathbb{R}^3_+$ under $u|_{x_1=0} = 0$
  Stability of constant state (stationary sol.) by **energy method**.
- **Kawashima–Nishibata–Zhu–N. ('09) :** Outflow problem $(u < 0)$ in $\mathbb{R}_+$.
  Existence & stability of boundary layer solution.
- **Huang–Li–Shi (to appear), Qin–Wang ('09), Nishibata–N. (2011) :**
  Inflow problem $(u > 0)$ in $\mathbb{R}_+$. Existence & stability of boundary layer solution.

$\diamondsuit$ Aim General theory on stability of boundary layer solution for (HPS) under stability condition.

• Application

$$
\text{Outflow problem for } \begin{cases} \dagger \text{ barotropic model (BM)} \\ \dagger \text{ heat-conductive model (HCM)} \\ \qquad \begin{cases} \bullet \text{ ideal gases} \\ \bullet \text{ general constitutive equations} \end{cases} \\ \vdots \end{cases}
$$

- Boundary layer solution $\cdots$

stationary solution under negative characteristics & viscosity.

$$\Uparrow$$

eigenvalues of $D_u f(u)$ for

$$f^0(u)_t + f(u)_x = 0.$$

Our aims are to show $\cdots$

- existence of boundary layer solution,
- nonlinear stability of (non-degenerate & degenerate) boundary layer solution under stability condition by energy method.

# §2. Stationary problem

Let $u = \begin{bmatrix} v \\ w \end{bmatrix}$, $\quad v \in \mathbb{R}^{m_1} \cdots$ hyperbolic part, $w \in \mathbb{R}^{m_2} \cdots$ parabolic part.

$f(u) = \begin{bmatrix} f_1(v,w) \\ f_2(v,w) \end{bmatrix}$, $\quad f_1(v,w) \in \mathbb{R}^{m_1}$, $f_2(v,w) \in \mathbb{R}^{m_2}$.

$\tilde{u}(x) = \begin{bmatrix} \tilde{v}(x) \\ \tilde{w}(x) \end{bmatrix}$ : stationary solution.

Equations

$$f(\tilde{u})_x = (G(\tilde{u})\tilde{u}_x)_x \iff \begin{cases} f_1(\tilde{v},\tilde{w})_x = 0, \\ f_2(\tilde{v},\tilde{w})_x = (G_2(\tilde{u})\tilde{w}_x)_x. \end{cases} \qquad (x \in \mathbb{R}_+) \qquad \text{(SE)}$$

Boundary conditions

$$\tilde{w}(0) = w_{\mathsf{b}}, \quad \lim_{x\to\infty} \tilde{u}(x) = u_+ = \begin{bmatrix} v_+ \\ w_+ \end{bmatrix}.$$

$$\text{(SE)} \begin{cases} f_1(\tilde{v},\tilde{w})_x = 0, \\ f_2(\tilde{v},\tilde{w})_x = (G_2(\tilde{u})\tilde{w}_x)_x. \end{cases} \qquad \tilde{w}(0) = w_{\mathsf{b}}, \quad \lim_{x\to\infty}(\tilde{v}(x),\tilde{w}(x)) = (v_+,w_+).$$

Integrate (SE) over $(x,\infty) \Rightarrow$

$$f_1(\tilde{v},\tilde{w}) = f_1(v_+,w_+), \tag{1}$$

$$G_2(\tilde{u})\tilde{w}_x = f_2(\tilde{v},\tilde{w}) - f_2(v_+,w_+). \tag{2}$$

**Assumption (A2)** : $\det D_v f_1(v_+,w_+) \neq 0.$

Solve (1) by implicit function theorem $\Rightarrow$

$$^{\exists}V(\tilde{w}) \quad \text{s.t.} \quad f_1(V(\tilde{w}),\tilde{w}) = f_1(v_+,w_+), \quad v_+ = V(w_+).$$

Substitute $\tilde{v} = V(\tilde{w})$ in (2) $\Rightarrow$

System of 1st order ODE for $\tilde{w}$ :

$$\tilde{w}_x = G_2(V(\tilde{w}),\tilde{w})^{-1}(H(\tilde{w}) - H(w_+)),$$

$$H(\tilde{w}) := f_2(V(\tilde{w}),\tilde{w}).$$

$$\tilde{w}_x = G_2(V(\tilde{w}),\tilde{w})^{-1}(H(\tilde{w}) - H(w_+)), \tag{3}$$
$$H(\tilde{w}) := f_2(V(\tilde{w}),\tilde{w}).$$

Let

$$\tilde{A}(w) := G_2(u_+)^{-1} D_w H(w), \quad D_w H = -D_v f_2 \cdot (D_v f_1)^{-1} \cdot D_w f_1 + D_w f_2.$$

(3) $\Longrightarrow$

$$\tilde{w}_x = \tilde{A}(w_+)(\tilde{w} - w_+) + \frac{1}{2} G_2(u_+)^{-1} D_w^2 H(w_+)(\tilde{w} - w_+)^2 + (\text{remainder}),$$
$$\tilde{w}(0) = w_{\mathrm{b}}, \quad \lim_{x\to\infty} \tilde{w}(x) = w_+.$$

Consider solvability of this ODE system under following assumption.

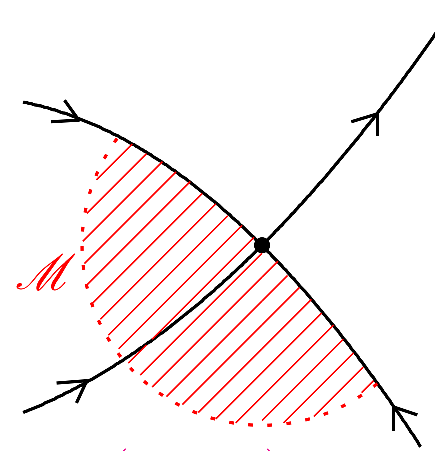
**Assumption (A3)** :

(i) Eigenvalues of $\tilde{A}(w)$ are distinct : $\mu_1(w) > \mu_2(w) > \cdots > \mu_{m_2}(w)$.

(ii) $\mu_1(w_+) \leq 0$ holds.

- $\mu_1(w_+) < 0 \ \cdots$ non-degenerate case.
- $\mu_1(w_+) = 0 \ \cdots$ degenerate case.

## ◇ Existence of stationary solution




$\delta := |w_{\mathrm{b}} - w_+|$,

$r_j(w)$ : eigenvector of $\tilde{A}(w)$ corresponding to $\mu_j(w)$.

**Theorem 1** [Nishibata–N.]

Assume $\delta \ll 1$.

(I) Non-degenerate case : $\mu_1(w_+) < 0$

$^\exists$stationary solution $\tilde{u}(x)$, s.t., $|\partial_x^k(\tilde{u}(x) - u_+)| \le Ce^{-cx}$.

(II) Degenerate case : $\mu_1(w_+) = 0$

If $w_{\mathrm{b}} \in \mathscr{M}$ & $D_w\mu_1(w_+) \cdot r_1(w_+) \neq 0 \Rightarrow$

$^\exists$stationary solution $\tilde{u}(x)$, s.t., $|\partial_x^k(\tilde{u}(x) - u_+)| \le C\dfrac{\delta^{k+1}}{(1+\delta x)^{k+1}} + Ce^{-cx}$.

## ◇ Entropy & Symmetrization

**Def.** $\eta(z) \in \mathbb{R}$ $(z = f^0(u) \in \mathbb{R}^m)$ is entropy $\iff$

(i) $\eta(z)$ is smooth and strictly convex, i.e., $D_z^2\eta(z) > 0$ for $z \in f^0(\mathscr{O})$.

(ii) ${}^\exists$entropy flux $q(u) \in \mathbb{R}$ s.t. $D_u q(u) = D_z\eta(f^0(u))D_u f(u)$.

(iii) $B(u) := {}^\mathsf{T}D_u f^0(u)D_z^2\eta(f^0(u))G(u)$ is real symmetric and non-negative.

**Assumption (A4)** : Entropy $\eta(z)$ exists.

$$f^0(u)_t + f(u)_x = (G(u)u_x)_x. \tag{HPS}$$

$\Downarrow$ Friedrichs & Lax ('71), Kawashima ('83)

Symmetric system

$$A^0(u)u_t + A(u)u_x = B(u)u_{xx} + g(u, u_x),$$

$$A^0 > 0,\ A,\ B \geq 0 : \text{real symmetric.}$$

Decompose symmetric system into hyperbolic part and parabolic part $\Longrightarrow$

$$A_1^0 v_t + A_{11} v_x + A_{12} w_x = g_1,$$
$$A_2^0 w_t + {}^{\mathsf{T}}\!A_{12} v_x + A_{22} w_x = B_2 w_{xx} + g_2,$$

$$A_1^0 > 0,\ \ A_2^0 > 0,\ \ A = \begin{bmatrix} A_{11} & A_{12} \\ {}^{\mathsf{T}}\!A_{12} & A_{22} \end{bmatrix},\ \ B_2 > 0 \ : \ \text{real symmetric.}$$

$$u = \begin{bmatrix} v \\ w \end{bmatrix}, \quad v \in \mathbb{R}^{m_1} \cdots \text{hyperbolic}, \quad w \in \mathbb{R}^{m_2} \cdots \text{parabolic}.$$

**Assumption (A5)** : $A(u) < 0$ and $A_{11}(u) < 0$ hold.

- Initial condition

$$\begin{bmatrix} v \\ w \end{bmatrix}\Bigg|_{t=0} = \begin{bmatrix} v_0 \\ w_0 \end{bmatrix} \xrightarrow{x\to\infty} \begin{bmatrix} v_+ \\ w_+ \end{bmatrix}.$$

- Boundary condition

$$w\big|_{x=0} = w_{\mathsf{b}}.$$

## ◇ Dissipative property

- Condition (K) $\,{}^{\exists}K$ : $m \times m$ real matrix, s.t.,

$$KA^0(u_+) \text{ is skew-symmetric } \ \& \ [KA(u_+)] + B(u_+) > 0. \tag{K}$$

$[A] := \frac{1}{2}(A + {}^{\mathsf{T}}A)$ : symmetric part of $A$.

- Stability condition

$$\begin{aligned} &\lambda A^0(u_+)\phi = A(u_+)\phi \ \text{ and } \ B(u_+)\phi = 0 \quad ({}^{\exists}\lambda \in \mathbb{R} \ \& \ {}^{\exists}\phi \in \mathbb{R}^m) \\ &\Longrightarrow \phi = 0. \end{aligned} \tag{SK}$$

† Shizuta–Kawashima ('85) : Condition (K) $\Longleftrightarrow$ Condition (SK).

† Kawashima ('83) : Asymptotic stability of **constant state in full space** $\mathbb{R}^n$
under condition (SK) (or (K)).

Aim : To show the stability of **stationary solution in half space** $\mathbb{R}_+$
under stability condition (SK).

**Assumption (A6)** : Stability condition (SK) holds.

## ◇ Asymptotic stability of non-degenerate stationary solution

**Theorem 2** [Nishibata–N.]

Let $\tilde{u}$ be a non-degenerate stationary solution.

Assume that

$$\|u_0 - \tilde{u}\|_{H^2} + \delta \ll 1.$$

$\Longrightarrow$ $^{\exists 1}$time global solution $u(t,x)$ to (HPS), s.t.,

$$u - \tilde{u} \in C([0,\infty); H^2(\mathbb{R}_+)),$$

$$\lim_{t\to\infty} \|u(t) - \tilde{u}\|_{L^\infty} = 0.$$

† We essentially use the condition $A_{11} < 0$. ($A < 0$ is not necessary.)

† Stability of degenerate stationary solution is also proved under suitable conditions.

## §4. Outline of proof

### ◇ Problem for perturbation

$\begin{bmatrix} \varphi \\ \psi \end{bmatrix} := \begin{bmatrix} v - \tilde{v} \\ w - \tilde{w} \end{bmatrix}$ $\cdots$ perturbation from stationary station.

$\varphi \in \mathbb{R}^{m_1} \cdots$ hyperbolic part, $\psi \in \mathbb{R}^{m_2} \cdots$ parabolic part.

- Equation

$$A_1^0 \varphi_t + A_{11} \varphi_x + A_{12} \psi_x = \tilde{g}_1, \tag{PE$_1$}$$

$$A_2^0 \psi_t + {}^{\mathsf{T}}A_{12} \varphi_x + A_{22} \psi_x = B_2 \psi_{xx} + \tilde{g}_2. \tag{PE$_2$}$$

- Initial condition

$$\begin{bmatrix} \varphi \\ \psi \end{bmatrix}\Bigg|_{t=0} = \begin{bmatrix} \varphi_0 \\ \psi_0 \end{bmatrix} := \begin{bmatrix} v_0 - \tilde{v} \\ w_0 - \tilde{w} \end{bmatrix}.$$

- Boundary condition

$$\psi|_{x=0} = 0.$$

## ◇ A priori estimate

$$N(t) := \sup_{0\le\tau\le t} \|(\varphi,\psi)(\tau)\|_{H^2}, \quad (\varphi,\psi) := (v,w) - (\tilde{v},\tilde{w}) \ : \ \text{perturbation}$$

**Proposition 4**

$N(t) + \delta \ll 1 \Longrightarrow$

$$\|(\varphi,\psi)(t)\|_{H^2}^2 + \int_0^t \left(\|\varphi_x(\tau)\|_{H^1}^2 + \|\psi_x(\tau)\|_{H^2}^2\right) d\tau \le C\|(\varphi_0,\psi_0)\|_{H^2}^2. \qquad \text{(A)}$$

† Local existence $\oplus$ Estimate (A) $\Longrightarrow$ Global existence & $\|(\varphi,\psi)(t)\|_{L^\infty} \xrightarrow{t\to\infty} 0$

Proof $\cdots$ $\boxed{\text{energy method}}$

- basic $L^2$ estimate by using **energy form**.
- estimate for higher order derivatives by using **Matsumura–Nishida's method in half space**.
- dissipative estimate of $\varphi_x$ under **stability condition**.

Thank you for your attention!