On $\mathcal{R}$-sectoriality of the Stokes equations with first order boundary condition in a general domain

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Problem

$$\begin{cases} \lambda u - \mathrm{Div}\, S(u,p) = f, \quad \mathrm{div}\, u = g & \text{in } \Omega, \\ S(u,p)\mathbf{n}|_\Gamma = h|_\Gamma \quad u|_{\Gamma_0} = 0. \end{cases} \tag{1}$$

- $S(u,p) = -pI + \mu D(u)$, $D(u) = \nabla u + {}^T\nabla u$
- Ω: uniform $C^{1,1}$ domain in $\mathbb{R}^N$ with boundaries Γ and Γ_0.
- $\mathrm{dist}\,(\Gamma, \Gamma_0) > 0$, $\mathbf{n}$: unit outer normal to Γ

If the surface tension is taken into account, then the bc. on Γ is:

$$\begin{cases} \lambda\eta - u\cdot\mathbf{n} = d & \text{on } \Gamma, \\ S(u,p)\mathbf{n} + (c_g - c_\sigma\Delta_\Gamma)\eta\mathbf{n} = h & \text{on } \Gamma. \end{cases} \tag{2}$$

- η: corresponding height function, Δ_Γ: Laplace-Beltrami op. on Γ.

Background of my talk: analytic semigroup

$$u'(t) - Au(t) = 0 \ \ (t > 0), \ \ u(0) = u_0 \text{ on some Banach space } X. \qquad (3)$$

If A has a resolvent estimate:

$$\|\lambda(\lambda I - A)^{-1}\| \leq C$$

for $\lambda \in \Sigma_{\epsilon,\lambda_0}$ with some $0 < \epsilon < \pi/2$ and $\lambda_0 \geq 0$, where

$$\Sigma_{\epsilon,\lambda_0} = \{\lambda \in \mathbb{C} \mid |\arg\lambda| \leq \pi - \epsilon, \ \ |\lambda| > \lambda_0\},$$

then

$$e^{At} = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda I - A)^{-1}\, d\lambda$$

gives an analytic semigroup associated with (3).

Backgroud of my talk: maximal regularity

$$u'(t) - Au(t) = f(t) \ \ (t > 0), \ \ u(0) = 0. \tag{4}$$

Letting $f_0(t)$ be 0 extension of $f(t)$ to $t < 0$,

$$u(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\lambda t}(\lambda I - A)^{-1}\hat{f}_0(\lambda)\,d\tau \quad (\lambda = \gamma + i\tau)$$

must be a solution to (4). Since

$$u'(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\lambda t}\lambda(\lambda I - A)^{-1}\hat{f}_0(\lambda)\,d\tau$$

If $\lambda(\lambda I - A)^{-1}$ is $\mathcal{R}$-bounded, then Weis's operator valued Fourier multiplier thoerem yields the maximal L_p regularity:

$$\|e^{\gamma t}u'\|_{L_p(\mathbb{R},X)} + \|e^{\gamma t}Au\|_{L_p(\mathbb{R},X)} \le C\|e^{\gamma t}f_0\|_{L_p(\mathbb{R},X)}.$$

$\mathcal{R}$-sectoriality is the notion that yields the generation of analyticity and the maximal L_p regularity at the same time.

Weis' operator valued Fourier multiplier theorem

$\mathcal{R}$-boundedness

$p \in [1, \infty)$, $\{\gamma_j\}_{j\in\mathbb{N}}$ are sequences of independent, symmetric $\{-1, 1\}$-valued random variables on $[0, 1]$. $\mathcal{A} \subset \mathcal{B}(X, Y)$ is $\mathcal{R}$-bounded $\Longleftrightarrow$

$$\int_0^1 \|\sum_{j=1}^N \gamma_j(u) T_j f_j\|_Y^p \, du \leqq C \int_0^1 \|\sum_{j=1}^N \gamma_j(u) f_j\|_X^p \, du \quad \text{for all } T_j \in \mathcal{A},\ f_j \in X,\ N \in \mathbb{N}.$$

Theorem (Weis)

Let X and Y be two UMD Banach spaces and $1 < p < \infty$. Let M be a function in $C^1(\mathbb{R} \setminus \{0\}, \mathcal{L}(X, Y))$ such that

$$\mathcal{R}(\{M(\rho) \mid \rho \in \mathbb{R} \setminus \{0\}\}) = \kappa_0 < \infty, \ \ \mathcal{R}(\{\rho M'(\rho) \mid \rho \in \mathbb{R} \setminus \{0\}\}) = \kappa_1 < \infty.$$

Set $T_M\phi = \mathcal{F}^{-1}[M\mathcal{F}[\phi]]$ ($\mathcal{F}[\phi] \in \mathcal{D}(\mathbb{R}, X)$). Then, we have

$$\|T_M\|_{\mathcal{L}(L_p(\mathbb{R},X),L_p(\mathbb{R},Y))} \leq C(\kappa_0 + \kappa_1)$$

for some positive constant C depending on p, X and Y.

Assumption 1 for domains

Assumption 1

Let Ω be a uniform $C^{1,1}$ domain.

$\underset{\text{def.}}{\Longleftrightarrow}$ $\exists\alpha,\ \beta > 0$ and $K > 0$ s.t.

- $\forall x_0 \in \Gamma \cup \Gamma_0$, $\exists h(x') \in C^{1,1}$ on $B_\alpha(x_0')$ with $\|h\|_{C^{\ell_0,1}(B_\alpha(x_0'))} \leq K$

$$\Omega \cap B_\beta(x_0) = \{x \in \mathbb{R}^N \mid x_j > h(x')\} \cap B_\beta(x_0),$$
$$\partial\Omega \cap B_\beta(x_0) = \{x \in \mathbb{R}^N \mid x_j = h(x')\} \cap B_\beta(x_0).$$

$C^{1,1}(B_{\alpha'}(x_0'))$ denotes the set of all $h \in C^1(B_{\alpha'}(x_0'))$ s.t.

$$\sum_{|\alpha'|=1} \sup_{\substack{x \neq y \\ x,y \in B_{\alpha'}(x_0')}} \frac{|\partial_{x'}^{\alpha'} h(x') - \partial_{y'}^{\alpha'} h(y')|}{|x' - y'|}$$

Assumption 2 for domains:

Let

$$W^1_{q,\Gamma}(\Omega) = \{\theta \in W^1_q(\Omega) \mid \theta|_\Gamma = 0\},$$
$$\hat W^1_{q,\Gamma}(\Omega) = \{\theta \in L_{q,\mathrm{loc}}(\Omega) \mid \nabla\theta \in L_q(\Omega)^N,\ \ \theta|_\Gamma = 0\}.$$

Definition of the space for the pressure

Let $\mathcal{W}^1_q(\Omega)$ be a closed subspace of $\hat W^1_{q,\Gamma}(\Omega)$ that contains $W^1_{q,\Gamma}(\Omega)$.

For any $f \in L_q(\Omega)^N$, we consider the weak Dirichlet-Neumann problem:

$$(\nabla\theta, \nabla\varphi) = (f, \nabla\varphi) \quad \forall\, \varphi \in \mathcal{W}^1_{r'}(\Omega). \tag{5}$$

Assumption 1

Let $1 < q < \infty$ and let $q' = q/(q-1)$. Then, there exist an operator $\mathcal{K}_r \in \mathcal{L}(L_r(\Omega)^N, \mathcal{W}^1_r(\Omega))$ with $r = q$ and $r = q'$ such that for any $f \in L_q(\Omega)^N$, $\mathcal{K}(f)$ is a unique solution to (5).

The space for the divergence data

$g \in W^1_{q,\Gamma}(\Omega) \cap W^{-1}_{q,\Gamma}(\Omega)$ means that there exists a constant $C_g > 0$ such that for any $\varphi \in W^1_{q',\Gamma}(\Omega)$

$$|(g, \varphi)| \leq C_g \|\nabla\varphi\|_{L_{q'}(\Omega)}.$$

By the Hahn-Banach theorem, there exists a $\mathcal{G}(g) \in L_q(\Omega)^N$ such that

$$\|\mathcal{G}(g)\|_{L_q(\Omega)} \leq \sup\{|(g, \varphi)| \mid \varphi \in W^1_{q',\Gamma}(\Omega), \ \|\nabla\varphi\|_{L_{q'}(\Omega)} = 1\},$$
$$(\mathcal{G}(g), \nabla\varphi) = (g, \varphi) \quad \forall\, \varphi \in W^1_{q',\Gamma}(\Omega).$$

Main Theorem

$1 < q < \infty$, $0 < \epsilon < \pi/2$. Ω satisfies Assumptions 1 and 2.

$$Y_q(\Omega) = L_q(\Omega) \times (W_q^1(\Omega) \cap W_{q,\Gamma}^{-1}(\Omega)) \times W_q^1(\Omega),$$
$$X_q(\Omega) = L_q(\Omega)^N \times L_q(\Omega)^N \times L_q(\Omega) \times L_q(\Omega) \times L_q(\Omega)^{N^2} \times L_q(\Omega)^N,$$
$$F_\lambda(f, g, h) = (f, \nabla g, \lambda^{1/2} g, \lambda \mathcal{G}(g), \nabla h, \lambda^{1/2} h) \in X_q(\Omega)$$

$\exists \gamma_0 \geq 1$ and $\Phi(\lambda) \in \mathcal{L}(Y_q(\Omega), W_q^2(\Omega)^N)$ $(\lambda \in \Sigma_{\epsilon,\gamma_0})$ such that $\forall \lambda \in \Sigma_{\epsilon,\gamma_0}$, $\forall (f, g, h) \in Y_q(\Omega)$, $u = \Phi(\lambda)(f, g, h)$ is a unique solution to (1) with some $p \in W_q^1(\Omega) + \mathcal{W}_q^1(\Omega)$.

Moreover, there exists an operator $\Psi(\lambda) \in \mathcal{L}(X_q(\Omega), L_q(\Omega)^{\tilde N})$ $(\lambda \in \Sigma_{\epsilon,\gamma_0})$ with $\tilde N = 2N + N^2$ such that

$$(\lambda u, |\lambda|^{1/2} \nabla u, \nabla^2 u) = \Psi(\lambda) F_\lambda(f, g, h),$$
$$\mathcal{R}_{\mathcal{L}(X_q^2(\Omega), L_q(\Omega)^{\tilde N})}(\{\left(\lambda \frac{d}{d\lambda}\right)^\kappa \Psi(\lambda) \mid \lambda \in \Sigma_{\epsilon,\gamma_0}\}) \leq {}^{\exists} \sigma_1, \quad (\kappa = 0, 1).$$

Reduced Stokes equation

Let $K(u) = \mu < D(u)\mathbf{n}, \mathbf{n} > -\mathrm{div}\, u + \theta_u \in W^1_q(\Omega) + \mathcal{W}^1_q(\Omega)$, where $\theta_u \in \mathcal{W}^1_q(\Omega)$ is a unique solution to

$$(\nabla\theta_u, \nabla\varphi) = \mu(\mathrm{Div}\, D(u) - \nabla < D(u)\mathbf{n}, \mathbf{n} >, \nabla\varphi) \quad {}^\forall\varphi \in \mathcal{W}^1_{q'}(\Omega).$$

$K(u)$ satisfies:

$$(\nabla K(u), \nabla\varphi) = (\mu\mathrm{Div}\, D(u) - \nabla\mathrm{div}\, u, \nabla\varphi) \quad {}^\forall\varphi \in \mathcal{W}^1_{q'}(\Omega),$$

subject to $K(u)|_\Gamma = (\mu < D(u)\mathbf{n}, \mathbf{n} > -\mathrm{div}\, u)|_\Gamma$.

The reduced Stokes equation:

$$\begin{cases} \lambda u - \mathrm{Div}\, S(u, K(u)) = f & \text{in } \Omega, \\ S(u, K(u))\mathbf{n} = h \text{ on } \Gamma, \quad u|_{\Gamma_0} = 0. \end{cases} \tag{6}$$

Equivalence between (1) and (6)

Let $\lambda \in \Sigma_{\epsilon,\lambda_0}$ with very large λ_0. Then, we have

(1) If $W^1_{q,\Gamma}(\Omega)$ is dense in $\mathcal{W}^1_q(\Omega)$, then (1) $\Longrightarrow$ (6)

(2) (6) $\Longrightarrow$ (1).

($\Longrightarrow$) Let $g \in W^1_q(\Omega) \cap W^{-1}_q(\Omega)$ be a solution to

$$-(f, \nabla\varphi) = \lambda(g, \varphi) + \mu(\nabla g, \nabla\varphi) \quad {}^\forall\varphi \in W^1_{q,\Gamma}(\Omega)$$

subject to $g|_\Gamma =< h, \mathbf{n} > |_\Gamma$. Let $u \in W^2_q(\Omega)^N$ and $p \in W^1_q(\Omega) + \mathcal{W}^1_q(\Omega)$ be a solution to (1). Since $\mathrm{div}\, u = g$, for any $\varphi \in W^1_{q,\Gamma}(\Omega)$ we have

$$\begin{aligned}(f, \nabla\varphi) &= \lambda(u, \nabla\varphi) - \mu(\mathrm{Div}\, D(u) - \nabla K(u), \nabla\varphi) + (\nabla(p - K(u)), \nabla\varphi)\\ &= -\{\lambda(\mathrm{div}\, u, \varphi) + \mu(\nabla\mathrm{div}\, u, \nabla\varphi)\} + (\nabla(p - K(u)), \nabla\varphi).\end{aligned}$$

$\Longrightarrow$ $(\nabla(p - K(u)), \nabla\varphi) = 0 \ {}^\forall\varphi \in W^1_{q,\Gamma}(\Omega)$. Since

$$\begin{aligned}p - K(u) &= \mu < D(u)\mathbf{n}, \mathbf{n} > - < h, \mathbf{n} > -(\mu < D(u)\mathbf{n}, \mathbf{n} > -\mathrm{div}\, u)\\ &= - < h, \mathbf{n} > +g = 0\end{aligned}$$

on Γ, the denseness of $W^1_{q,\Gamma}(\Omega)$ in $\mathcal{W}^1_q(\Omega)$ yields that $p = K(u)$.

$(6) \implies (1)$

Let $J_q(\Omega)$ be the solenoidal space defined by

$$J_q(\Omega) = \{f \in L_q(\Omega)^N \mid (f, \nabla\varphi) = 0 \ {}^\forall \varphi \in \mathcal{W}^1_{q'}(\Omega)\}.$$

To solve (1), let $\theta_{f,h} \in W^1_q(\Omega) + \mathcal{W}^1_q(\Omega)$ be a solution to

$$(\nabla\theta_{f,h}, \nabla\varphi) = (f, \nabla\varphi) \ {}^\forall \varphi \in \mathcal{W}^1_{q'}(\Omega)$$

subject to $\theta|_\Gamma =< h, \mathbf{n} > |_\Gamma$. Then, we have

$$\begin{cases} \lambda u - \mathrm{Div}\, S(u, p - \theta_{f,g}) = f - \nabla\theta_{f,g}, \ \ \mathrm{div}\, u = g & \text{in } \Omega, \\ S(u, p - \theta_{f,g})\mathbf{n}|_\Gamma = (h- < h, \mathbf{n} > \mathbf{n})|_\Gamma, \ \ u|_{\Gamma_0} = 0. \end{cases}$$

Thus, we may assume that $f \in J_q(\Omega)$ and $< h, \mathbf{n} >= 0$ on Γ. Let $K(\lambda, g) \in W^1_q(\Omega) + \mathcal{W}^1_q(\Omega)$ be a solution to

$$(\nabla K(\lambda, g), \nabla\varphi) = -(\lambda\mathcal{G}(g) + \nabla g, \nabla\varphi) \ {}^\forall \varphi \in \mathcal{W}^1_q(\Omega) \tag{7}$$

subject to $K(\lambda, g) = g$ on Γ. Let $u \in W^2_q(\Omega)^N$ be a solution to

$$\begin{cases} \lambda u - \mathrm{Div}\, S(u, K(u)) = f + \nabla K(\lambda, g) & \text{in } \Omega, \\ S(u, K(u))\mathbf{n}|_\Gamma = (h + K(\lambda, g))\mathbf{n}|_\Gamma, \ \ u|_{\Gamma_0} = 0. \end{cases}$$

Continuation

$$
\begin{aligned}
(\nabla K(\lambda, g), \nabla\varphi) &= (\lambda u - \mathrm{Div}\, S(u, K(u)), \nabla\varphi) \\
&= (\lambda u, \nabla\varphi) - \mu(\nabla \mathrm{div}\, u, \nabla\varphi) \quad {}^\forall\varphi \in \mathcal{W}^1_q(\Omega).
\end{aligned} \tag{8}
$$

Let $\varphi \in W^1_{q,\Gamma}(\Omega)$, and then

$$
\begin{aligned}
&(\nabla K(\lambda, g), \nabla\varphi) = -(\lambda\mathcal{G}(g) + \nabla g, \nabla\varphi) = -\{\lambda(g, \varphi) + (\nabla g, \nabla\varphi)\} \\
&(7), (8) \implies = -\{\lambda(\mathrm{div}\, u, \varphi) + \mu(\nabla \mathrm{div}\, u, \nabla\varphi)\}.
\end{aligned} \tag{9}
$$

And, $\mathrm{div}\, u = \mu < D(u)\mathbf{n}, \mathbf{n} > -K(u) = K(\lambda, g) = g$ on Γ
$\implies \mathrm{div}\, u = g \implies$ u and $p = K(u) - K(\lambda, g)$ solve (1).

Especially, if $u \in W^2_q(\Omega)$ solves

$$
\begin{cases}
\lambda u - \mathrm{Div}\, S(u, K(u)) = f & \text{in } \Omega, \\
S(u, K(u))\mathbf{n}|_\Gamma = h|_\Gamma, \quad u|_{\Gamma_0} = 0,
\end{cases}
$$

with $f \in J_q(\Omega)$ and $< h, \mathbf{n} >= 0$ on Γ, then $K(\lambda, g) = 0$, (8), (9) $\implies$

$$
\mathrm{div}\, u = 0 \quad \text{and} \quad u \in J_q(\Omega). \tag{10}
$$

Example of Domains

- Ω is a bounded domain $\Rightarrow$ $\mathcal{W}^1_q(\Omega) = \hat{W}^1_{q,\Gamma}(\Omega)$,
- Ω is an exterior domain with $\Gamma_0 = 0$ $\Rightarrow$ $\mathcal{W}^1_q(\Omega) =$ the closure of $W^1_{q,\Gamma}(\Omega)$ with seminorm: $\|\nabla \cdot\|_{L_q(\Omega)}$.
- Let $\varphi_i \in C^{1,1}(\mathbb{R}^{N-1})$ $(i = 1, 2)$ with $\varphi_1(x') < a_1 < a_2 < \varphi_2(x')$ $(x' \in \mathbb{R}^{N-1})$, where a_i $(i = 1, 2)$ are constants. Let
$$\Omega = \{x \in \mathbb{R}^N \mid \varphi_1(x') < x_N < \varphi(x') \ \ (x' \in \mathbb{R}^{N-1})\},$$
$$\Gamma_0 = \{x \in \mathbb{R}^N \mid x_N = \varphi_1(x') \ \ (x' \in \mathbb{R}^{N-1}\},$$
$$\Gamma = \{x \in \mathbb{R}^N \mid x_N = \varphi_2(x') \ \ (x' \in \mathbb{R}^{N-1}\}.$$
- Set $H_{a_1,a_2} = \{x \in \mathbb{R}^N \mid x_N < a_1\} \cup \{x \in \mathbb{R}^N \mid x_N > a_2\}$. Let Ω be an aperture domain such that $B^R \cap \Omega = B^R \cap H_{a_1,a_2}$ for some $R > 0$ with $B^R = \{x \in \mathbb{R}^N \mid |x| > R\}$.

(1) $\Gamma = \emptyset \quad \Longrightarrow \quad \mathcal{W}^1_q(\Omega) = \hat{W}^1_q(\Omega)$.

(2) $\Gamma_0 = \emptyset \quad \Longrightarrow \quad \mathcal{W}^1_q(\Omega) = \overline{W^1_{q,\Gamma}(\Omega)}^{\|\nabla\cdot\|_{L_q(\Omega)}} = \hat{W}^1_{q,\Gamma}(\Omega)$.

(3) $\Gamma_0 \neq \emptyset$, $\Gamma \neq \emptyset$. So far $\mathcal{W}^1_q(\Omega)$?

Corollary - Generation of Analytic Semigroup

We consider the reduced Stokes equations:

$$\begin{cases} \partial_t u - \mathrm{Div}\, S(u, K(u)) = 0 \\ \qquad S(u, K(u))\nu|_\Gamma = 0, \quad u|_{\Gamma_0} = 0 \\ \qquad u(0) = u_0. \end{cases}$$

Let $\mathcal{A}$ be an operator defined by

$$\mathcal{A}u = \mathrm{Div}\, S(u, K(u)) \quad (u \in \mathcal{D}_q(\mathcal{A}))$$
$$\mathcal{D}_q(\mathcal{A}) = \{u \in W_q^2(\Omega) \mid S(u, K(u))|_\Gamma = 0, \quad u|_{\Gamma_0} = 0\}.$$

The underlying space is

$$\mathcal{H}_q(\Omega) = \{u \in L_q(\Omega) \mid (u, \nabla\varphi)_\Omega = 0 \quad \forall\, \varphi \in \hat{W}^1_{q',\Gamma}(\Omega)\}.$$

$\mathcal{A}$ generates an analytic semigroup on $\mathcal{H}$.

Corollary - Maximal L_p-regularity

$$\begin{cases} \partial_t u - \mathrm{Div}\, S(u,\theta) = f, \quad \mathrm{div}\, u = g \quad \text{in } \Omega,\ t>0, \\ S(u,\theta)\nu|_\Gamma = h|_\Gamma, \quad u|_{\Gamma_0} = 0, \quad t>0, \\ u(0) = 0. \end{cases} \tag{11}$$

$$u(x,t) = \mathcal{L}^{-1}[\Phi(\lambda)(f,g,h)](t)$$

solves (11) and satisfies the estimate:

$$\begin{aligned} &\|e^{-\gamma t}(\partial_t u, \gamma u, \Lambda_\gamma^{\frac12}\nabla u, \nabla^2 u, \nabla\theta)\|_{L_p(\mathbb{R}, L_q(\Omega))} \\ &\quad \le C(\|e^{-\gamma t}(f, \Lambda_\gamma^{\frac12} g, \nabla g, \Lambda_\gamma^{\frac12} h, \nabla h)\|_{L_p(\mathbb{R}, L_q(\Omega))} + \|e^{-\gamma t}\partial_t \mathcal{G}(g)\|_{L_p(\mathbb{R}, W^{-1}_{q,\Gamma}(\Omega))}). \end{aligned}$$

Sketch of Proof of Main Theorem. Instead of (1), we consider (6).

Uniqueness: Let $u \in W^2_q(\Omega)^N$ satisfy the homogeneous equation:

$$\lambda u - \mathrm{Div}\, S(u, K(u)) = 0 \ \text{ in } \Omega, \quad S(u, K(u))\mathbf{n}|_\Gamma = 0, \quad u|_{\Gamma_0} = 0.$$

For any $\varphi \in C^\infty_0(\Omega)^N$, we shall prove that $(u, \varphi) = 0$. Let $\theta_\varphi \in \mathcal{W}^1_{q'}(\Omega)$ be a solution to

$$(\nabla\theta_\varphi, \nabla\psi) = (\varphi, \nabla\psi) \quad {}^\forall\psi \in \mathcal{W}^1_q(\Omega).$$

By (10) we have $\mathrm{div}\, u = 0$ and $(u, \nabla\psi) = 0 \ {}^\forall\psi \in \mathcal{W}^1_{q'}(\Omega)$. $\Longrightarrow$
$(u, \varphi) = (u, \varphi - \nabla\theta_\varphi)$. Set $\tilde\varphi - \varphi - \nabla\theta_\varphi$, and then $\tilde\varphi \in J_{q'}(\Omega)$. Let $v \in W^2_{q'}(\Omega)^N$:

$$\lambda v - \mathrm{Div}\, S(v, K(v)) = \tilde\varphi \ \text{ in } \Omega, \quad S(v, K(v))\mathbf{n}|_\Gamma = 0, \quad v|_{\Gamma_0} = 0.$$

By (10) we have $\mathrm{div}\, v = 0$ and $(v, \nabla\psi) = 0 \ {}^\forall\psi \in \mathcal{W}^1_q(\Omega)$. Set
$K(v) = \psi_1 + \psi_2 \in W^1_{q'}(\Omega) + \mathcal{W}^1_{q'}(\Omega)$, we have

$$\begin{aligned}(u, \tilde\varphi) &= (u, \lambda v - \mathrm{Div}\, S(v, K(v)) = \lambda(u, v) - (u, \mathrm{Div}\, S(v, \psi_1)) + (u, \nabla\psi_2) \\ &= \lambda(u, v) - (u, S(v, K(v)\mathbf{n})_\Gamma + \frac{1}{2}(\nabla u, \nabla v) + (\mathrm{div}\, u, \psi_1) \\ &= \lambda(u, v) + \frac{1}{2}(\nabla u, \nabla v), \quad \text{where, we have used: } K(v) = \psi_1 \text{ on } \Gamma.\end{aligned}$$

In the same way, we see $0 = (\lambda u - \mathrm{Div}\, S(u, K(u)), v) = \lambda(u, v) + \frac{1}{2}(\nabla D(u), D(v))$
$\Longrightarrow 0 = (u, \varphi)$.

Existence Proof

$$S(u, K(u))\mathbf{n} = S(u, K(u))\mathbf{n} - < S(u, K(u))\mathbf{n}, \mathbf{n} > \mathbf{n} + < S(u, K(u))\mathbf{n}, \mathbf{n} > \mathbf{n}$$
$$= \mu(D(u)\mathbf{n} - < D(u)\mathbf{n}, \mathbf{n} > \mathbf{n}) + (\mu < D(u)\mathbf{n}, \mathbf{n} > -K(u))\mathbf{n} = \mu\mathcal{T}D(u)\mathbf{n} + \operatorname{div} u\mathbf{n}$$

with $\mathcal{T}D(u)\mathbf{n} = D(u)\mathbf{n} - < D(u)\mathbf{n}, \mathbf{n} > \mathbf{n}$. (6) is written by

$$\begin{cases} \lambda u - \operatorname{Div} S(u, K(u)) = f & \text{in } \Omega, \\ \mu\mathcal{T}D(u)\mathbf{n}|_\Gamma = \tilde{h}|_\Gamma, \ \ \operatorname{div} u|_\Gamma = h_N|_\Gamma, \ \ u|_{\Gamma_0} = 0. \end{cases} \tag{12}$$

Theorem $1 < q < \infty$, $0 < \epsilon < \pi/2$,
$Y_q(\Omega) = \{(f, h) \mid f \in L_q(\Omega)^N, \ \ \tilde{h} \in W^1_q(\Omega), \ \ h_N \in W^1_q(\Omega) \cap W^{-1}_{q,\Gamma}(\Omega)\}$. *Then, there exist* $\lambda_1 >> 1$, $A(\lambda) \in \mathcal{L}(Y_q(\Omega), W^2_q(\Omega)^N)$ *and* $\mathcal{B}(\lambda) \in \mathcal{L}(X_q(\Omega), L_q(\Omega)^{\tilde{N}})$ $(\lambda \in \Sigma_{\epsilon,\lambda_1})$ *such that*
(i) *For any* $\lambda \in \Sigma_{\epsilon,\lambda_1}$, $(f, h) \in Y_q(\Omega)$, $u = A(\lambda)(f, h)$ *is a unique solution to* (12).
(ii) $G_\lambda A(\lambda)(f, h) = \mathcal{B}(\lambda)F_\lambda(f, h)$ *and*

$$\mathcal{R}_{\mathcal{L}(X_q(\Omega), L_q(\Omega)^{\tilde{N}})}(\{(\lambda\frac{d}{d\lambda})^\ell\mathcal{B}(\lambda) \mid \lambda \in \Sigma_{\epsilon,\lambda_1}\}) \leq {}^\exists\gamma_1 \ \ (\ell = 0, 1).$$

Here, $G_\lambda u = (\lambda u, \lambda^{1/2}\nabla u, \nabla^2 u)$, $F_\lambda(f, h) = (f, \nabla(\tilde{h}, h_N), \lambda^{1/2}(\tilde{h}, h_N), \mathcal{G}(h_N))$.

Half-Space case

Proof is proceeded in the following cases:
1. Whole space, 2. Half-Space, 3. Bent Half-Spcae, 4. General Domain.

$$\lambda u - \mathrm{Div} S(u,p) = f, \ \ \mathrm{div}\, u = g \ \text{ in } \mathbb{R}^N_+, \ \ S(u,p)\mathbf{n}_0|_{\partial\mathbb{R}^N_+} = h \tag{13}$$

with $\mathbb{R}^N_+ = \{x_N > 0\}$ and $\mathbf{n}_0 = (0, \ldots, 0, -1)$.

- $\overline{W^1_{q,0}(\mathbb{R}^N_+)}^{\|\nabla\cdot\|_{L_q(\mathbb{R}^N_+)}} = \hat W^1_{q,0}(\mathbb{R}^N_+)$ with $W^1_{q,0}(\mathbb{R}^N_+) = \{u \in W^1_q(\mathbb{R}^N_+) \mid u|_{x_N=0} = 0\}$ and $\hat W^1_{q,0}(\mathbb{R}^N_+) = \{u \in \hat W^1_q(\mathbb{R}^N_+) \mid u|_{x_N=0} = 0\}$.
- $^\exists\ V \in \mathcal{L}(W^1_q(\mathbb{R}^N_+) \cap W^{-1}_q(\mathbb{R}^N_+), W^2_q(\mathbb{R}^N_+)^N)$ s.t. $\mathrm{div}\, V(g) = g$ in $\mathbb{R}^N_+$ and

$$\nabla^2 V(g) = V_1(\nabla g), \ \ \nabla V(g) = V_0(g), \ \ V(g) = V_{-1}(\mathcal{G}(g)),$$

  with $V_1 \in \mathcal{L}(L_q(\mathbb{R}^N_+)^N, L_q(\mathbb{R}^N_+)^{N^3})$, $V_0 \in \mathcal{L}(L_q(\mathbb{R}^N_+), L_q(\mathbb{R}^N_+)^{N^2})$, $V_{-1} \in \mathcal{L}(L_q(\mathbb{R}^N_+)^N)$.
- $u = V(g) + v \Longrightarrow$

$$\begin{cases} \lambda v - \mathrm{Div} S(v,p) = f + \mu \mathrm{Div}\, D(V(g)), \ \ \mathrm{div}\, v = 0 & \text{in } \mathbb{R}^N_+, \\ S(v,p)\mathbf{n}_0|_{\partial\mathbb{R}^N_+} = (h - \mu D(V(g))\mathbf{n})|_{\partial\mathbb{R}^N_+}. \end{cases}$$

Theorem in the half-space

Let $1 < q < \infty$ *and* $0 < \epsilon < \pi/2$. *Set*

$$Y_q(\mathbb{R}^N_+) = \{(f, g, h) \mid f \in L_q(\mathbb{R}^N_+)^N,\ \ g \in W^1_q(\mathbb{R}^N_+) \cap W^{-1}_q(\mathbb{R}^N_+),\ \ h \in W^1_q(\mathbb{R}^N_+)^N\},$$

$$X_q(\mathbb{R}^N_+) = \{(F_1, \ldots, F_6) \mid F_1, F_2, F_4, F_6 \in L_q(\mathbb{R}^N_+)^N,\ \ F_3 \in L_q(\mathbb{R}^N_+),\ \ F_5 \in L_q(\mathbb{R}^N_+)^{N^2}\},$$

$$\Sigma_\epsilon = \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg \lambda| \leq \pi - \epsilon\}.$$

Then, there exist $V \in \mathcal{L}(W^1_q(\mathbb{R}^N_+) \cap W^{-1}_q(\mathbb{R}^N_+), W^2_q(\mathbb{R}^N_+)^N)$ *and* $\mathcal{T}(\lambda) \in \mathcal{L}(X_q(\mathbb{R}^N_+), W^2_q(\mathbb{R}^N_+)^N)$ $(\lambda \in \Sigma_\epsilon)$ *s.t.*

(i) *For any* $\lambda \in \Sigma_\epsilon$ *and* $(f, g, h) \in Y_q(\mathbb{R}^N_+)$, $u = V(g) + \mathcal{T}(\lambda)F_\lambda(f, g, h)$ *is a unique solution to* (13), *where* $F_\lambda(f, g, h) = (f, \nabla g, \lambda^{1/2}g, \lambda\mathcal{G}(g), \nabla h, \lambda^{1/2}h) \in X_q(\mathbb{R}^N_+)$.

(ii) *Set* $G_\lambda u = (\lambda u, \lambda^{1/2}\nabla u, \nabla^2 u)$. *Then, there exists a* $U(\lambda) \in \mathcal{L}(X_q(\mathbb{R}^N_+), L_q(\mathbb{R}^N_+)^{\tilde{N}})$ $(\lambda \in \Sigma_\epsilon)$ *s.t.*

- $G_\lambda u = U(\lambda)F_\lambda(f, g, h)$
- $\mathcal{R}_{\mathcal{L}(X_q(\mathbb{R}^N_+), L_q(\mathbb{R}^N_+)^{\tilde{N}})}(\{(\lambda \frac{d}{d\lambda})^\ell U(\lambda) \mid \lambda \in \Sigma_\epsilon\}) \leq {}^\exists\gamma_0$ $(\ell = 0, 1)$.

Bent Half-Space

$\Phi : \mathbb{R}^N \to \mathbb{R}^N$: $C^{1,1}$ diffeomorphism s.t. $\nabla\Phi = \mathcal{A}_1 + B_1(x)$, $\nabla\Phi^{-1} = \mathcal{A}_{-1} + B_{-1}(x)$, $\mathcal{A}$ and $\mathcal{A}_{-1}$ are constant orthonormal matrices, B, $B_{-1} \in C^{0,1}$ and

$$\|B_i\|_{L_\infty(\mathbb{R}^N_+)} \leq M_1, \;\; \|\nabla B_i\|_{L_\infty(\mathbb{R}^N_+)} \leq M_2 \;\; (i = 1, -1).$$

Here, $0 < M_1 \leq 1 \leq M_2$ and M_1 is chosen so small that every arguments will be done nicely.

$$\Omega_+ = \Phi(\mathbb{R}^N_+), \;\; \Gamma_+ = \Phi(\partial\mathbb{R}^N_+), \;\; \mathbf{n}_+ : \text{unit outer normal to } \Gamma_+.$$

We consider the equation:

$$\begin{cases} \lambda u - \mathrm{Div}\, S(u,p) = f, \;\; \mathrm{div}\, u = g & \text{in } \Omega_+, \\ S(u,p)\mathbf{n}_+|_{\Gamma_+} = h|_{\Gamma_+}. \end{cases} \tag{14}$$

$y = \Phi(x)$, $u(y) = {}^T\mathcal{A}v(x)$, $p(y) = \theta(x) \;\; \Longrightarrow$

$$\begin{cases} \lambda v - \mathrm{Div}\, S(v,\theta) + \mathcal{F}(v) = f & \text{in } \mathbb{R}^N_+, \\ \mathrm{div}\, v + \mathcal{G}(v) = g & \text{in } \mathbb{R}^N_+, \\ S(v,\theta)\mathbf{n}_0 + \mathcal{H}(v) = h & \text{on } \partial\mathbb{R}^N_+. \end{cases} \tag{15}$$

Continuation of the bent half-space case

$\mathcal{F}(v) = \mathcal{F}_1(\lambda v) + \mathcal{F}_2(\nabla^2 v) + \mathcal{F}_3(\nabla v)$, $\mathcal{G}(v) = \mathcal{G}_1(\nabla v) = \mathrm{div}\,(\mathcal{G}_2 v)$, $\mathcal{H}(v) = \mathcal{H}_1(\nabla v)$,
$\|\mathcal{F}_i\|_{L_\infty(\mathbb{R}^N_+)} \leq C_N M_1$, $\|\mathcal{G}_i\|_{L_\infty(\mathbb{R}^N_+)} \leq C_N M_1$ $(i = 1, 2)$, $\|\mathcal{H}_1\|_{L_\infty(\mathbb{R}^N_+)} \leq C_N M_1$,
$\|\mathcal{F}_3\|_{L_\infty(\mathbb{R}^N_+)} \leq C_N M_2$, $\|\nabla\mathcal{F}_i\|_{L_\infty(\mathbb{R}^N_+)} \leq C_N M_2 (i = 1, 2)$, $\|\nabla\mathcal{G}_1\|_{L_\infty(\mathbb{R}^N_+)} \leq C_N M_2$,
$\|\nabla\mathcal{H}_1\|_{L_\infty(\mathbb{R}^N_+)} \leq C_N M_2$.

$v = V(g) + \mathcal{T}(\lambda)F_\lambda(f, g, h) \implies$

$$\begin{cases} \lambda v - \mathrm{Div}\, S(v, \theta) + \mathcal{F}(v) = f + \mathcal{V}_{\mathcal{F}}(\lambda)(f, g, h) & \text{in } \mathbb{R}^N_+, \\ \mathrm{div}\, v + \mathcal{G}(v) = g + \mathcal{V}_{\mathcal{G}}(\lambda)(f, g, h) & \text{in } \mathbb{R}^N_+, \\ S(v, \theta)\mathbf{n}_0 + \mathcal{H}(v) = h + \mathcal{V}_{\mathcal{H}}(\lambda)(f, g, h) & \text{on } \partial\mathbb{R}^N_+ \end{cases} \tag{16}$$

with

$$\begin{aligned} \mathcal{V}_{\mathcal{F}}(\lambda)(f, g, h) &= \mathcal{F}_1(\lambda V(g)) + \mathcal{F}_2(\nabla^2 V(g)) + \mathcal{F}_3(\nabla V(g)) \\ &\quad + \mathcal{F}_1(\lambda\mathcal{T}(\lambda)F_\lambda(f, g, h)) + \mathcal{F}_2(\nabla^2\mathcal{T}(\lambda)F_\lambda(f, g, h)) + \mathcal{F}_3(\nabla\mathcal{T}(\lambda)F_\lambda(f, g, h)) \\ \mathcal{F}_{\mathcal{G}}(\lambda)(v) &= \mathcal{G}_1(\nabla V(g)) + \mathcal{G}_1(\nabla\mathcal{T}(\lambda)F_\lambda(f, g, h)) \\ &= \mathrm{div}\,(\mathcal{G}_2(V(g) + \mathcal{T}(\lambda)F_\lambda(f, g, h))), \\ \mathcal{V}_{\mathcal{H}}(\lambda)(f, g, h) &= \mathcal{H}_1(\nabla\mathcal{T}(\lambda)F_\lambda(f, g, h)), \end{aligned}$$

Continuation of the bent half-space case

- For any $\varphi \in W^1_{q,0}(\mathbb{R}^N_+)$, $(\mathcal{V}_{\mathcal{G}}(\lambda(f,g,h),\varphi) = -(\mathcal{G}_2(V(g) + \mathcal{T}(\lambda)F_\lambda(f,g,h), \nabla\varphi)$ $\Longrightarrow$ $\mathcal{V}_{\mathcal{G}}(\lambda)(f,g,h) \in W^1_q(\mathbb{R}^N_+) \cap W^{-1}_q(\mathbb{R}^N_+)$ and

  $$|\lambda|\|\mathcal{V}_{\mathcal{G}}(\lambda)(f,g,h)\|_{\hat{W}^{-1}_q(\mathbb{R}^N_+)} \leq C_N M_1(|\lambda|\|\mathcal{G}(g)\|_{L_q(\mathbb{R}^N_+)} + |\lambda|\|\mathcal{T}(\lambda)F_\lambda(f,g,h)\|_{L_q(\mathbb{R}^N_+)}).$$

- Set $\mathcal{V}(\lambda)(f,g,h) = (\mathcal{V}_{\mathcal{F}}(\lambda), \mathcal{V}_{\mathcal{G}}(\lambda), \mathcal{V}_{\mathcal{H}}(\lambda))(f,g,h)$. Then, $F_\lambda\mathcal{V}(\lambda)(f,g,h) = \mathcal{W}(\lambda)F_\lambda(f,g,h)$ with

  $$\mathcal{R}_{\mathcal{L}(X_q(\Omega))}(\{(\lambda\frac{d}{d\lambda}\mathcal{W}(\lambda) \mid \lambda \in \Sigma_{\epsilon,\lambda_1}\}) \leq C_N(M_1 + \lambda^{-1/2}M_2) \quad (\ell = 0,1).$$

- Since $\|F_\lambda(f,g,h)\|_{X_q(\mathbb{R}^N_+)}$ gives an equivalent norm to $\|(f,g,h)\|_{Y_q(\mathbb{R}^N_+)}$ and since $\|F_\lambda\mathcal{V}(\lambda)(f,g,h)\|_{X_q(\Omega)} = \|\mathcal{W}(\lambda)F_\lambda(f,g,h)\|_{X_q(\Omega)} \leq (1/2)\|F_\lambda(f,g,h)\|_{X_q(\mathbb{R}^N_+)}$ with small M_1 and large λ_1.
- ${}^\exists(I + \mathcal{V}(\lambda))^{-1} \in \mathcal{L}(Y_q(\Omega))$
- $u = V((I + \mathcal{V}(\lambda))^{-1}(f,g,h)) + \mathcal{T}(\lambda)F_\lambda(I + \mathcal{V}(\lambda))^{-1}(f,g,h)$ solves (16).
- $G_\lambda u = \tilde{V}(\lambda)F_\lambda(I + \mathcal{V}(\lambda))^{-1} + G_\lambda\mathcal{T}(\lambda)F_\lambda(I + \mathcal{V}(\lambda))^{-1}(f,g,h)$.
- $F_\lambda(I + \mathcal{V}(\lambda))^{-1}F_\lambda(I + \sum_{-1})^j\mathcal{V}(\lambda)^1) = (I + \sum_{j=1}^\infty(-1)^j\mathcal{W}(\lambda)^j)F_\lambda = (I + \mathcal{W}(\lambda))^{-1}F_\lambda$.

Theorem in the bent half-space

Let $1 < q < \infty$ *and* $0 < \epsilon < \pi/2$. *Then, there exist* $M_1 \in (0, 1)$ *and* $\lambda_1 \geq 1$ $V \in \mathcal{L}(W^1_q(\Omega_+) \cap W^{-1}_q(\Omega_+), W^2_q(\mathbb{R}^N_+)^N)$ *and* $\mathcal{T}(\lambda) \in \mathcal{L}(X_q(\Omega_+), W^2_q(\Omega_+)^N)$ $(\lambda \in \Sigma_{\epsilon,\lambda_1})$ *s.t.*

(i) *For any* $\lambda \in \Sigma_\epsilon$ *and* $(f, g, h) \in Y_q(\Omega_+)$, $u = V(g) + \mathcal{T}(\lambda)F_\lambda(f, g, h)$ *is a unique solution to* (15), *where* $F_\lambda(f, g, h) = (f, \nabla g, \lambda^{1/2}g, \lambda\mathcal{G}(g), \nabla h, \lambda^{1/2}h) \in X_q(\Omega_+)$.

(ii) *There exists a* $U(\lambda) \in \mathcal{L}(X_q(\Omega_+), L_q(\Omega_+)^{\tilde{N}})$ $(\lambda \in \Sigma_{\epsilon,\lambda_1})$ *s.t.*

- $G_\lambda u = U(\lambda)F_\lambda(f, g, h)$
- $\mathcal{R}_{\mathcal{L}(X_q(\Omega_+), L_q(\Omega_+)^{\tilde{N}})}(\{(\lambda \frac{d}{d\lambda})^\ell U(\lambda) \mid \lambda \in \Sigma_{\epsilon,\lambda_1}\}) \leq {}^\exists\gamma_0 \;\; (\ell = 0, 1)$.

Properties of uniform $C^{1,1}$ domains

For any $M_1 \in (0,1)$, $\exists$ countably numbers $\Phi_j \in C^{1,1}(\mathbb{R}^N)^N$, $\Psi_j \in C^{1,1}(\mathbb{R}^N)^N$ s.t.

- The maps: $x \to \Phi_j^1(x)$ and $x \to \Phi_j^2(x)$ are bijective.
- $0 < \exists d^0, d^1, d^2 \leq 1$ and $x_j^0 \in \Omega$, $x_j^1 \in \Gamma$ and $x_j^2 \in \Gamma_0$ s.t.

$$\Omega = \Big(\bigcup_{j=1}^\infty B_{d^0}(x_j^0)\Big) \cup \Big(\bigcup_{j=1}^\infty \Phi_j^1(\mathbb{R}^N_+) \cap B_{d^1}(x_j^1)\Big) \cup \Big(\bigcup_{j=1}^\infty \Phi_j^2(\mathbb{R}^N_+) \cap B_{d^2}(x_j^2)\Big),$$

$$B_{d^0}(x_j^0) \subset \Omega, \quad \Phi_j^i(\mathbb{R}^N_+) \cap B_{d^i}(x_j^i) = \Omega \cap B_{d^i}(x_j^i) \quad (i = 1,2, \quad j = 1,2,3,\ldots).$$

- $\nabla \Phi_j = \mathcal{T}_j + T_j, \quad \nabla(\Phi_j)^{-1} = \mathcal{T}_{-1,j} + T_{-1,j},$
  $\nabla \Psi_j = \mathcal{S}_j + S_j, \quad \nabla(\Psi_j)^{-1} = \mathcal{S}_{-1,j} + S_{-1,j},$
  $\mathcal{T}_j$, $\mathcal{T}_{-1,j}$, $\mathcal{S}_j$, $\mathcal{S}_{-1,j}$ are constant orthonomal matrices,

$$\|T_j, T_{-1,j}, S_j, S_{-1,j}\|_{L_\infty} \leq M_1 << 1,$$
$$\|\nabla T_j, \nabla T_{-1,j}\|_{L_\infty}, \|\nabla S_j, \nabla S_{-1,j}\|_{L_\infty} \leq M_2.$$

- There exists an positive integer L such that any $L+1$ distinct sets of the covering sets $\{B_{d^i}(x_j^i) \mid i = 0,1,2, \quad j = 1,2,3,\ldots\}$ have an empty intersection.

General Domain Case

- Let ζ_j^i and $\tilde\zeta_j^i \in C_0^\infty(B_j^i)$ with$B_j^i = B_{d^i}(x_j^i)$ such that

$$\sum_{i=0}^2 \sum_{j=1}^\infty \zeta_j^i = 1 \text{ on } \overline{\Omega}, \quad \tilde\zeta_j^i = 1 \text{ on } \operatorname{supp} \zeta_j^i,$$

$$\|\zeta_j^i\|_{W^2_\infty(\mathbb{R}^N)}, \quad \|\tilde\zeta_j^i\|_{W^2_\infty(\mathbb{R}^N)} \leq {}^\exists c_0 \quad (i = 0, 1, 2, \quad j = 1, 2, 3, \ldots,).$$

- Let u_j^i satisfy the equations:

$$\begin{aligned}
&\lambda u_j^0 - \mathrm{Div}\, S(u_j^0, K^0(u_j^0)) = \tilde\zeta_j^0 f && \text{in } \mathcal{H}_j^0 = \mathbb{R}^N,\\
&\lambda u_j^1 - \mathrm{Div}\, S(u_j^1, K_j^1(u_j^1)) = \tilde\zeta_j^1 f && \text{in } \mathcal{H}_j^1 = \Phi_j^1(\mathbb{R}^N_+),\\
&\mu \mathcal{T} D((u_j^1)\mathbf{n}_j = \tilde\zeta_j^1 \tilde h, \ \ \mathrm{div}\, u_j^1 = \tilde\zeta_j^1 h_N && \text{on } \partial\mathcal{H}_j^1,\\
&\lambda u_j^2 - \mathrm{Div}\, S(u_j^2, K_j^2(u_j^2)) = \tilde\zeta_j^2 f && \text{in } \mathcal{H}_j^2 = \Phi_j^2(\mathbb{R}^N_+),\\
&u_j^2 = 0 && \text{on } \partial\mathcal{H}_j^2.\\
&(\nabla K^0(u), \nabla\varphi)_{\mathbb{R}^N} = (\mu \mathrm{Div}\, D(u) - \nabla \mathrm{div}\, u, \nabla\varphi)_{\mathbb{R}^N} && {}^\forall\varphi \in \hat W_q^1(\mathbb{R}^N);\\
&(\nabla K_j^1(u), \nabla\varphi)_{\mathcal{H}_j^1} = (\mu \mathrm{Div}\, D(u) - \nabla \mathrm{div}\, u, \nabla\varphi)_{\mathcal{H}_j^1} && {}^\forall\varphi \in \hat W_{q,0}^1(\mathcal{H}_j^1),\\
&K_j^1(u) = \mu < D(u)\mathbf{n}_j, \mathbf{n}_j > -\mathrm{div}\, u \ \text{ on } \partial\mathcal{H}_j^1;\\
&(\nabla K_j^2(u), \nabla\varphi)_{\mathcal{H}_j^2} = (\mu \mathrm{Div}\, D(u) - \nabla \mathrm{div}\, u, \nabla\varphi)_{\mathcal{H}_j^2} && {}^\forall\varphi \in \hat W_q^1(\mathcal{H}_j^2).
\end{aligned}$$

General Domain Case

- $u - \sum_{i=0}^{2}\sum_{j=1}^{\infty} u_j^i$

$$\begin{cases} \lambda u - \mathrm{Div}\, S(u, K(u)) = f + \mathcal{R}_{\mathcal{F}}(f,h) & \text{in } \Omega, \\ \mu \mathcal{T} D(u)\mathbf{n} = \tilde h + \tilde{\mathcal{R}}_{\mathcal{H}}(f,h), \ \mathrm{div}\, u = h_N + \mathcal{R}_{\mathcal{H}N}(f,h) & \text{on } \Gamma, \quad u|_{\Gamma_0} = 0. \end{cases}$$

$$\mathcal{R}_{\mathcal{F}}(f,h) = \sum_{i=0}^{2}\sum_{j=1}^{\infty}\{ u(\mathrm{Div}\, D(\zeta_j^i u_j^i) - \zeta_j^i \mathrm{Div}\, D(u_j^i)) - \nabla K(\zeta_j^i u_j^i) - \zeta_j^i K_j^i(u\ i_j))\},$$

$$\tilde{\mathcal{R}}_{\mathcal{H}}(f,h) = -\mu \sum_{j=1}^{\infty} \mathcal{T} D(\zeta_j^1 u_j^1)\mathbf{n}_j - \zeta_j^1 \mathcal{T} D(\zeta_j^1 u_j^1)\mathbf{n}_j),$$

$$\mathcal{R}_{\mathcal{H}N}(f,h) = \sum_{j=1}^{\infty}(\nabla \zeta_j^1)\cdot u_j^1 \tag{17}$$

- The main task in how to treat the error term, especially non-local terms. To give you a rough idea, let me consider v_j^1 satisfying:

$$\begin{cases} \lambda v_j^1 - \mathrm{Div}\, S(v_j^1, K_j^1(v_j^1)) = 0 & \text{in } \mathcal{H}_j^1, \\ \mu \mathcal{T} D(v_j^1)\mathbf{n}_j = 0, \quad \mathrm{div}\, v_j^1 = \tilde\zeta_j^1 h_N \text{ on } \partial\mathcal{H}_j^1 \end{cases}$$

General Domain Case

- How to estimate: $(\nabla\zeta_j^1)I_j^1$ with $I_j^1 = K_j^1(u_j^1) - (\mu < D(u_j^1)\mathbf{n}_j, \mathbf{n}_j > -\mathrm{div}\, u_j^1)$.
- For any $\varphi \in L_{q'}(\Omega)$, let $\Phi_j \in \hat W^2_{q',0}(\mathcal{H}_j^1)$ such that

$$(\nabla\Phi_j, \nabla\psi)_{\mathcal{H}_j^1} = ((\nabla\zeta_j^1)\varphi, \psi)_{\mathcal{H}_j^1} \quad {}^\forall\psi \in \hat W^1_q(\mathcal{H}_j^1),$$
$$\|\nabla\Phi_j\|_{W^1_q(\mathcal{H}_j^1)} \leq \delta_1\|(\nabla\zeta_j^1)\varphi\|_{L_{q'}(\mathcal{H}_j^1)}$$

with some δ_1 independent of $j = 1, 2, 3, \ldots$. Especially, $\Delta\Phi_j = -(\nabla\zeta_j^1)\varphi$ in $\mathcal{H}_j^1$ and $\Phi_j^1|_{\partial\mathcal{H}_j^1} = 0$.

- Since $I_j^1 = K_j^1(v_j^1) - (\mu < D(v_j^1)\mathbf{n}_j, \mathbf{n}_j > -\mathrm{div}\, v_j^1) \in \hat W^1_{q,0}(\mathcal{H}_j^1)$, we have

$$(I_j^1, (\nabla\zeta_j^1)\varphi)_{\mathcal{H}_j^1} = (\nabla I_j^1, \nabla\Phi_j^1)_{\mathcal{H}_j^1} = \mu(\mathrm{Div}\, D(v_j^1) - \nabla < D(v_j^1)\mathbf{n}_j, \mathbf{n}_j >, \nabla\Phi_j^1)_{\mathcal{H}_j^1}$$
$$= -\mu(D(v_j^1), \nabla^2\Phi_j)_{\mathcal{H}_j^1} + (< D(v_j^1)\mathbf{n}_j, \mathbf{n}_j >, \Delta\Phi_j)_{\mathcal{H}_j^1}$$
$$= -\mu(D(v_j^1), \nabla^2\Phi_j)_{\mathcal{H}_j^1} + (< D(v_j^1)\mathbf{n}_j, \mathbf{n}_j >, (\nabla\zeta_j^1)\varphi)_{\mathcal{H}_j^1}.$$

$$|\sum_{j=1}^\infty (I_j^1, \nabla\varphi)_\Omega| \leq \sum_{j=1}^\infty \|\nabla v_j^1\|_{\mathcal{H}_j^1}\|\varphi\|_{L_{q'}(\Omega\cap B_j^1)}$$
$$\leq (\sum_{j=1}^\infty \|\nabla v_j^1\|^q_{\mathcal{H}_j^1})^{1/q}(\sum_{j=1}^\infty \|\varphi\|^{q'}_{L_{q'}(\Omega\cap B_j^1)})^{1/q'} \leq C_{q,\Omega}(\sum_{j=1}^\infty \|\nabla v_j^1\|^q_{\mathcal{H}_j^1})^{1/q}\|\varphi\|_{L_{q'}(\Omega)}.$$

General Domain Case

- There exists a $\mathcal{I}(f,h) \in L_q(\Omega)$ such that $\mathcal{I}(f,h) = \sum_{j=1}^\infty I_j(f,h)$ and

$$|\mathcal{I}(f,h)\|_{L_q(\Omega)}^q \leq C_{q,\Omega}^q \sum_{j=1}^\infty \|\nabla v_j^1\|_{\mathcal{H}_j^1}^q.$$

- For each v_j^1, we use the local properties of v_j^1.
- Set $\mathcal{V}(\lambda)(f,g) = (\mathcal{R}_{\mathcal{F}}, \tilde{\mathcal{R}}_{\mathcal{H}}, \mathcal{R}_{\mathcal{H}N})(f,h)$. There exists a $\mathcal{W}(\lambda) \in \mathcal{L}(X_q(\Omega))$ such that $F_\lambda \mathcal{V}(\lambda)(f,h) = \mathcal{W}(\lambda) F_\lambda(f,h)$ and

$$\mathcal{R}_{\mathcal{L}(X_q(\Omega))}(\{(\lambda \frac{d}{d\lambda})^\ell \mathcal{W}(\lambda) \mid \lambda \in \Sigma_{\epsilon,\lambda_2}\}) \leq 1/2 \quad (\ell = 0, 1)$$

provided we choose $\lambda_2 >> 1$ large enough. Employing the same argument as in the bent-half space case, we can show the existence theorem.