Asymptotic stability of boundary layers in plasma physics with fluid-boundary interaction

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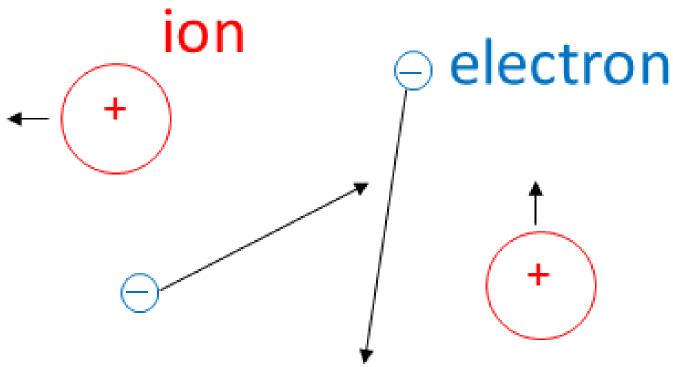
1. Physical background of the problem

Plasma in the Whole Space



m : mass

u : velocity

ρ : density

ϕ : electric potential

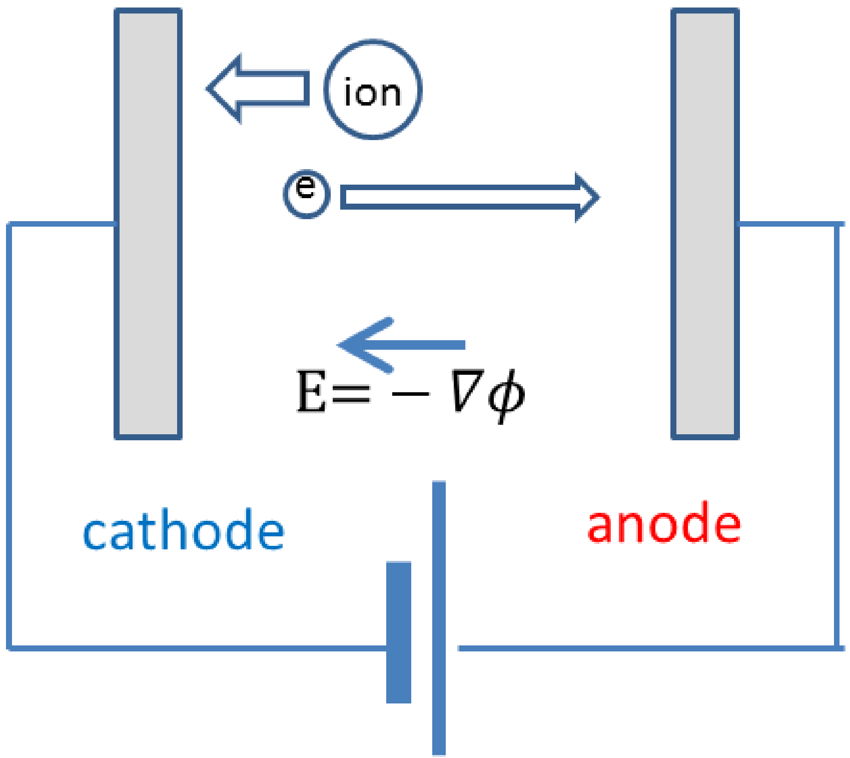
$$u_e \gg u_i$$
$$(\because m_e \ll m_i)$$

Nearly neutral : $\rho_e \fallingdotseq \rho_i$
$\phi \fallingdotseq 0$

subscripts
i : ion
e : electron

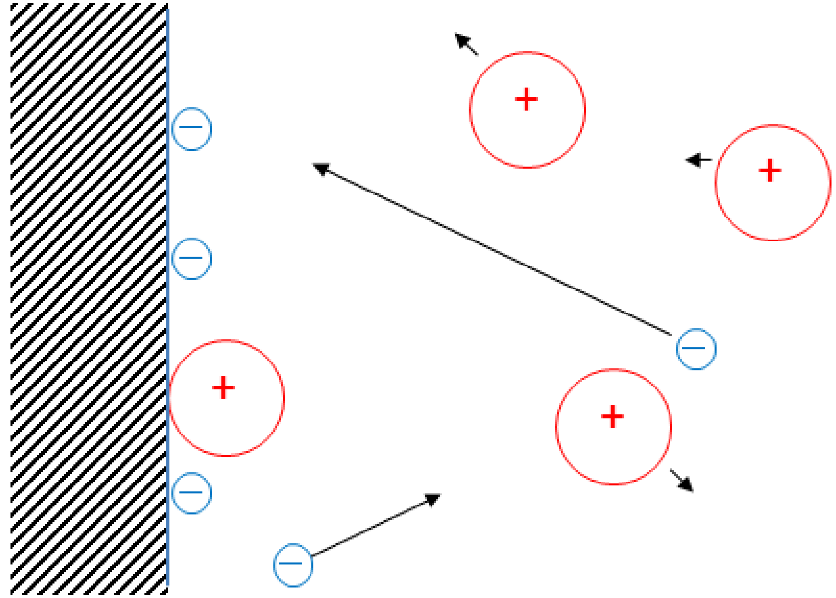
Plasma with structures

Externally applied voltage



Voltage difference accelerates ions toward cathode, and electrons toward anode.

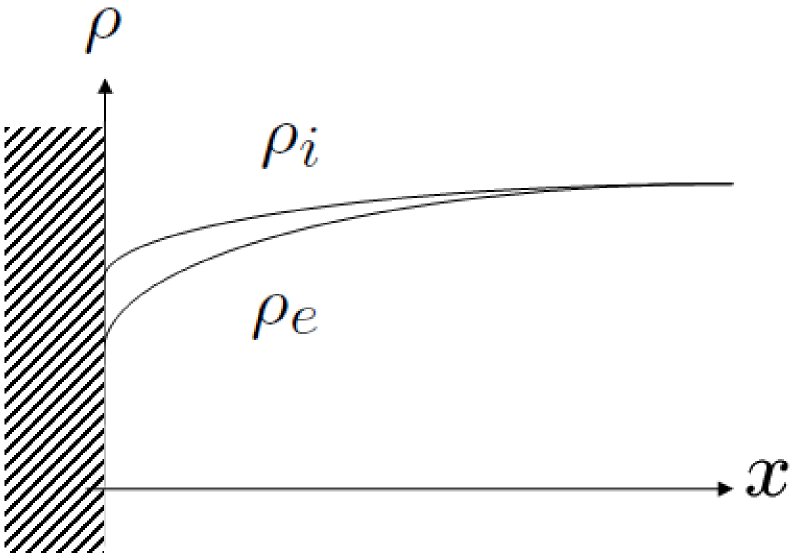
Insulated wall (particles accumulate)



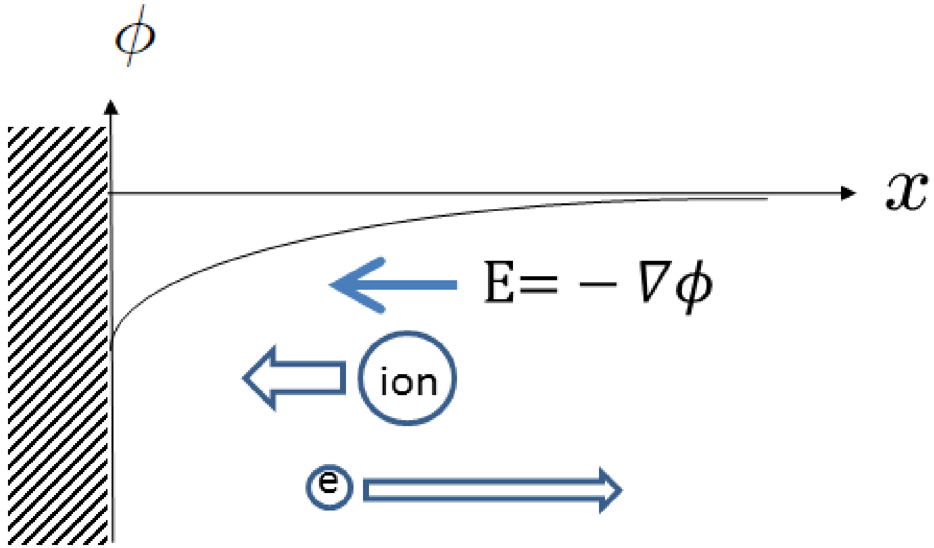
Wall is negatively charged due to the flux difference, causing potential gradient in the space

Process of Sheath Formation

Density distribution



Potential distribution



On the wall, electrons gather.
Elsewhere, ions dominate.

→ Lower potential on the wall.

↓

Toward the wall,
ions are accelerated
electrons are decelerated.

← In the end, both flux to the wall coincide and a steady state is attained.

↓

This <u>stationary boundary layer</u> is called a SHEATH.

Bohm's Sheath Criterion

For the sheath formation, physical observation requires
the **Bohm sheath criterion** (BSC):

$$\boldsymbol{u_+^2 \geq K + 1, \quad u_+ < 0,} \tag{BSC}$$

u_+ : Ion's velocity at the interface between boundary layer and inner region
K : Const. proportional to abs. temperature (= (Acoustic Velocity)^2)

$$(p(\rho) = K\rho, \quad K > 0, \quad \text{Isothermal})$$

Remark : (BSC) $\Rightarrow$ Supersonic condition : $u_+^2 > K$.

We aim to validate BSC from the mathematical point of view.

2. Mathematical formulation of the problem

- Governing equations

$$\rho_t + (\rho u)_x = 0, \tag{E.a}$$

$$(\rho u)_t + (\rho u u)_x + p(\rho)_x + \rho \phi_x = 0, \tag{E.b}$$

$$-\phi_{xx} = \rho - \boldsymbol{\rho_e}. \tag{E.c}$$

$x \in \mathbb{R}_+$

ρ : Ion density , u : Ion velocity, ϕ : Electrostatic potential

$p(\rho) = K\rho$ $(K > 0)$ (Isothermal) : Pressure

$\boldsymbol{\rho_e} = e^{\phi} > 0$ (Boltzmann relation) : Electron density

- Initial values

$$(\rho, u)(0, x) = (\rho_0, u_0)(x), \tag{I.a}$$

$$\inf_{x \in \mathbb{R}_+} \rho_0(x) > 0,$$

$$\lim_{x \to \infty} \rho_0(x) = \rho_+ > 0, \quad \lim_{x \to \infty} u_0(x) = u_+ < 0. \tag{I.b}$$

- Reference value of potential

$$\lim_{x \to \infty} \phi(t, x) = 0, \tag{R}$$

• Boundary Conditions (No need for ρ, u due to supersonic outflow)

Either one of the following two types of BCs.

(1) Dirichlet BC

$$\phi(t, 0) = \phi_b, \tag{DBC}$$

where ϕ_b is a given constant.

(2) Fluid-Boundary Interactive BC

$$\phi_{xt}(t, 0) = \left[\rho u + e^{\phi} u_e\right](t, 0), \quad u_e := \sqrt{\frac{m_i}{2\pi m_e}}. \tag{IBC}$$

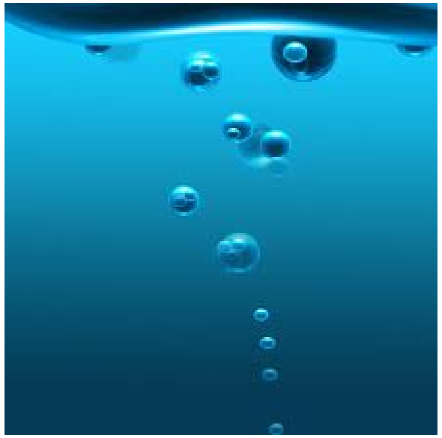
(m_i, m_e : mass of ion, electron, u_e : thermal velocity of electrons)

($-\phi_x(t, 0) \propto$ quantity of charged particles on the wall)

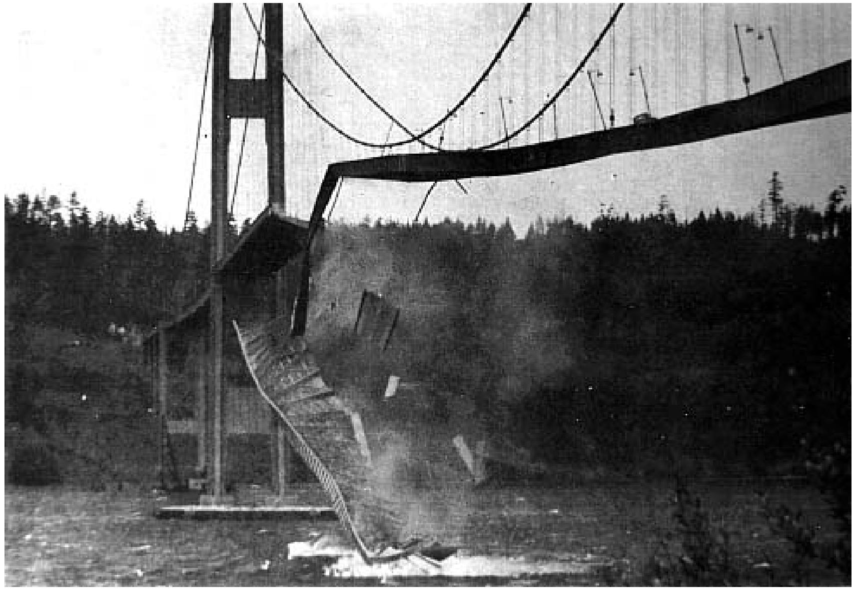
◇ Additional Initial Value
(IBC) $\Rightarrow$ either one of $q_0 = \phi_x(0, 0)$ or $p_0 = \phi(0, 0)$ is required.

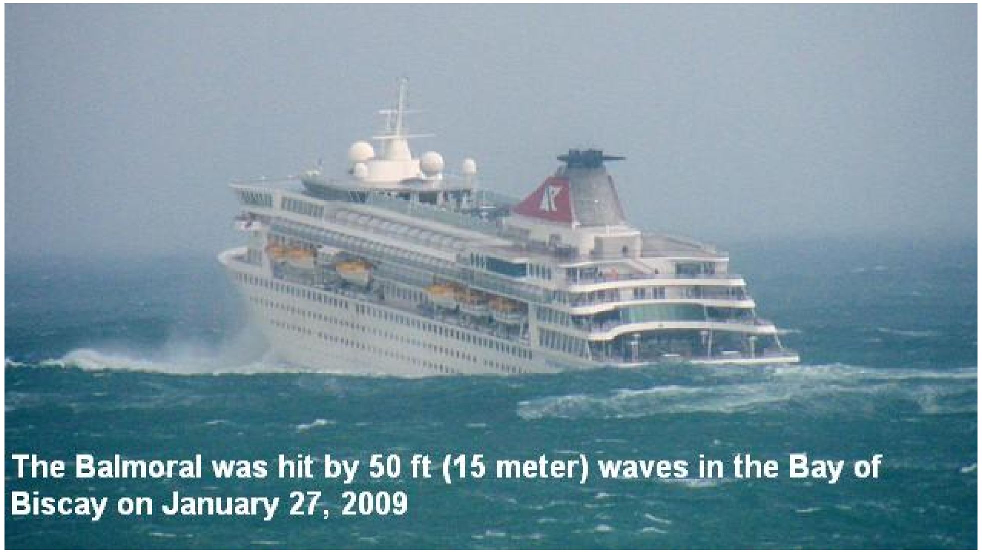
Related problems include ...

- Free boundary problems



- Fluid-Structure interaction problems





3. Known results under Dirichlet BC

[M.Suzuki(KRM '11)] & [S.Nishibata, M.O., M.Suzuki (SIAM '12)] validated (BSC) mathematically:
They prove the existece of stationary sol. under (BSC) and its stability.

Stationary problem

We define sheath by a stationary solution $(\tilde{\rho}, \tilde{u}, \tilde{\phi})(x)$ to (E)

$$(\tilde{\rho}\tilde{u})_x = 0, \tag{S.a}$$

$$\Big(\tilde{\rho}\tilde{u}^2 + p(\tilde{\rho})\Big)_x + \tilde{\rho}\tilde{\phi}_x = 0, \tag{S.b}$$

$$-\tilde{\phi}_{xx} = \tilde{\rho} - e^{\tilde{\phi}}, \tag{S.c}$$

with conditions (I.b), (R) and (DBC)

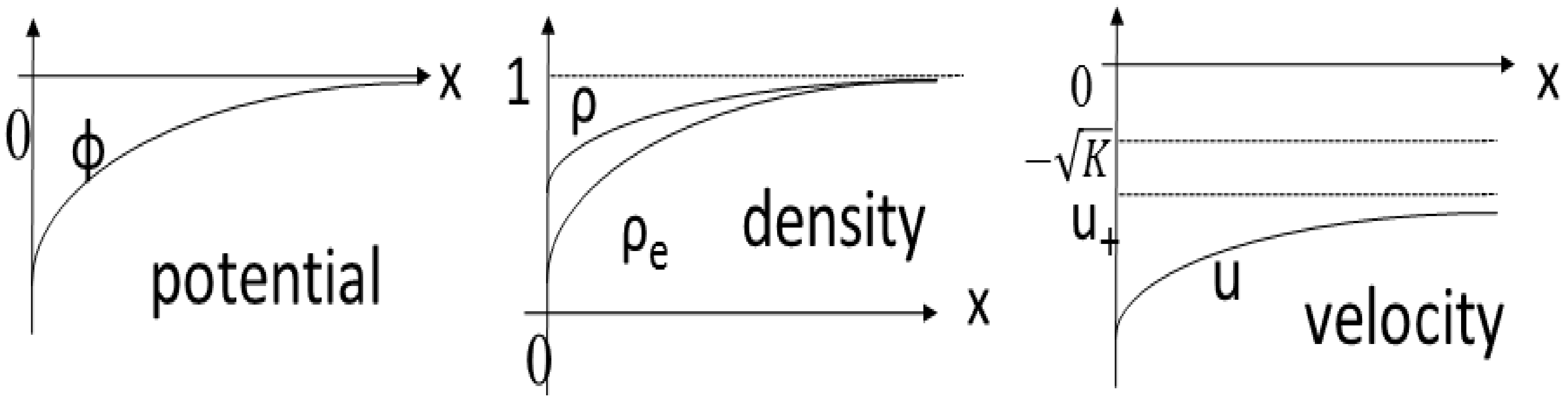
$$\inf_{x\in\mathbb{R}_+} \tilde{\rho}(x) > 0, \quad \lim_{x\to\infty} (\tilde{\rho}, \tilde{u}, \tilde{\phi})(x) = (\rho_+, u_+, 0), \quad \tilde{\phi}(0) = \phi_b.$$

Conditions for the existence of stationary solutions $\cdots$ [M.Suzuki, '11]

Theorem 1
Stationary solution exist $\Leftrightarrow \phi_b \le f(|u_+|/\sqrt{K})$ *and* $V(\phi_b) \ge 0$

$$f(\tilde{\rho}) := \frac{u_+^2}{2}\left(1 - \frac{1}{\tilde{\rho}^2}\right) - K \log \tilde{\rho}, \quad V(\tilde{\phi}) := \int_0^{\tilde{\phi}} [e^{\eta} - f^{-1}(\eta)]\, d\eta. \quad \left(\begin{array}{c}\text{Sagdeev} \\ \text{potential}\end{array}\right)$$

$\Rightarrow u_+^2 > K + 1$ (nondegenerate Bohm's criterion) and $|\phi_b| \ll 1$
or $u_+^2 = K + 1$ (degenerate Bohm's criterion) and $\phi_b < 0$
are sufficient for the existence of monotone stationary solution.



Asymptotic stability under **Dirichlet BC** (1D with exp weight)

[S.Nishibata, M.O., M.Suzuki, '12]

Perturbation $(\psi, \eta, \sigma)(t,x) := (\log \rho, u, \phi)(t,x) - (\log \tilde{\rho}, \tilde{u}, \tilde{\phi})(x)$.

Theorem 2 *Assume* $u_+ < -\sqrt{K+1}$, $K > 0$. *(nondegenerate case)*

If $(e^{\lambda x/2}\psi_0, e^{\lambda x/2}\eta_0) \in H^2(\mathbb{R}_+)$, $^\exists\lambda > 0$ *and*

$$\lambda + \left(|\phi_b| + \|(e^{\lambda x/2}\psi_0, e^{\lambda x/2}\eta_0\|_{H^2}\right)/\lambda \ll 1,$$

then $^\exists 1$ *time global solution* (ψ, η, σ) *s.t.*

$$e^{\lambda x/2}\psi,\ e^{\lambda x/2}\eta \ \in \ \bigcap_{i=0}^{2} C^i\left([0,\infty); H^{2-i}(\mathbb{R}_+)\right),$$

$$e^{\lambda x/2}\sigma \ \in \ \bigcap_{i=0}^{2} C^i\left([0,\infty); H^{4-i}(\mathbb{R}_+)\right).$$

$$^\exists C, \gamma > 0 \quad s.t. \quad \|(e^{\lambda x/2}\psi, e^{\lambda x/2}\eta)(t)\|_{H^2} + \|e^{\lambda x/2}\sigma(t)\|_{H^4}$$
$$\leq C\|(e^{\lambda x/2}\psi_0, e^{\lambda x/2}\eta_0)\|_{H^2}\, e^{-\gamma t}.$$

Difficulty to show asymptotic stability

System of linearized equations of (P) (governing eqs. for perturbation) around asymptotic state $(\rho, u, \phi) = (\rho_+, u_+, 0)$ is

$$\begin{pmatrix} \sqrt{K}\psi \\ \eta \end{pmatrix}_t + \begin{pmatrix} u_+ & \sqrt{K} \\ \sqrt{K} & u_+ \end{pmatrix} \begin{pmatrix} \sqrt{K}\psi \\ \eta \end{pmatrix}_x + \begin{pmatrix} 0 \\ \sigma \end{pmatrix}_x = 0, \quad -\sigma_{xx} = \psi - \sigma. \tag{L}$$

Spectrums of (L) are given by

$$\mu(i\xi) = i\left(-\xi u_+ \pm |\xi|\sqrt{K + \frac{1}{1+|\xi|^2}}\right), \quad \xi \in \mathbb{R}.$$

Real parts of all spectrums are ZERO.

To resolve this difficulty, we employ weighted energy method.

♢ All characteristics go into boundary. $(\lambda = u \pm \sqrt{K} < 0)$

♢ Decay of (ψ_0, η_0) as $x \to \infty \Rightarrow$
convergence of solution towards stationary solution as $t \to \infty$.

Weighted energy method

Introduce new variables $(\Psi, H, \Sigma) := (e^{\beta x/2}\psi, e^{\beta x/2}\eta, e^{\beta x/2}\sigma)$.

Rewrite systems of equation (P) w.r.t. $(\Psi, H, \Sigma) \Rightarrow (P')$.

Linearize (P') around asymptotic state $(\rho, u, \phi) = (\rho_+, u_+, 0) \Rightarrow (L')$.

Spectrums of (L') are given by

$$\mu(i\xi) = \frac{\beta u_+}{2} + i\left(-\xi u_+ \pm \sqrt{K\zeta - \frac{1}{\zeta} + 1 - K}\right),$$

$$\text{where} \quad \zeta = 1 + |\xi|^2 - \frac{\beta^2}{4} + i\beta\xi \quad \text{for } \xi \in \mathbb{R}.$$

Linearly Stable $\Leftrightarrow \sup_{\xi\in\mathbb{R}} Re\,(\mu(i\xi)) < 0 \Leftrightarrow u_+^2 > K + \dfrac{1}{1-\beta^2/4}, \ \beta > 0.$ ($\natural$)

$$\left(\because \quad \sup_{\xi\in\mathbb{R}} Re\,(\mu(i\xi)) = \max Re\,(\mu(0)) = \frac{\beta}{2}\left(u_+ \sqrt{K + \frac{1}{1-\beta^2/4}}\right)\right)$$

$\therefore$ If $u_+^2 > K + 1$, setting $0 < \beta \ll 1$ ensures ($\natural$).

4. Main result 1 (Asympt. stability of sheath under **IBC**) [M.O.]

$$\text{IBC}: \phi_{xt}(t,0) = [\rho u + e^{\phi} u_e](t,0) \Rightarrow \tilde{\phi}(0) = \log\left(|u_+|/u_e\right)$$

Theorem 3 *Assume* $u_+ < -\sqrt{K+1}$

and set $\phi_b = \log\left(|u_+|/u_e\right), \quad r_0 = \phi_x(0,0) - \tilde{\phi}_x(0).$

If $(e^{\lambda x/2}\psi_0, e^{\lambda x/2}\eta_0) \in H^2(\mathbb{R}_+), \quad {}^{\exists}\lambda > 0$

and $\lambda + \left(|\phi_b| + |r_0| + \|(e^{\lambda x/2}\psi_0, e^{\lambda x/2}\eta_0)\|_{H^2}\right)/\lambda \ll 1,$

then ${}^{\exists}1$ *time global solution* (ψ, η, σ) *s.t.*

$$e^{\lambda x/2}\psi,\ e^{\lambda x/2}\eta \ \in\ \bigcap_{i=0}^{2} C^i\left([0,\infty); H^{2-i}(\mathbb{R}_+)\right),$$

$$e^{\lambda x/2}\sigma \ \in\ \bigcap_{i=0}^{2} C^i\left([0,\infty); H^{4-i}(\mathbb{R}_+)\right).$$

${}^{\exists}C, \gamma > 0$ *s.t.*

$$\|(\psi,\eta)(t)\|_{H^2} + \|\sigma(t)\|_{H^4} \ \le\ C\left(\|(e^{\lambda x/2}\psi_0, e^{\lambda x/2}\eta_0)\|_{H^2} + |r_0|\right) e^{-\gamma t}.$$

Outline of proof of Main results (for exp. weight case)

(Local existence) + (A-priori estimate) $\Rightarrow$ (Global existence)

Lemma 4 (Local existence)
$(e^{\lambda x/2}\psi_0, e^{\lambda x/2}\eta_0) \in H^2(\mathbb{R}_+)$ *with*

$$(\eta_0 + \tilde{u})(0) + \sqrt{K} < 0, \quad \inf_{x\in\mathbb{R}_+} \rho_0(x) > 0.$$

$\Rightarrow$ $^{\exists}T > 0$, *s.t.*, $^{\exists}1$ *solution* (ψ, η, σ) *as*

$$(e^{\lambda x/2}\psi, e^{\lambda x/2}\eta) \in C([0,T]; H^2), \quad e^{\lambda x/2}\sigma \in C([0,T]; H^4).$$

Lemma 5 (A-priori estimate)
$N(T) := \sup_{0\le t\le T} \left(\|(e^{\lambda x/2}\psi, e^{\lambda x/2}\eta)(t)\|_{H^2} + |\sigma_x(t,0)| \right).$

$\lambda + (N(T) + |\phi_b|)/\lambda \ll 1 \Rightarrow {}^{\exists}\nu, C > 0$ *s.t.*

$$\|(e^{\lambda x/2}\psi, e^{\lambda x/2}\eta)(t)\|_{H^2} + \|e^{\lambda x/2}\sigma(t)\|_{H^4} \le Ce^{-\nu t} \left(\|(e^{\lambda x/2}\psi_0, e^{\lambda x/2}\eta_0)\|_{H^2} + |r_0| \right).$$

Local existence (without weight functions)

$$\begin{pmatrix} \sqrt{K}\psi \\ \eta \end{pmatrix}_t + \begin{pmatrix} u & \sqrt{K} \\ \sqrt{K} & u \end{pmatrix} \begin{pmatrix} \sqrt{K}\psi \\ \eta \end{pmatrix}_x + \eta \begin{pmatrix} \sqrt{K}\tilde{v}_x \\ \tilde{u}_x \end{pmatrix} + \begin{pmatrix} 0 \\ \sigma_x \end{pmatrix} = 0,$$

$$-\sigma_{xx} + e^{\tilde{\phi}} \int_0^1 \exp(\theta\sigma)\, d\theta\sigma = \rho - \tilde{\rho},$$

$$\sigma_{tx}(t,0) = \left[\rho u - \tilde{\rho}\tilde{u} + |u_+|(e^{\sigma} - 1)\right](t,0), \qquad \lim_{x\to\infty} \sigma(t,x) = 0.$$

† Construct solutions s.t. $\psi, \eta \in H^2, \sigma \in H^4$.

<u>Solvability of hyperbolic equation</u>

No boundary data ($\because$ supersonic outflow)

$\Rightarrow$ Extend problem over $\mathbb{R}_+$ to that over $\mathbb{R}$ and apply Kato's theory.

Solvability of elliptic equation

For given ρ and $\sigma_x|_{x=0}$ (or $\sigma|_{x=0}$),

$$\begin{array}{rlll} \text{solve} & -\phi_{xx} = \rho - e^{\phi}, & \phi_x(0) = g, & \phi(x) \to 0 \ (x \to \infty) \\ \text{around} & -\tilde{\phi}_{xx} = \tilde{\rho} - e^{\tilde{\phi}}, & \tilde{\phi}_x(0) = \tilde{g}, & \tilde{\phi}(x) \to 0 \ (x \to \infty). \end{array}$$

(stationary solution)

Lemma 6 *If $\rho \in L^\infty$ satisfies $\inf_x \rho(x) > 0$ & $\rho - \tilde{\rho} \in L^2$*

(i.e. $\psi \in L^2 \cap L^\infty$ satisfies $\inf_x \psi(x) > -\infty$),

$$-\sigma_{xx} + e^{\tilde{\phi}}(e^{\sigma} - 1) = \rho - \tilde{\rho}, \quad \sigma_x(0) = g - \tilde{g}, \quad \sigma(x) \to 0 \ (x \to \infty) \quad \text{(E0)}$$

is uniquely solvable for $\sigma \in H^2$. ($\phi = \tilde{\phi} + \sigma$)

(Apply Schauder's fixed point theorem
using Stampacchia's method of truncation)

Remark We can define the Neumann-Dirichlet map F s.t.

$$\sigma|_{x=0} = F(\sigma_x|_{x=0}, \psi) \qquad .$$

- Construction of approximate sequence $(\psi,\eta,\sigma)^{(n)}$

First step $(\psi,\eta)^{(0)}\equiv(\psi_0,\eta_0)$. Iteration $(\psi,\eta)^{(n)} \to \sigma_x|_{x=0}^{(n)} \to \sigma^{(n)} \to (\psi,\eta)^{(n+1)}$:

(1) ODE for BC $(\sigma_x|_{x=0}^{(n)}(t))$

$$\text{Solve}\quad \begin{aligned}\frac{d}{dt}\sigma_x|_{x=0}(t) &= (\rho u)^{(n)}|_{x=0}(t) - (\tilde\rho\tilde u)|_{x=0} + u_e e^{\tilde\phi(0)}(\exp\sigma|_{x=0}(t)-1)\\ &= (\rho u)^{(n)}|_{x=0}(t) - (\tilde\rho\tilde u)|_{x=0} + |u_+|(\exp F(\sigma_x|_{x=0},\psi^{(n)})(t)-1)\\ &= G(\sigma_x|_{x=0}(t),t)\end{aligned}$$

for $\sigma_x|_{x=0}(t)$ $(t\geq 0)$ with $\sigma_x|_{x=0}(0)=r_0$. G is Lipschitz continuous!

(2) Elliptic eq.

$$\text{Solve}\quad -\sigma_{xx}^{(n)} + e^{\tilde\phi}\int_0^1 \exp(\theta\sigma^{(n)})\,d\theta\sigma^{(n)} = \rho^{(n)} - \tilde\rho$$

with $\lim_{x\to\infty}\sigma^{(n)}(t,x)=0$ and $\sigma_x|_{x=0}^{(n)}(t)$ determined in (1) for $\sigma^{(n)}$.

(3) Hyperbolic eq.

$$\begin{pmatrix}\sqrt{K}\psi\\ \eta\end{pmatrix}_t^{(n+1)} + \begin{pmatrix}u^{(n)} & \sqrt{K}\\ \sqrt{K} & u^{(n)}\end{pmatrix}\begin{pmatrix}\sqrt{K}\psi\\ \eta\end{pmatrix}_x^{(n+1)} + \eta^{(n+1)}\begin{pmatrix}\sqrt{K}\tilde v_x\\ \tilde u_x\end{pmatrix} + \begin{pmatrix}0\\ \sigma_x^{(n)}\end{pmatrix} = 0$$

with $(\psi,\eta)^{(n+1)}(0,x)=(\psi_0,\eta_0)(x)$.

$(\psi,\eta,\sigma)^{(n)}$ makes a Cauchy sequence for $[0,{}^\exists T]$. (Energy method)

- A-Priori Estimates for IBC (0-th and first order derivatives)

$$\frac{d}{dt}\left(\int_0^\infty W_1\,\rho\left[\frac{e^{\tilde{\phi}}}{2}(K\psi^2+\eta^2)+\frac{1}{2}(K\psi_x^2+\eta_x^2)+\tilde{\rho}(e^\psi-\psi-1)\right]dx\right)$$
$$+\ c\int_0^\infty W_1'\left[\psi^2+\eta^2+\psi_x^2+\eta_x^2+\sigma_x^2\right]dx+cW_1\left[\psi^2+\eta^2+\psi_x^2+\eta_x^2\right]_{x=0}$$
$$\leq\ CW_1\sigma_x|_{x=0}^2. \qquad (1)$$

$$\frac{d}{dt}\left(\int_0^\infty W_2\left[\rho(K\psi^2+\eta^2)+(e^\phi\sigma^2+\sigma_x^2)\right]dx\right)+C\frac{d}{dt}\left[W_2'(e^\sigma-\sigma-1)|_{x=0}\right]$$
$$+\ c\int_0^\infty W_2'\left[\sigma^2+\sigma_x^2+\sigma_t^2\right]dx+cW_2\left[\psi^2+\eta^2+\sigma^2+\sigma_x^2\right]_{x=0}$$
$$\leq\ C\int_0^\infty W_2'\left[\psi^2+\eta^2+\psi_x^2+\eta_x^2\right]dx \qquad (2)$$

Let $W_1=e^{\beta_1 x}$ and $W_2=\gamma e^{\beta_2 x}$ with $\gamma\gg 1\gg\beta_1,\beta_2$ and $\beta_1\gg\gamma\beta_2$.

$\Rightarrow W_1\ll W2$ on $x=0$ while $W_1'\gg W_2'$ for $x>0$.

(1)+(2) for $W_1 = e^{\beta_1 x}$ and $W_2 = \gamma e^{\beta_2 x}$ $\Rightarrow$

$$\begin{aligned} &\frac{d}{dt}\Big[E_{W_1}(\psi,\eta,\psi_x,\eta_x) + E_{W_2}(\psi,\eta,\sigma,\sigma_x)\Big] + C\frac{d}{dt}\Big[W_2'\left(e^{\sigma} - \sigma - 1\right)|_{x=0}\Big] \\ + \;\; & c\int_0^\infty W_1'\left[\psi^2 + \eta^2 + \psi_x^2 + \eta_x^2 + \sigma^2 + \sigma_x^2\right] dx \\ + \;\; & cW_2\left[\psi^2 + \eta^2 + \psi_x^2 + \eta_x^2 + \sigma^2 + \sigma_x^2\right]_{x=0} \quad \leq \quad 0 \end{aligned} \tag{3}$$

$\int_0^t e^{\nu\tau}(3)d\tau$ with $\nu \ll \beta_1 =: \beta$ and $\beta_2\nu \ll 1$ and <u>elliptic estimates</u> $\Rightarrow$

$$\begin{aligned} &\|(e^{\beta x/2}\psi, e^{\beta x/2}\eta)(t)\|_{H^1}^2 + \|e^{\beta x/2}\sigma(t)\|_{H^1}^2 \\ &\qquad\qquad \leq Ce^{-\nu t}\left(\|(e^{\beta x/2}\psi_0, e^{\beta x/2}\eta_0)\|_{H^1}^2 + \sigma_x^2\big|_{x=0}(0)\right). \end{aligned}$$

<u>elliptic estimates</u>

$$\begin{aligned} \|\sqrt{W}\sigma\|_{H^1}^2 &\leq C\|\sqrt{W}\psi\|_{H^1}^2 + CW\sigma_x^2\big|_{x=0}, \\ \sigma_x^2\big|_{x=0} &\leq C\|\psi\|_{H^1}^2 + C\sigma^2\big|_{x=0}, \\ \sigma^2\big|_{x=0} &\leq C\|\psi\|_{H^1}^2 + C\sigma_x^2\big|_{x=0}, \cdots \end{aligned}$$

Main result 2 (Asympt. stability of sheath under **IBC**) [M.O.]

$$W_{\alpha,\beta}(x) := (1+\beta x)^{\alpha} \text{ for } \alpha, \beta > 0$$

Theorem 7 *Assume* $u_+ < -\sqrt{K+1}$
and set $\phi_b = \log\left(|u_+|/u_e\right)$, $r_0 = \phi_x(0,0) - \tilde{\phi}_x(0)$.
Suppose $(W_{\lambda/2,\beta}\psi_0, W_{\lambda/2,\beta}\eta_0) \in H^2(\mathbb{R}_+)$, ${}^{\exists}\lambda \geq 2,\ \beta > 0$.
$\Rightarrow {}^{\forall}\alpha \in (0,\lambda]$, ${}^{\exists}\delta = \delta(\alpha) > 0$ *s.t. if*

$$\beta + \left(|\phi_b| + |r_0| + \|(W_{\lambda/2,\beta}\psi_0, W_{\lambda/2,\beta}\eta_0)\|_{H^2}\right)/\beta \leq \delta,$$

then ${}^{\exists}1$ *time global solution* (ψ, η, σ) *s.t.*

$$W_{\alpha/2,\beta}\psi,\ W_{\alpha/2,\beta}\eta \in \bigcap_{i=0}^{2} C^i([0,\infty); H^{2-i}(\mathbb{R}_+)),$$

$$W_{\alpha/2,\beta}\sigma \in \bigcap_{i=0}^{2} C^i([0,\infty); H^{4-i}(\mathbb{R}_+)).$$

$${}^{\exists}C(\alpha) > 0 \text{ s.t. } \|(W_{\alpha/2,\beta}\psi, W_{\alpha/2,\beta}\eta)(t)\|_{H^2}^2 + \|W_{\alpha/2,\beta}\sigma(t)\|_{H^4}^2 \leq C(\|(W_{\lambda/2,\beta}\psi_0, W_{\lambda/2,\beta}\eta_0)\|_{H^4}^2 + r_0^2)\ (1+\beta t)^{-(\lambda-\alpha)}.$$

Main result 3 (Asympt. stability of sheath under **IBC**) [M.O.] (degenerate case)

Theorem 8 *Assume* $u_+ = -\sqrt{K+1}$
and set $\phi_b = \log\left(|u_+|/u_e\right),\quad r_0 = \phi_x(0,0) - \tilde{\phi}_x(0).$
Suppose $(W_{\lambda/2,\beta}\psi_0, W_{\lambda/2,\beta}\eta_0) \in H^2(\mathbb{R}_+)$, $\quad {}^\exists\lambda \in [4,\lambda_0),\ \beta > 0.$
$\Rightarrow {}^\forall\alpha \in (0,\lambda],\ {}^\forall\theta \in (0,1],\ {}^\exists\delta = \delta(\alpha,\theta) > 0$ *s.t. if*

$\phi_b \in [-\delta, 0),\ \ \beta/\Gamma|\phi_b|^{1/2} \in [\theta, 1],\ \ \|(W_{\lambda/2,\gamma}\psi_0, W_{\lambda/2,\gamma}\eta_0)\|_{H^m}/\beta^3 \le \delta,$
then ${}^\exists 1$ *time global solution* (ψ,η,σ) *s.t.*

$$W_{\alpha/2,\beta}\psi,\ W_{\alpha/2,\beta}\eta \in \bigcap_{i=0}^{2} C^i([0,\infty); H^{2-i}(\mathbb{R}_+)),$$
$$W_{\alpha/2,\beta}\sigma \in \bigcap_{i=0}^{2} C^i([0,\infty); H^{4-i}(\mathbb{R}_+)).$$

$${}^\exists C(\alpha) > 0 \ s.t.\ \|(W_{\alpha/2,\beta}\psi, W_{\alpha/2,\beta}\eta)(t)\|_{H^2}^2 + \|W_{\alpha/2,\beta}\sigma(t)\|_{H^4}^2 \le C(\|(W_{\lambda/2,\beta}\psi_0, W_{\lambda/2,\beta}\eta_0)\|_{H^4}^2 + r_0^2)\ (1+\beta t)^{-(\lambda-\alpha)/3}.$$

Concluding remarks

- Boundary layer is stable also under Interactive BC.
- Convergence rate reflects decay structure of the initial perturbation
- Up to const., decay rate is the same as that under Dirichlet BC.
- Neumann or Dirichlet data on the boundary goes to zero ($\to$ DBC).
- Stabilized by repulsion? Time delay by hysteresis?

Thank you very much for your kind attention !

Elliptic Equation

- Uniqueness in H^1 is easy.

- Existence is shown in two steps.

i) $^{\forall} i \in \mathbb{N}$, set $\chi_i \in C_0^\infty(\mathbb{R})$ s.t. $0 \le \chi_i \le 1$ and $\chi_i(x) = \begin{cases} 1 & \text{if } |x| \le i \\ 0 & \text{if } |x| \ge 2i \end{cases}$.

Then solve

$$-\partial_x^2 \sigma_i + e^{\tilde{\phi}}(e^{\sigma_i} - 1) = \chi_i(\rho - \tilde{\rho}), \ \partial_x \sigma_i(0) = g - \tilde{g}, \ \lim_{x \to \infty} \sigma_i(x) = 0. \text{ (E1)}$$

ii) Show $\{\sigma_i\}_{i \in \mathbb{N}}$ is a Cauchy seq. in H^2 and it actually solves (E0).

<u>Step i)</u> : Leray-Schauder's fixed point theorem

Linearize (E1) $\Rightarrow$

$$-\partial_x^2\sigma + e^{\tilde{\phi}}\int_0^1 e^{s\hat{\sigma}}ds\,\sigma = \chi_i(\rho-\tilde{\rho}),\ \partial_x\sigma(0) = g-\tilde{g},\ \lim_{x\to\infty}\sigma(x) = 0.\ \text{(E2)}$$

Solving for σ for given $\hat{\sigma}$, we define $T:\hat{\sigma}\to\sigma$.

Show the following a)–d), and we have a sol. to (E1).

a) T is well-defined as a map on $B^0(\overline{\mathbb{R}_+})$.

b) T is continuous.

c) T is compact.

d) $^\exists C>0$ s.t. if $\sigma=\lambda T\sigma$ is satisfied for certain $\lambda\in(0,1]$ and $\sigma\in B^0$, then $\|\sigma\|_{B^0}\le C$.

• Proof of d)

Suppose $\sigma = \lambda T\sigma,\ \lambda \in (0,1] \Leftrightarrow$

$$-\sigma_{xx} + e^{\tilde{\phi}} \int_0^1 e^{s\sigma} ds\sigma = \lambda\chi_i(\rho - \tilde{\rho}), \quad \sigma_x(0) = g - \tilde{g} = \lambda(g - \tilde{g}). \qquad \text{(E3)}$$

Add $-\tilde{\phi}_{xx} + e^{\tilde{\phi}} = \tilde{\rho}$ and $-a_{xx}$ to (E3). $\quad \left(a(x) := \varepsilon^{-1} g e^{-\varepsilon x}\right)$

$$-(\phi + a)_{xx} + e^{\phi} = \lambda\chi_i(\rho - \tilde{\rho}) + \tilde{\rho} - a_{xx}, \quad (\phi + a)_x(0) = 0. \qquad \text{(E4)}$$

Let $\varepsilon \ll 1$ s.t. $\ m_0 := \inf_x \lambda\chi_i(\rho - \tilde{\rho}) + \tilde{\rho} - \|a_{xx}\| > 0.$

$^\forall \varepsilon' > 0,\ m_1 := \min\{\log m_0, -|\phi_b| - \varepsilon'\}, \quad m_2 := m_1 - 2\|a\|_\infty = m_1 - \frac{2|g|}{\varepsilon}.$

<u>claim</u> : $\phi(x) \geq m_2, \quad {}^\forall x \in \mathbb{R}_+.$

proof pf the claim

Set $G(r) := \begin{cases} (r - \|a\|_\infty)^2 & r < \|a\|_\infty \\ 0 & r \geq \|a\|_\infty \end{cases}$ and $F(x) := G(\phi(x) + a(x) - m_2)$.

Note $F(x)$ has compact support and $F \in H^1 \cap L^1$.

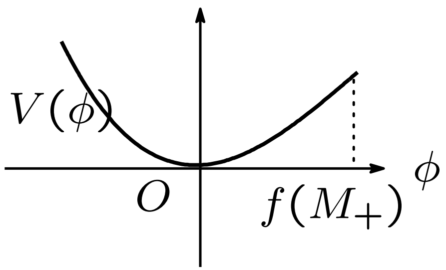
$$\int_0^\infty ((\mathrm{E4}) - e^{m_1})\, F(x) dx$$

$$\Rightarrow \quad \int_0^\infty (\phi + a)_x^2\, G'(\phi + a - m_2) dx + \int_0^\infty G(\phi + a - m_2)(e^\phi - e^{m_1}) dx$$
$$= \int_0^\infty \{\lambda \chi_i (\rho - \tilde{\rho}) + \tilde{\rho} - a_{xx} - e^{m_1}\}\, G(\phi + a - m_2) dx$$

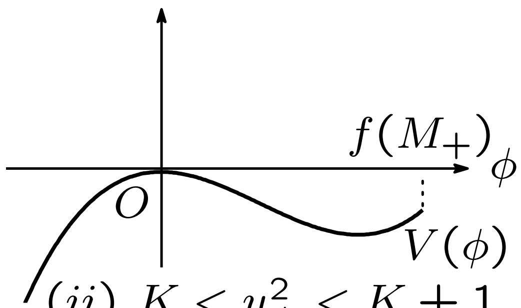
$\mathrm{LHS}_1 \leq 0$ and RHS $\geq 0 \Rightarrow \mathrm{LHS}_2 \geq 0$.

While $G(\phi + a - m_2)(e^\phi - e^{m_1}) \begin{cases} = 0 & \text{if } \phi \geq m_1 \\ \leq 0 & \text{if } m_2 \leq \phi < m_1 \\ < 0 & \text{if } \phi < m_2 \end{cases}$. $\quad \therefore \phi(x) \geq m_2$.

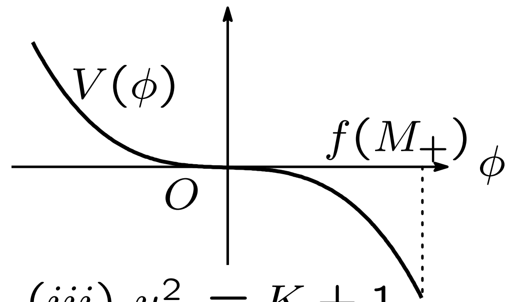
Graph of V



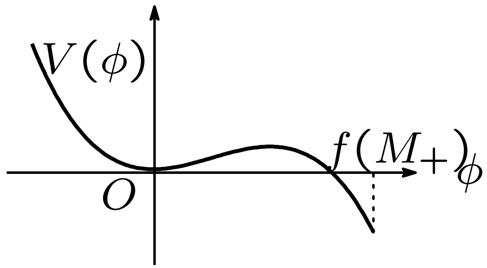
$(i)\ u_+^2 \leq K$



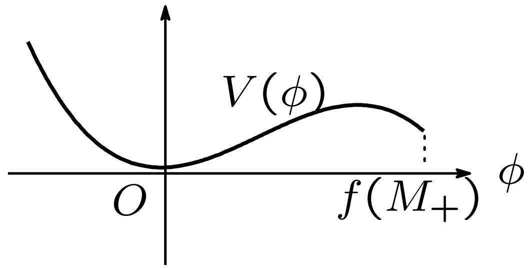
$(ii)\ K < u_+^2 < K+1$



$(iii)\ u_+^2 = K+1$



$\boldsymbol{(iv)\ K+1 < u_+^2 \leq \omega}$



$\boldsymbol{(v)\ \omega < u_+^2}$

$$f(\tilde{\rho}) := \frac{u_+^2}{2}\left(1 - \frac{1}{\tilde{\rho}^2}\right) - K \log \tilde{\rho}, \quad V(\tilde{\phi}) := \int_0^{\tilde{\phi}} [e^{\eta} - f^{-1}(\eta)]\, d\eta. \quad \left(\begin{array}{c}\text{Sagdeev} \\ \text{potential}\end{array}\right)$$

$$\tilde{\phi}_{x_1}{}^2 = 2V(\tilde{\phi}), \quad \tilde{\phi}(0) = \phi_b, \quad \lim_{x_1 \to \infty} \tilde{\phi}(x_1) = 0,$$

Proposition 9 (Decay rate of stationary solution)

$u_+^2 > K+1 \Rightarrow$

$$\left(|\partial_{x_1}^j(\tilde{\rho}-\rho_+)| + |\partial_{x_1}^j(\tilde{u}-u_+)| + |\partial_{x_1}^j\tilde{\phi}|\right)(x_1) \leq C|\phi_b|e^{-cx_1}.$$

$u_+^2 = K+1 \Rightarrow$

$$^\forall\delta_0 > 0,\ ^\exists C = C(\delta_0) > 0 \textit{ s.t. } ^\forall\phi_b \in [\delta_0, 0)$$

$$\left|\ \partial_{x_1}^i\tilde{\phi}(x_1) \times G(x_1)^{i+2} - c_i\right| \leq C|\phi_b|,$$

$$\left|\ \partial_{x_1}^i\left(\tilde{\rho}(x_1)-\rho_+\right) \times G(x_1)^{i+2} - c_i\right| \leq C|\phi_b|,$$

$$\left|\ \partial_{x_1}^i\left(\tilde{u}(x_1)-u_+\right)/u_+ \times G(x_1)^{i+2} - c_i\right| \leq C|\phi_b|,$$

for $i = 0, 1, 2, 3\ldots$ *and* $^\forall x_1 \geq 0$,

where $\quad G(x_1) := \Gamma x_1 + |\phi_b|^{-1/2}, \quad \Gamma := \sqrt{(K+1)/6}$

and $\quad c_0 := -1,\ c_1 := 2\Gamma,\ c_2 := -(K+1),\ c_3 := 4\Gamma(K+1), \ldots$

• Physical ansatz of the Bohm criterion

Rewrite (E.a) and (E.b) by deviding by ρ. ($v := \log \rho$, 1D)

$$v_t + uv_x + u_x = 0, \qquad \text{(E.a}'\text{)}$$

$$u_t + uu_x + Kv_x + \phi_x = 0, \qquad \text{(E.b}'\text{)}$$

$$-\phi_{xx} = e^v - e^\phi. \qquad \text{(E.c}'\text{)}$$

Under "quasi-neutrality" assumption, drop ϕ_{xx} in (E.c$'$) to have $\phi = v$ and

$$v_t + uv_x + u_x = 0, \qquad \text{(E.a}'\text{)}$$

$$u_t + uu_x + (K+1)v_x = 0. \qquad \text{(E.b}''\text{)}$$

Characteristic of this system are $u \pm \sqrt{K+1}$.

$\sqrt{K+1}$ is the phase velocity of the wave supported in the linearized system of (E.a$'$) and (E.b$''$). This wave is called the ion acoustic wave.

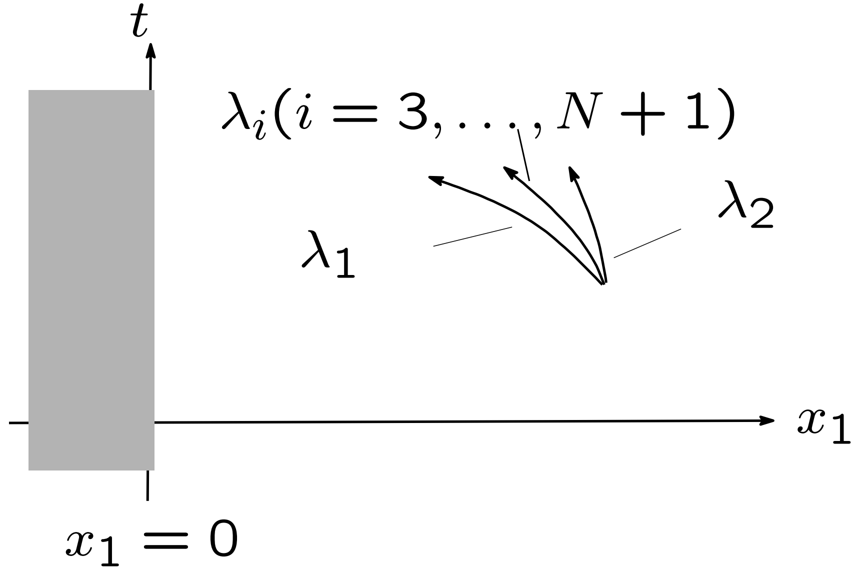
Wellposedness of the problem

If perturbation is small enough, characteristics in x_1 direction are

$$\begin{aligned} \lambda_1 &:= u_1 - \sqrt{K} < 0, \\ \lambda_2 &:= u_1 + \sqrt{K} < 0, \\ \lambda_i &:= u_1 < 0 \quad (i = 3, \ldots, N+1). \end{aligned}$$

$$\left(\because u_1 = u_+ + (\tilde{u} - u_+) + \eta_1, \quad (\text{BSC}) : u_+ \leq -\sqrt{K+1}\right)$$

- For hyperbolic equation (P.a), no boundary condition is necessary.
- For elliptic equation (P.b), one boundary condition is necessary.



$\Rightarrow$ Well-posed with 1 boundary condition (PB),

$$\sigma(t, 0, x') = 0, \quad {}^\forall x' \in \mathbb{R}^{N-1}.$$

• A-Priori Estimates for DBC (N=1,2,3) (0-th & first order derivatives)

$$
\begin{aligned}
(a)\quad & \frac{d}{dt}\left(\int_{\mathbb{R}^N_+} W\ \left[(K+1)\,\psi^2+\eta^2+K\,(\nabla\psi)^2+(\mathrm{div}\eta)^2\right]dx\right)\\
+\quad & c\int_{\mathbb{R}^N_+} W'\left[\psi^2+\eta^2+(\nabla\psi)^2+(\mathrm{div}\eta)^2+(\nabla\sigma)^2\right]dx+cW\left[\psi^2+\eta^2+(\nabla\psi)^2+(\mathrm{div}\eta)^2\right]_{x_1=0}\\
\le\quad & C\delta\int_{\mathbb{R}^N_+} W'(\nabla'\eta)^2dx.\\
(b)\quad & \frac{d}{dt}\left(\int_{\mathbb{R}^N_+} W\ \left[(K+1)\,\psi^2+\eta^2+K\,(\nabla\psi)^2+(\nabla\eta)^2\right]dx\right)\\
+\quad & c\int_0^\infty W'\left[\psi^2+\eta^2+(\nabla\psi)^2+(\nabla\eta)^2\right]dx+cW\left[\psi^2+\eta^2+(\nabla\psi)^2+(\nabla\eta)^2\right]_{x_1=0}\\
\le\quad & C\int_0^\infty W'(\nabla\sigma)^2dx+CW\sigma_{x_1}^2\big|_{x_1=0}.\\
(c)\quad & \frac{d}{dt}\left(\int_0^\infty W\left[K\psi^2+\eta^2+\sigma^2+(\nabla\sigma)^2\right]dx\right)\\
+\quad & c\int_0^\infty W'\left[\psi^2+\eta^2+\sigma^2+(\nabla\sigma)^2\right]dx+cW\left[\psi^2+\eta^2+\sigma_{x_1}^2\right]_{x_1=0}\\
\le\quad & C\int_0^\infty W'\left[(\nabla\psi)^2+(\mathrm{div}\eta)^2+\eta^2\right]dx
\end{aligned}
$$

(a) $+\,\varepsilon\times$ ((c) $+\,\varepsilon\times$ (b)) & $\varepsilon+\delta\ll 1$ completes a-priori estimates up to first order.

- Take constants $\epsilon \ll 1$ and set $\delta = N_W(T) + |\phi_b| + \beta^2 \ll \beta \Rightarrow$

$^\exists c_0, \cdots, c_7 > 0$ s.t. (a) $+ \epsilon \times$ ((c) $+ \epsilon \times$ (b)) $\Rightarrow$

$$\frac{d}{dt}\left(\int_{\mathbb{R}^N_+} dxW\left[c_1\psi^2 + c_2\eta^2 + c_3(\nabla\psi)^2 + c_4(\nabla\eta)^2 + c_5(\mathrm{div}\eta)^2 + c_6\sigma^2 + c_7(\nabla\sigma)^2\right]\right)$$
$$+ c_0\left(\int_{\mathbb{R}^N_+} dxW'\left[\psi^2 + \eta^2 + (\nabla\psi)^2 + (\nabla\eta)^2 + (\mathrm{div}\eta)^2 + \sigma^2 + (\nabla\sigma)^2\right]\right) \leq 0$$

- For exp. weight $W = e^{\beta x_1}$, $\int_0^t d\tau\,[e^{\gamma\tau}\ \cdot\ ]$ $(\gamma \ll \beta) \Rightarrow {}^\exists C > 0$ s.t.

$$e^{\gamma t}\|\sqrt{W}(\psi,\eta)(t)\|^2_{H^1} + \int_0^t e^{\gamma\tau}\|\sqrt{W'}(\psi,\eta)(\tau)\|^2_{H^1}\,d\tau \leq C\|\sqrt{W}(\psi_0,\eta_0)\|^2_{H^1}$$

- For alg. weight $W = (1+\beta x_1)^\alpha$, $\int_0^t d\tau\,[(1+\beta\tau)^\gamma\ \cdot\ ] \Rightarrow {}^\exists C > 0$ s.t.

$$(1+\beta t)^\gamma\|\sqrt{W}(\psi,\eta)(t)\|^2_{H^1} + \int_0^t (1+\beta\tau)^\gamma\|\sqrt{W'}(\psi,\eta)(\tau)\|^2_{H^1}\,d\tau$$
$$\leq C\|\sqrt{W}(\psi_0,\eta_0)\|^2_{H^1} + C\beta\gamma\int_0^t (1+\beta\tau)^{\gamma-1}\|\sqrt{W}(\psi,\eta)(\tau)\|^2_{H^1}\,d\tau$$

Asymptotic stability under **Dirichlet BC** (nondegenerate case)

(with algebraic weight) $\cdots$ [S.Nishibata, M.O., M.Suzuki, '12]

$$W_{\alpha,\beta} := (1+\beta x_1)^{\alpha} \quad \text{for} \quad \alpha > 0,\ \beta > 0$$

Theorem 10 $(N,m) = (1,2), (2,3), (3,3)$. $u_+ < -\sqrt{K+1},\ K > 0$.

Suppose $(W_{\lambda/2,\beta}\psi_0, W_{\lambda/2,\beta}\eta_0) \in H^m(\mathbb{R}^N_+)$ *for* $^\exists\lambda \geq 2,\ \beta > 0$.

$\Rightarrow$ $^\forall\alpha \in (0,\lambda]$, $^\exists\delta = \delta(\alpha) > 0$ *s.t. if*

$$\beta + (|\phi_b| + \|(W_{\lambda/2,\beta}\psi_0, W_{\lambda/2,\beta}\eta_0)\|_{H^m})/\beta \leq \delta$$

then $^\exists 1$ *time global solution* (ψ,η,σ) *s.t.*

$$W_{\alpha/2,\beta}\psi,\ W_{\alpha/2,\beta}\eta \ \in \ \bigcap_{i=0}^{m} C^i([0,\infty); H^{m-i}(\mathbb{R}^N_+)),$$

$$W_{\alpha/2,\beta}\sigma \ \in \ \bigcap_{i=0}^{m} C^i([0,\infty); H^{m+2-i}(\mathbb{R}^N_+)).$$

$$^\exists C(\alpha) > 0 \ s.t.\ \|(W_{\alpha/2,\beta}\psi, W_{\alpha/2,\beta}\eta)(t)\|^2_{H^m} + \|W_{\alpha/2,\beta}\sigma(t)\|^2_{H^{m+2}} \leq C\|(W_{\lambda/2,\beta}\psi_0, W_{\lambda/2,\beta}\eta_0)\|^2_{H^m}(1+\beta t)^{-(\lambda-\alpha)}.$$

Asymptotic stability under **Dirichlet BC** (degenerate case)

$\cdots$ [S.Nishibata, M.O., M.Suzuki, '12]

Theorem 11 $(N,m)=(1,2),(2,3),(3,3)$. $u_+=-\sqrt{K+1}$, $K>0$.

Suppose $(W_{\lambda/2,\beta}\psi_0, W_{\lambda/2,\beta}\eta_0)\in H^m(\mathbb{R}^N_+)$, $\exists\lambda\in[4,\lambda_0)$,

where $\lambda_0=5.56\cdots\in\mathbb{R}$ *satisfy* $\lambda_0(\lambda_0-1)(\lambda_0-2)-12(\lambda_0+2)=0$.

$\Rightarrow {}^\forall\alpha\in(0,\lambda]$, ${}^\forall\theta\in(0,1]$, ${}^\exists\delta=\delta(\alpha,\theta)>0$ *s.t. if*

$$\phi_b\in[-\delta,0),\quad \beta/\Gamma|\phi_b|^{1/2}\in[\theta,1],\quad \|(W_{\lambda/2,\gamma}\psi_0, W_{\lambda/2,\gamma}\eta_0)\|_{H^m}/\beta^3\le\delta,$$

then ${}^\exists 1$ *time global solution* (ψ,η,σ) *s.t.*

$$W_{\alpha/2,\beta}\psi,\ W_{\alpha/2,\beta}\eta\ \in\ \bigcap_{i=0}^{m} C^i([0,\infty);H^{m-i}(\mathbb{R}^N_+)),$$

$$W_{\alpha/2,\beta}\sigma\ \in\ \bigcap_{i=0}^{m} C^i([0,\infty);H^{m+2-i}(\mathbb{R}^N_+)).$$

$${}^\exists C(\alpha,\theta)>0\ \ s.t.\ \|(W_{\alpha/2,\beta}\psi, W_{\alpha/2,\beta}\eta)(t)\|^2_{H^m}+\|W_{\alpha/2,\beta}\sigma(t)\|^2_{H^{m+2}}$$

$$\le C\|(W_{\lambda/2,\beta}\psi_0, W_{\lambda/2,\beta}\eta_0)\|^2_{H^m}(1+\beta t)^{-(\lambda-\alpha)/3}.$$

- Derivation of the basic estimate (a) $\left(\omega = \begin{pmatrix} \psi \\ \eta \end{pmatrix} \in \mathbb{R}^{N+1} \quad D_t := \partial_t + (u \cdot \nabla)\right)$

$$\omega_t + \sum_{j=1}^{N} M_j \omega_{x_j} + \begin{pmatrix} 0 \\ \nabla\sigma \end{pmatrix} + \eta_1 \begin{pmatrix} \tilde{v} \\ \tilde{u} \end{pmatrix}_{x_1} = 0 \qquad \text{(P.a)}$$

$$-\Delta\sigma = e^{\psi+\tilde{v}} - e^{\tilde{v}} - e^{\sigma+\tilde{\phi}} + e^{\tilde{\phi}} \qquad \text{(P.b)}$$

$$\int dx \quad \left[e^{\beta x_1}\ {}^t\omega \cdot (\text{P.a})\right] \Rightarrow \qquad \left(\delta := N_{e^{\beta x_1}}(T) + |\phi_b| + \beta^2\right)$$

$$\frac{d}{dt}\left(\int_{\mathbb{R}^N_+} dx e^{\beta x_1} \frac{K\psi^2 + \eta^2}{2}\right) + \beta \int_{\mathbb{R}^N_+} dx e^{\beta x_1} \left[{}^t\omega \frac{-M_1}{2}\omega - \eta_1\sigma\right]$$

$$+ \int_{x=0} dx' \left[{}^t\omega \frac{-M_1}{2}\omega\right] - \int_{\mathbb{R}^N_+} dx e^{\beta x_1} [\sigma \text{div}\eta] \le C\delta \|e^{\beta x_1/2}\omega\|_1^2.$$

$$\text{(P.b)} \quad \Rightarrow \quad |-\Delta\sigma - (\psi - \sigma)| \le C\delta |(\psi, \sigma)|$$

$$\text{div}\left((\text{P.a})_2 \cdots (\text{P.a})_{N+1}\right) \quad \Rightarrow \quad |D_t(\text{div}\eta) + K\Delta\psi + \Delta\sigma| \le C\delta|(\eta, \nabla\eta)|$$

$$\therefore \quad |D_t(\text{div}\eta) + K\Delta\psi - \psi + \sigma| \le C\delta|(\omega, \nabla\omega, \sigma)|$$

$$\therefore \quad -\int_{\mathbb{R}^N_+} dx e^{\beta x_1} [\sigma \mathrm{div}\eta]$$

$$\geq \int_{\mathbb{R}^N_+} dx e^{\beta x_1} [(D_t (\mathrm{div}\eta) + K\Delta\psi - \psi) \mathrm{div}\eta] - C\delta \| e^{\frac{\beta x_1}{2}} \omega\|_1^2 \qquad (\because \text{elliptic estimate})$$

$$\geq \int_{\mathbb{R}^N_+} dx e^{\beta x_1} [D_t (\mathrm{div}\eta) \mathrm{div}\eta] + \int_{\mathbb{R}^N_+} dx e^{\beta x_1} [(K\Delta\psi - \psi)(-D_t\psi)] - C\delta \| e^{\frac{\beta x_1}{2}} \omega\|_1^2 \qquad (\because (\mathrm{P.a})_1)$$

$$\geq \frac{d}{dt}\left(\int_{\mathbb{R}^N_+} dx e^{\beta x_1} \frac{(\mathrm{div}\eta)^2 + K(\nabla\psi)^2 + \psi^2}{2}\right) + \beta \int_{\mathbb{R}^N_+} dx \left[e^{\beta x_1} f(t, x_1, x')\right] + \int_{x_1=0} dx' \left[f(t, 0, x')\right]$$

$$-C\delta \| e^{\beta x_1/2} \omega\|_1^2. \qquad f(t, x_1, x') = \frac{-u_1}{2}(\mathrm{div}\eta)^2 - K\psi_x \mathrm{div}\eta + \frac{-Ku_1}{2}(\nabla\psi)^2 + \frac{-u_1}{2}\psi^2$$

basic estimate (a) $\left(\begin{array}{c}\text{blue terms} > 0 \Leftrightarrow u_+^2 > K+1 \\ \text{green terms} > 0 \Leftarrow u_+^2 > K\end{array}\right)$

$$\frac{d}{dt}\left(\int_{\mathbb{R}^N_+} dx \frac{e^{\beta x_1}}{2}\left[(K+1)\psi^2 + \eta^2 + K(\nabla\psi)^2 + (\mathrm{div}\eta)^2\right]\right)$$

$$+ \quad \beta \int_{\mathbb{R}^N_+} dx e^{\beta x_1}\left[{}^t\omega \frac{-M_1}{2}\omega - \eta_1\sigma + \frac{-u_1}{2}\psi^2 + \frac{-Ku_1}{2}(\nabla\psi)^2 - K\psi_x \mathrm{div}\eta + \frac{-u_1}{2}(\mathrm{div}\eta)^2\right]$$

$$+ \quad \int_{x=0} dy \left[{}^t\omega \frac{-M_1}{2}\omega + \frac{-u_1}{2}\psi^2 + \frac{-Ku_1}{2}(\nabla\psi)^2 - K\psi_x \mathrm{div}\eta + \frac{-u_1}{2}(\mathrm{div}\eta)^2\right]$$

$$\leq \quad C\delta \| e^{\beta x_1/2}\omega\|_1^2.$$

- Derivation of supplementary inequality (b)

$$\int dx \left[e^{\beta x_1}\ {}^t\omega \cdot (\mathrm{P.a}) \right] + \sum_{i=1}^{N} \int dx \left[e^{\beta x_1}\ {}^t\partial_i\omega \cdot \partial_i (\mathrm{P.a}) \right] \Rightarrow$$

$$\begin{aligned}
\text{(b)} \quad & \frac{d}{dt}\left(\int_{\mathbb{R}^N_+} dx \frac{e^{\beta x_1}}{2} \left[(K+1)\psi^2 + \eta^2 + K(\nabla\psi)^2 + (\nabla\eta)^2 \right] \right) \\
+ \quad & \beta \int_{\mathbb{R}^N_+} dx e^{\beta x_1} \left[{}^t\omega \frac{-M_1}{2}\omega - \eta_1\sigma + \frac{-u_1}{2}\psi^2 + \sum_{i=1}^{N} {}^t\partial_i\omega \frac{-M_1}{2}\partial_i\omega + \sigma_x\ \mathrm{div}\eta - \sum_{i=1}^{N} \sigma_i\ \partial_i\eta_1 \right] \\
+ \quad & \int_{x=0} dy \left[{}^t\omega \frac{-M_1}{2}\omega + \frac{-u_1}{2}\psi^2 + \sum_{i=1}^{N} {}^t\partial_i\omega \frac{-M_1}{2}\partial_i\omega + \sigma_x\ \mathrm{div}\eta - \sigma_x\ \partial_x\eta_1 \right] \leq C\delta \| e^{\beta x_1/2}\omega \|_1^2
\end{aligned}$$

$$\begin{aligned}
\because \quad & + \int_{\mathbb{R}^N_+} dx \left[e^{\beta x_1}\eta \cdot \nabla\sigma \right] + \sum_{i=1}^{N} \int_{\mathbb{R}^N_+} dx \left[e^{\beta x_1}\partial_i\eta \cdot \nabla\partial_i\sigma \right] \\
= \quad & - \int_{\mathbb{R}^N_+} dx \left[e^{\beta x_1}\mathrm{div}\eta\,(\sigma - \Delta\sigma) \right] + \cdots \\
= \quad & + \int_{\mathbb{R}^N_+} dx \left[e^{\beta x_1} D_t\psi\psi \right] + \cdots \qquad (\because (\mathrm{P.a})_1, (\mathrm{P.b}))
\end{aligned}$$

- <u>Derivation of supplementary inequality (c)</u>

$$\int dx \left[e^{\beta x_1}\ {}^t\omega \cdot (\mathrm{P.a}) \right] \Rightarrow$$

$$\text{(c)}\quad \frac{d}{dt}\left(\int_{\mathbb{R}^N_+} dx \frac{e^{\beta x_1}}{2}\left[K\psi^2 + \eta^2 + \sigma^2 + (\nabla\sigma)^2 \right] \right) + \int_{x=0} dy \left[{}^t\omega \frac{-M_1}{2}\omega + \frac{-u_1}{2}\sigma_x^2 \right]$$
$$+\beta \int_{\mathbb{R}^N_+} dx e^{\beta x_1} \left[{}^t\omega \frac{-M_1}{2}\omega + \eta_1\sigma + \frac{-u_1}{2}\sigma^2 - \sigma_x\ \sigma_t - u_1 \left(\frac{(\nabla\sigma)^2}{2} + \sigma_x^2 \right) \right] \leq C\delta \| e^{\beta x_1/2}\omega \|_1^2$$

$$\because \int_{\mathbb{R}^N_+} dx \left[e^{\beta x_1}\eta \cdot \nabla\sigma \right] = -\int_{\mathbb{R}^N_+} dx \left[e^{\beta x_1} \mathrm{div}\eta\sigma \right] + \cdots = \int_{\mathbb{R}^N_+} dx \left[e^{\beta x_1} D_t\psi\sigma \right] + \cdots$$
$$= \int_{\mathbb{R}^N_+} dx \left[e^{\beta x_1}\psi_t\sigma \right] + \int_{\mathbb{R}^N_+} dx \left[e^{\beta x_1} \left\{ (u\cdot\nabla)\psi \right\} \sigma \right] + \cdots \equiv (\mathrm{c}_1) + (\mathrm{c}_2) + \cdots$$
$$(\mathrm{c}_1) = -\int_{\mathbb{R}^N_+} dx \left[e^{\beta_1 x} (\Delta\sigma_t - \sigma_t)\sigma \right] + \cdots \left(\because -\Delta\sigma_t = \psi_t e^{v} - \sigma_t e^{-\phi} \Leftarrow \partial_t (\mathrm{P.b}) \right)$$
$$(\mathrm{c}_2) = -\int_{\mathbb{R}^N_+} dx \left[e^{\beta_1 x}\psi (u\cdot\nabla)\sigma \right] + \cdots = \int_{\mathbb{R}^N_+} dx \left[e^{\beta_1 x} (\Delta\sigma - \sigma)(u\cdot\nabla)\sigma \right] + \cdots$$

- Take constants $\epsilon \ll 1$ and set $\delta = N_{e^{\beta x_1}}(T) + |\phi_b| + \beta^2 \ll 1 \Rightarrow$

$^\exists c_0, \cdots, c_7 > 0$ s.t. $\quad$ (a) $+ \epsilon \times$ ((c) $+ \epsilon \times$ (b)) $\Rightarrow$

$$\frac{d}{dt}\left(\int_{\mathbb{R}^N_+} dx e^{\beta x_1}\left[c_1\psi^2 + c_2\eta^2 + c_3(\nabla\psi)^2 + c_4(\nabla\eta)^2 + c_5(\mathrm{div}\eta)^2 + c_6\sigma^2 + c_7(\nabla\sigma)^2\right]\right)$$
$$+ c_0\beta\left(\int_{\mathbb{R}^N_+} dx e^{\beta x_1}\left[\psi^2 + \eta^2 + (\nabla\psi)^2 + (\nabla\eta)^2 + (\mathrm{div}\eta)^2 + \sigma^2 + (\nabla\sigma)^2\right]\right) \leq 0$$

$\therefore \quad \int_0^t d\tau\, [e^{\gamma\tau}\ \cdot\ ] \quad (\gamma \ll \beta) \Rightarrow {}^\exists C > 0$ s.t.

$$e^{\gamma t}\|e^{\beta x_1/2}(\psi,\eta)(t)\|_1^2 + \int_0^t e^{\gamma\tau}\|e^{\beta x_1/2}(\psi,\eta)(\tau)\|_1^2\, d\tau \leq C\|e^{\beta x_1/2}(\psi,\eta)(0)\|_1^2 \ (\sharp)$$

For higher order derivatives,

$\partial_{pq}\{(a),(b),(c)\}$ $(p, q = t, x_2, \cdots, x_N)$ + (equivalence of norms) + ($\sharp$)

$\Rightarrow {}^\exists C > 0 \quad$ s.t.

$$e^{\gamma t}\|e^{\beta x_1/2}(\psi,\eta)(t)\|_3^2 + \int_0^t e^{\gamma\tau}\|e^{\beta x_1/2}(\psi,\eta)(\tau)\|_3^2\, d\tau \leq C\|e^{\beta x_1/2}(\psi,\eta)(0)\|_3^2.$$