Existence of Global Solutions for Unsteady Isentropic Gas Flow in a Laval Nozzle

Properties of Steady Solutions and Existence of Time Global Solutions

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A Girl is Blowing out Candles.

- The girl puckers up[1] her mouth to make her breath stronger.

Properties of Gas

By making the exit (her mouth) of a nozzle (her windpipe) narrow, we can make the gas (her breath) through the nozzle stronger.

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Jet Engine of Plane

- The exhaust port[1] of the jet engine is convergent[2] to make the exhaust gas[3] stronger.
- 

[1] 排気口
[2] 狭まっている
[3] 排気ガス

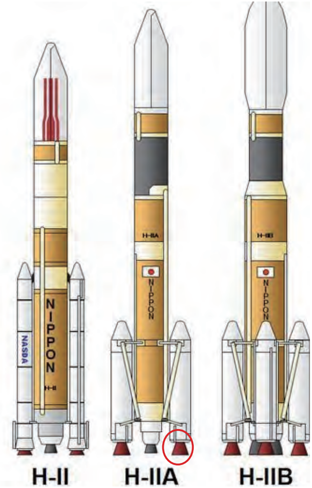
Jet Engine of Plane

- The exhaust port[1] of the jet engine is convergent[2] to make the exhaust gas[3] stronger.
- Is the exhaust port of every engine convergent?

[1]排気口
[2]狭まっている
[3]排気ガス

Rocket

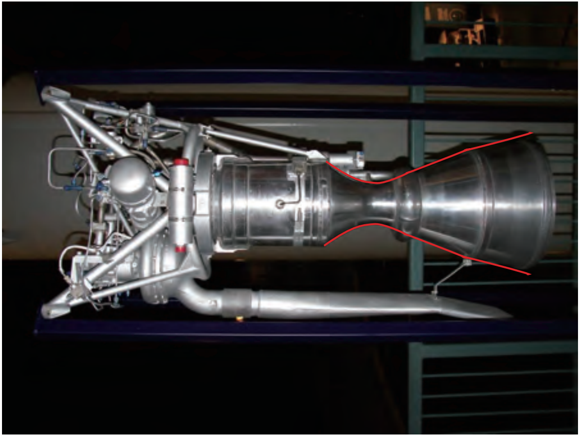
- The exhaust port of rocket engine is divergent[1].



[1] 広がっている

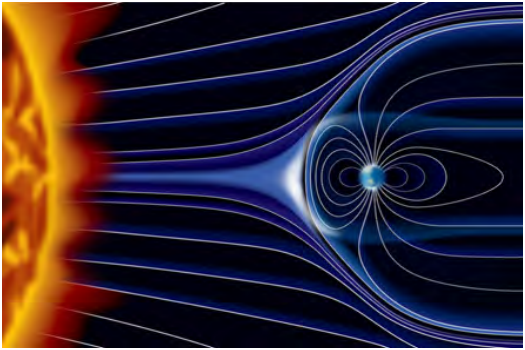
Rocket Engine

- Why does the rocket engine have convergent-divergent form?



Solar Wind

The solar wind is the stream of plasma ejected from the corona of the sun. It consists of electrons and positive ions. The solar wind collides with the magnetosphere[1] of the earth. As a result, influencing its magnetic field, the solar wind causes the outbreak of auroras and the electromagnetic interference[2].

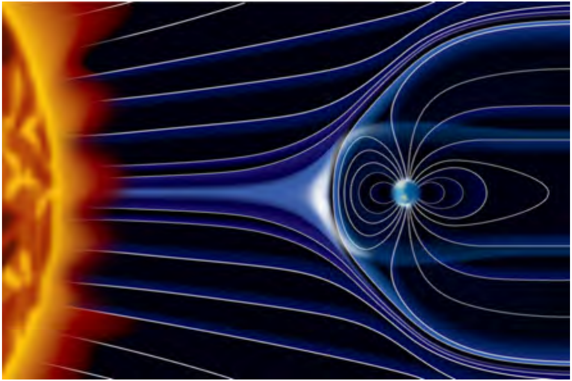


[1] 磁気圏
[2] 電波障害

Speed of Solar Wind near the Orbit of the Earth

The speed of the solar wind near the orbit of the earth is 300∼700 km/s.

Why is the speed of the solar wind so large?



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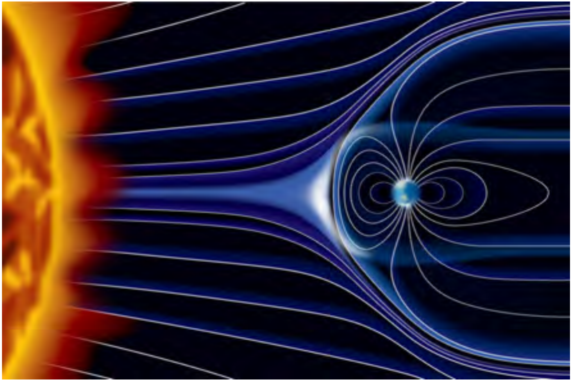

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Nozzle

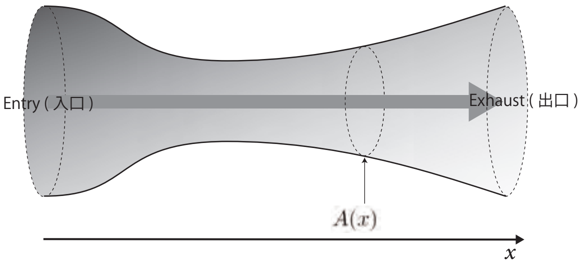


Figure: Nozzle

Remark

In this section, we consider the only gas which flows from left to right. Therefore, every velocity, momentum and Mach number are positive in this section. Moreover, we call the left and right of the nozzle the **entry** and **exhaust** section.

Isentropic Gas Flow through a Nozzle of (Slowly) Varying Cross Section $A(x)$

Compressible Euler Equations:The equations represent the motion of the inviscid and compressible gas through a nozzle.

$$\begin{cases} m' = -\dfrac{A'(x)}{A(x)}m, \\ \left(\dfrac{m^2}{\rho} + p(\rho)\right)' = -\dfrac{A'(x)}{A(x)}\dfrac{m^2}{\rho}, \end{cases} \qquad x \in \mathbf{R}, \quad ' = \frac{d}{dx}.$$

- ρ: the density of the gas
- v: the velocity of the gas
- $m = \rho v$: the momentum of the gas
- $\gamma \in (1, 5/3]$: the adiabatic exponent[1]
- $p(\rho) = \rho^\gamma/\gamma$: the pressure of the gas
- $A(x)$: the cross section[2] of the nozzle at x

[1] 比熱比

[2] 断面積

Outline

Mass Flow Conservation, the Bernoulli Equation

$$\begin{cases} m' = -\dfrac{A'(x)}{A(x)} m, \\ \left(\dfrac{m^2}{\rho} + p(\rho)\right)' = -\dfrac{A'(x)}{A(x)} \dfrac{m^2}{\rho}, \end{cases} \qquad x \in \mathbf{R}. \tag{1.1}$$

From (1.1), we obtain
Mass Flow Conservation

$$A(x)m = A(x)\rho v = C_1 \tag{1.2}$$

and

Bernoulli Equation:

$$\frac{v^2}{2} + \frac{\rho^{\gamma-1}}{\gamma - 1} = C_2. \tag{1.3}$$

Mach Number, Supersonic, Sonic, Subsonic, Thrust

- $c := \sqrt{p'(\rho)} = \rho^{\frac{\gamma-1}{2}}$:sonic speed (the speed of sound)
- $M := \dfrac{v}{c}$: Mach number
  - $M > 1$ supersonic$\Longrightarrow$The speed of gas is greater than that of sound.
  - $M = 1$ sonic$\Longrightarrow$The speed of gas is equal to that of sound.
  - $M < 1$ subsonic$\Longrightarrow$The speed of gas is lower than that of sound.
- Thrust[1] is the force that produced by an engine to push a plane and rocket forward.
  Thrust $T =$ the mass of the exhaust gas jetting per second $A(x)\rho v \times$the velocity of the exhaust gas v.

$$T = A(x)\rho v^2 \tag{1.4}$$

[1]推（進）力

Outline

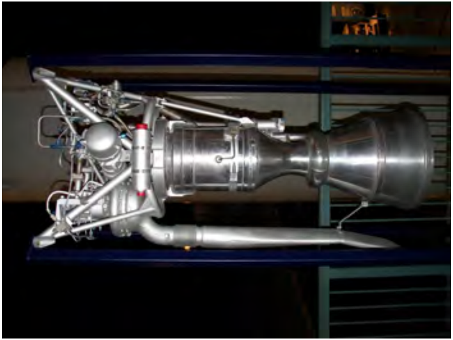
Gustaf de Laval

Figure: Gustaf de Laval (1845-1913)

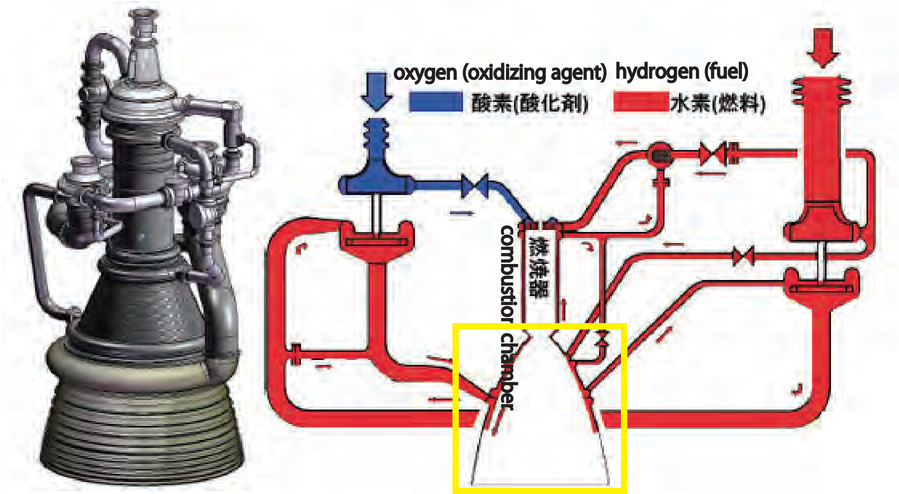
Shape of the Rocket Engine

Question 1

Why dose the exhaust port of the rocket engine have the convergent-divergent form?



Internal Constitution of the Rocket Engine



Laval Nozzle (Convergent-Divergent Nozzle)

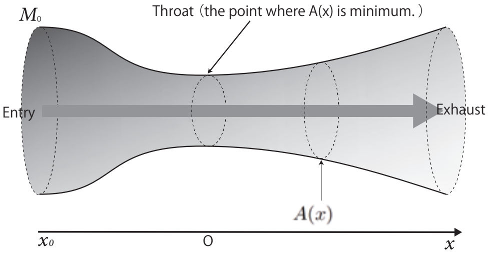


Figure: Laval Nozzle

We assume that $A(x)$ satisfies the following.

$$\begin{cases} A'(x) \leq 0, \ x < 0, \\ A'(x) \geq 0, \ x \geq 0, \\ A'(0) = 0. \end{cases} \tag{1.5}$$

Purpose

Purpose

If the Mach number of the exhaust gas at the entry section is lower than 1 (i.e. $M_0 < 1$), we want to make the thrust (of the exhaust gas at the exhaust section) T maximum.

From (1.2) and (1.3),

$$T = A(x)\rho v^2 = C_1\sqrt{(\gamma-1)C_2}\frac{1}{\sqrt{\frac{\gamma-1}{2}+\frac{1}{M^2}}} \tag{1.6}$$

$\Longrightarrow$ T is the increasing function of M.

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Properties of $F(M)$

From (1.1), we obtain the following

$$(A(x)F(M))' = 0 \Longrightarrow A(x)F(M) = A(x_0)F(M_0), \tag{1.7}$$

where

$$F(M) = M\left(\frac{\frac{\gamma+1}{2}}{1+\frac{\gamma-1}{2}M^2}\right)^{\frac{\gamma+1}{2(\gamma-1)}}.$$

Properties of $F(M)$

- $F(0) = 0,\ F(1) = 1.$
- $\begin{cases} F'(M) \geq 0, & 0 \leq M < 1, \\ F'(M) \leq 0, & M \geq 1. \end{cases}$
- $F(M) \to 0 \quad (M \to \infty).$

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Graph of $F(M)$

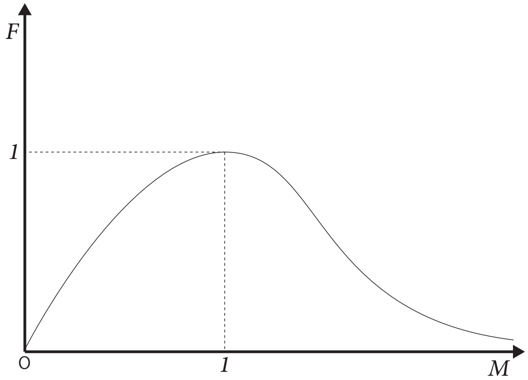
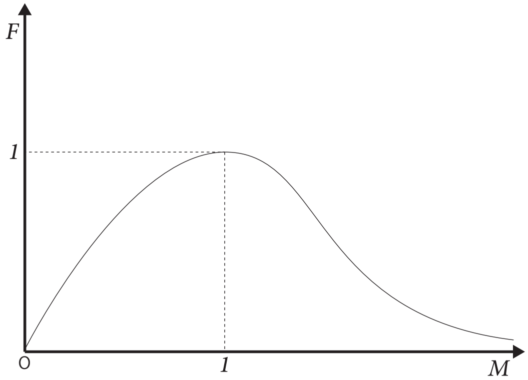


Figure: Graph of MF

Purpose

We investigate the increase and decrease of the Mach number M.

Graph of $F(M)$



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Value of M on $x_0 \le x \le 0$

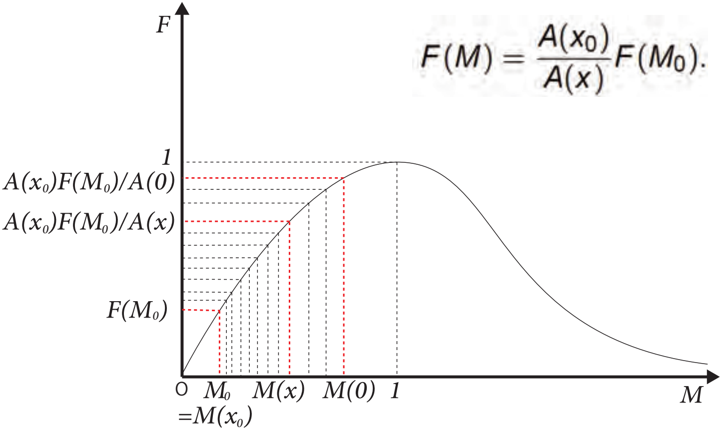

Figure: Graph of *MF*

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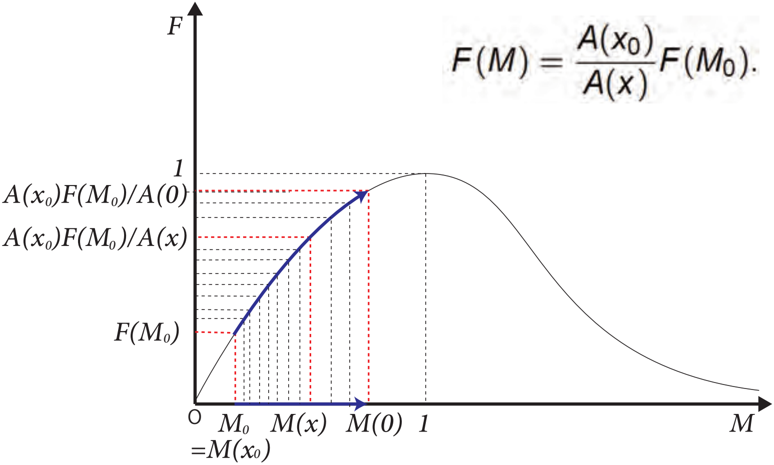

Figure: Graph of MF

Value of M on $x > 0$

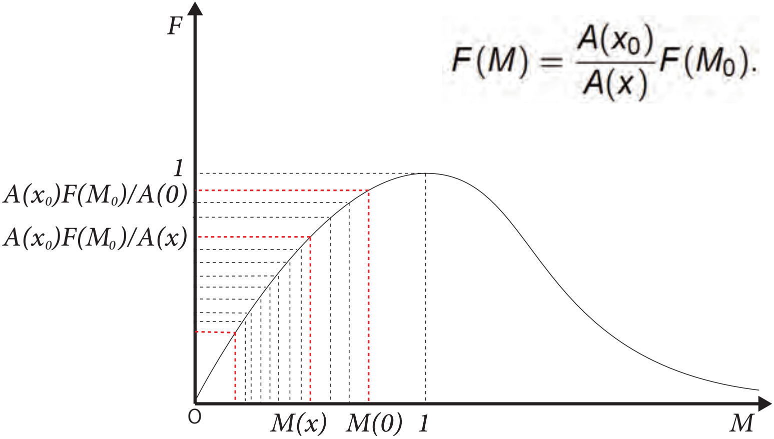

Figure: Graph of MF

Value of M on $x > 0$

$$F(M) = \frac{A(x_0)}{A(x)} F(M_0).$$

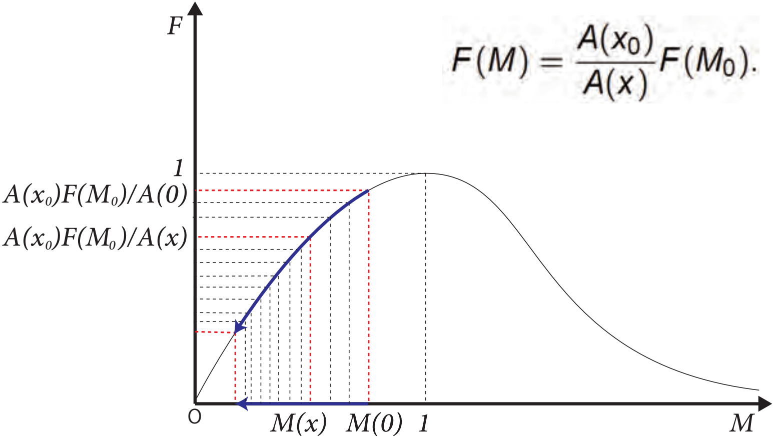

Figure: Graph of MF

Graph of xM

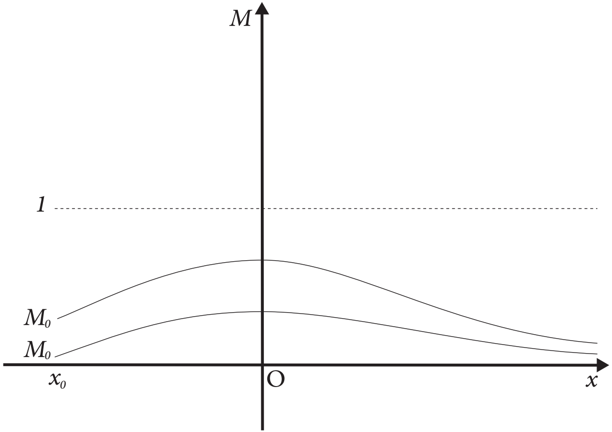

Figure: Graph of xM

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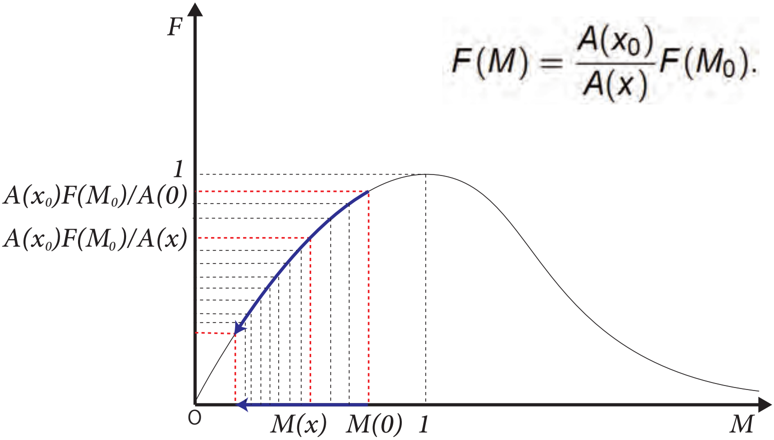

Figure: Graph of MF

Critical Value of M_0

A solution only exists, if the following holds

$$\frac{A(x_0)}{A(0)}F(M_0) \leq 1.$$

Varying M_0, we consider the case where the equal sign is valid.

$$\frac{A(x_0)}{A(0)}F(\hat{M}_0) = 1.$$

We denote M_0 in this case by $\hat{M}_0$. We call $\hat{M}_0$ the **critical value** of the Mach number.

Graph of $F(M)$ on $x_0 \le x \le 0$ in the Case where M_0 is the Critical Value

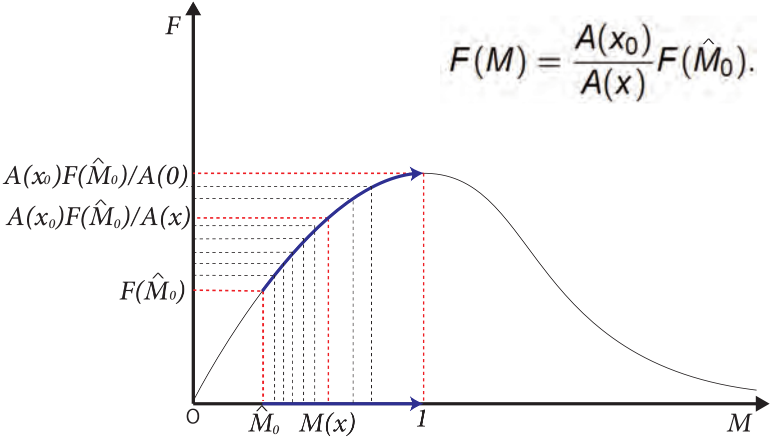

Figure: Graph of *MF*

Graph of $F(M)$ on $x > 0$ in the Case where M_0 is the Critical Value 1

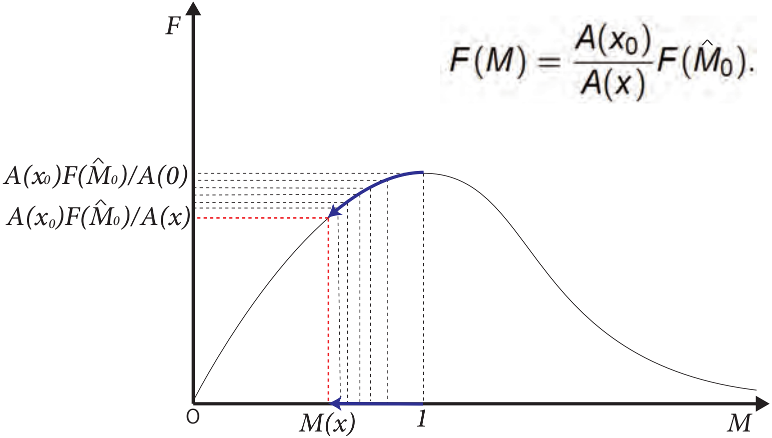

Figure: Graph of MF

Bifurcation 1 (Subsonic-Subsonic Flow)

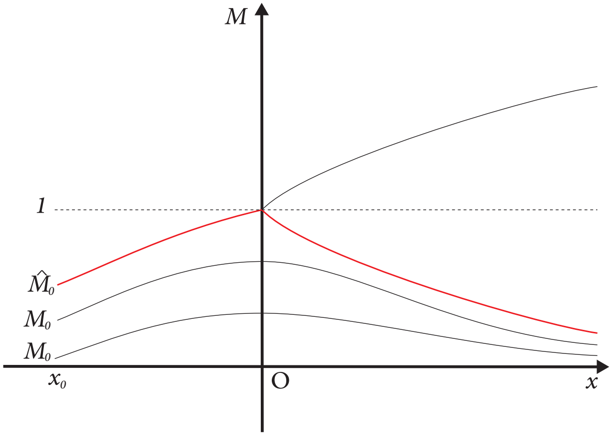

Figure: Graph of *xM*

Graph of $F(M)$ on $x > 0$ in the Case where M_0 is the Critical Value 2

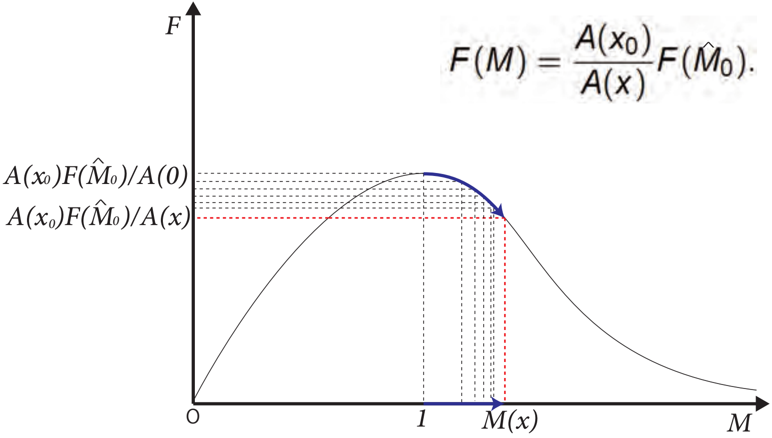

Figure: Graph of *MF*

Bifurcation 2 (Subsonic-Supersonic Flow)

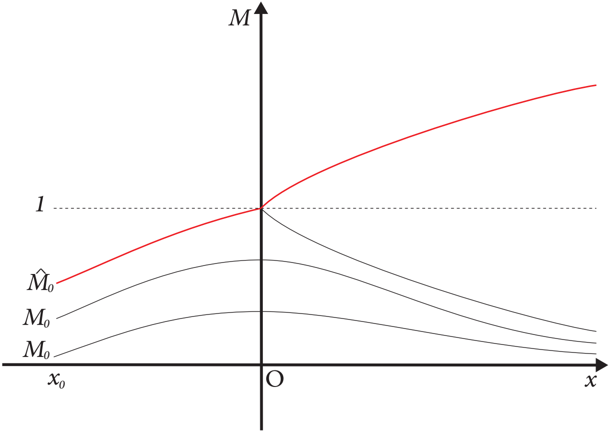

Figure: Graph of xM

Properties of Laval Nozzle

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By using the Laval nozzle, we can obtain the supersonic gas ($M > 1$) from the subsonic gas ($M < 1$). As a result, the gas has the maximal thrust at the exhaust section.

$\Longrightarrow$The Laval nozzle is essential for the rocket and the supersonic jet plane.

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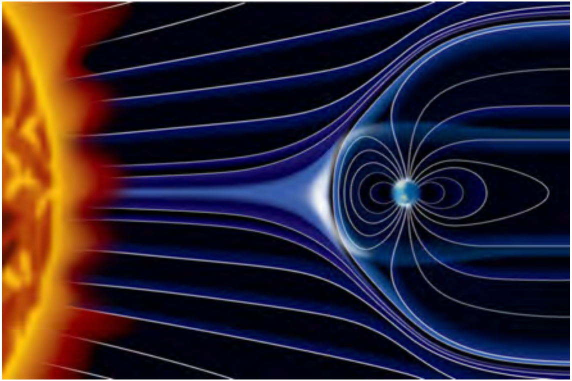
$\Longrightarrow$The Laval nozzle is essential for the rocket and the supersonic jet plane.

Outline

Solar Wind

Question 2

Why is the speed of the solar wind so large near the orbit of the earth?



Spherically Symmetric Flow around the Sun

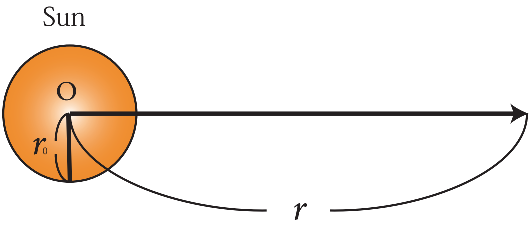


Figure: Spherical Symmetric Flow around the Sun

Spherical Symmetric Gas Flow with the Gravitational Source Term

We consider the isothermal case ($\gamma = 1$).

$$\begin{cases} m' = -\dfrac{2}{r}m, \\ \left(\dfrac{m^2}{\rho} + \rho\right)' = -\dfrac{2}{r}\dfrac{m^2}{\rho} - \dfrac{GM_s}{r^2}\rho, \quad r \geq r_0, \quad ' = \dfrac{d}{dr}. \end{cases} \tag{1.8}$$

- G: the gravitational constant
- M_s: the mass of the sun
- r: the distance from the center of the sun

Remark

From $p(\rho) = \rho$, since $c = 1$, we find $v = M$.

Comparison between Plasma and Laval Nozzle

From (1.8), we have the following.

$$(F_s(M)A_s(r))' = 0, \tag{1.9}$$

where

$$F_s(M) = Me^{-\frac{1}{2}M^2}, \quad A_s(r) = r^2 e^{\frac{GM_s}{r}}.$$

$F_s(M)$ and $A_s(r)$ satisfy the following.

$$\begin{cases} F_s'(M) \geq 0, \ 0 \leq M \leq 1, \\ F_s'(M) \leq 0, \ 1 < M, \\ F_s(0) = 0, \\ F_s(M) \to 0 \quad (M \to \infty), \end{cases} \qquad \begin{cases} A_s'(r) \leq 0, \ r_0 \leq r < GM_s/2, \\ A_s'(r) \geq 0, \ r \geq GM_s/2, \\ A_s'(GM_s/2) = 0. \end{cases} \tag{1.10}$$

Conclusion

(1.8) has the same structure as the Laval nozzle. Therefore, the motion of plasma is the same as that of the gas in the Laval nozzle. As a result, the plasma can be supersonic.

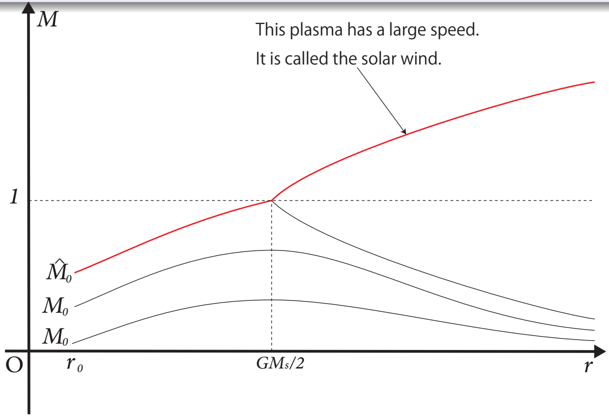


Figure: Graph of Solar Wind

Outline

Isentropic Gas Flow through a Nozzle of (Slowly) Varying Cross Section $A(x)$

$$\begin{cases} \rho_t + m_x = a(x)m, \\ m_t + \left(\dfrac{m^2}{\rho} + p(\rho)\right)_x = a(x)\dfrac{m^2}{\rho}, \end{cases} \quad x \in \mathbf{R}. \tag{2.1}$$

- ρ: the density of gas
- v: the velocity of gas
- $m = \rho v$: the momentum of gas
- $\gamma \in (1, 5/3]$: the adiabatic exponent
- $p(\rho) = \rho^\gamma/\gamma$: the pressure of gas
- $a(x) = -A'(x)/A(x)$, $\quad A(x)$: the cross section of the nozzle at x

For simplicity, we denote (2.1) by

$$u_t + f(u)_x = g(x, u), \quad u = \begin{pmatrix} \rho \\ m \end{pmatrix}. \tag{2.1}$$

Then, we consider the Cauchy problem (2.1) with initial data

$$u\big|_{t=0} = u_0(x) = (\rho_0(x), m_0(x)). \tag{2.2}$$

Riemann Invariants

We define Riemann invariants by

$$z = v - \frac{\rho^\theta}{\theta}, \quad w = v + \frac{\rho^\theta}{\theta}, \quad \theta = \frac{\gamma - 1}{2}. \tag{2.3}$$

Properties of Riemann Invariants

- If $v \geq 0$, $|w| \geq |z|$ and $w \geq 0$.
- If $v \leq 0$, $|w| \leq |z|$ and $z \leq 0$.
- z is bounded from below and w is bounded from above. $\Longleftrightarrow$ ρ and v are bounded.

Outline

Diagonalization, Characteristic Speed

One-dimensional Euler equations (i.e., $A'(x) = 0$):

$$\begin{cases} \rho_t + m_x = 0, \\ m_t + \left(\dfrac{m^2}{\rho} + p(\rho) \right)_x = 0, \end{cases} \qquad x \in \mathbf{R}. \tag{2.4}$$

If (2.4) has a smooth solution, we can diagonalize (2.4) into

$$\begin{cases} z_t + \lambda_1 z_x = 0, \\ w_t + \lambda_2 w_x = 0, \end{cases} \tag{2.5}$$

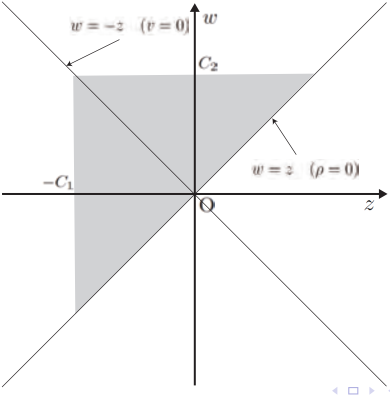
where λ_1 and λ_2 are characteristic speeds defined by

$$\lambda_1 = v - \rho^\theta, \quad \lambda_2 = v + \rho^\theta.$$

Existence of Invariant Regions

Theorem 1 (by Chueh, K. N., Conley, C. C., Smoller, J. A. in 1977)

For any fixed $C_1, C_2 > 0$, if initial data $u_0(x)$ are contained the region $\Delta = \{(\rho, v) \in \mathbf{R}^2; \rho \geq 0,\ z \geq -C_1,\ w \leq C_2\}$, solutions to the Cauchy problem of (2.4) are also contained in Δ.



Existence of Solutions for One-dimensional Case (2.4)

Theorem 2 (by (1) DiPerna, R. J. in 1983, (2) Ding, X., Chen, G.-Q., and Luo, P. in 1985)

If $\rho_0, v_0 \in L^\infty(\mathbf{R})$, the Cauchy problem of (2.4) has a weak solution.

Outline of the proof

1. We deduce from Theorem 1 the L^∞ estimate of approximate solutions (constructed by difference schemes and vanishing viscosity methods).
2. We deduce from the compactness of approximate solutions.
3. From compensated compactness, we derive the convergence of approximate solutions.

Remark 1

The invariant region for the nozzle flow (2.1) has not yet been obtained.

Outline

Relative Results (the Glimm Scheme)

1. Liu, T. P. (1979)
   Existence of time global solutions to $n \times n$ system of conservation laws with inhomogeneous terms including (2.1).: The Cauchy problem (2.1)–(2.2) has a time global solution, provided that
   - (C1) the total variation of initial data is small enough,
   - (C2) initial data are away from sonic states,
   - (C3) inhomogeneous terms are small enough in L^1 and L^∞.

Bifurcation 2(Subsonic-Supersonic Flow)

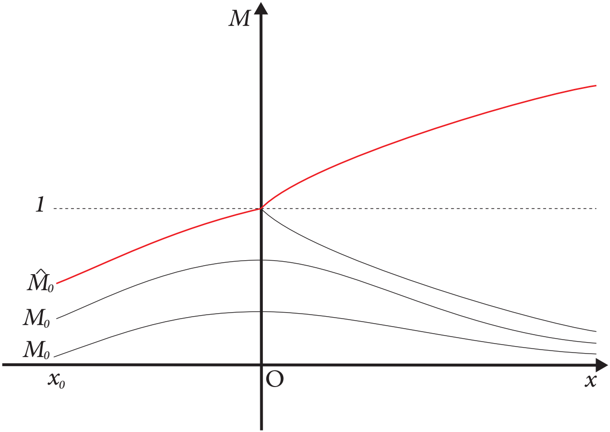


Figure: Graph of x and M

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   (C3) inhomogeneous terms are small enough in L^1 and L^∞.

Relative Results (Compensated Compactness)

5. Tsuge, N. (2006)
   Existence of time global solutions to an exterior problem with spherical symmetry (i.e., the initial-boundary value problem of (2.1) with $A(x) = x^2$ the boundary condition $m|_{x=1} = 0$ $x \geq 1$):
   For any fixed $C_1, C_2 > 0$, if initial data satisfy

$$\rho_0(x) \geq 0, \quad -C_1 x^{-\frac{2(\gamma-1)}{\gamma+1}} \leq z(u_0(x)), \quad w(u_0(x)) \leq C_2,$$

   the initial-boundary problem has a solution.
   method:the Godunov scheme
6. Lu, Y.-G. (2011)
   Existence of time global solutions to the Cauchy problem of (2.1) with the monotone cross section
   method: vanishing viscosity

Outline

Laval Nozzle

We hereafter consider the following Laval nozzle
$(A'(x) \leq 0,\ x \leq 0,\ A'(x) \geq 0,\ x \geq 0)$.

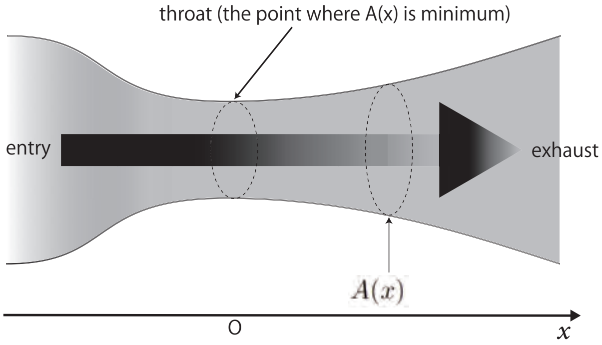

Figure: Laval nozzle

Main Theorem

Theorem 3 (published in ARMA)

For any fixed $M > 0$, if initial data satisfy

$$\begin{aligned} & -M\{A_{-}(x)\}^{-\frac{\gamma-1}{\gamma+1}} \leq z(u_0(x)), \quad w(u_0(x)) \leq M\{A_{+}(x)\}^{-\frac{\gamma-1}{\gamma+1}}, \\ & 0 \leq \rho_0(x), \end{aligned} \tag{2.6}$$

the Cauchy problem (2.1)–(2.2) has a weak entropy solution, where

$$A_{-}(x) = \begin{cases} A(x), & x \geq 0, \\ A(0), & x < 0, \end{cases} \qquad A_{+}(x) = \begin{cases} A(0), & x \geq 0, \\ A(x), & x < 0. \end{cases}$$

Remark for Theorem 3

1. If $A(x)$ is uniformly bounded, the condition (2.6) implies that we can give arbitrary L^∞ data, $\rho_0, v_0 \in L^\infty(\mathbf{R})$.
2. Our theorem contains sonic state, subsonic-supersonic flow.
3. Solutions $u(x,t)$ of Theorem 3 satisfy

$$-M\{A_-(x)\}^{-\frac{\gamma-1}{\gamma+1}} \le z(u(x,t)), \quad w(u(x,t)) \le M\{A_+(x)\}^{-\frac{\gamma-1}{\gamma+1}},$$
$$0 \le \rho(x,t).$$

That is, the region

$$\Delta_x = \{(z,w); \rho \ge 0,\ -M\{A_-(x)\}^{-\frac{\gamma-1}{\gamma+1}} \le z,\ w \le M\{A_+(x)\}^{-\frac{\gamma-1}{\gamma+1}}\}$$

is the invariant region for the Cauchy problem (2.1)–(2.2).

Invariant Region Δ_x

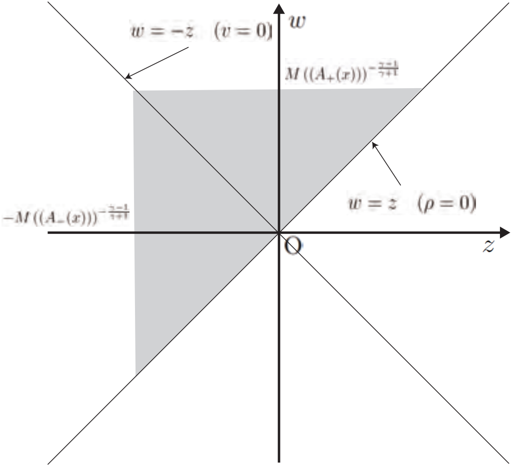

Figure: Invariant Region Δ_x

What Kinds of Nozzles Can We Apply the Method of Theorem 3 to?

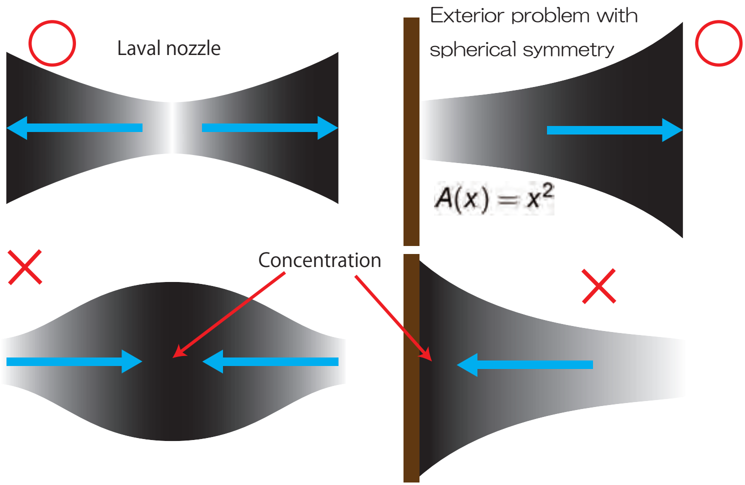

Figure: Four kinds of nozzles

Outline

Diagonalization of (2.1)

If (2.1) has a smooth solution, we can diagonalize (2.1) into

$$\begin{aligned} z_t + \lambda_1 z_x &= -a(x)\rho^\theta v, \\ w_t + \lambda_2 w_x &= a(x)\rho^\theta v. \end{aligned} \tag{2.7}$$

For symmetry, we consider the only case where $x \geq 0$. Then, we notice that $a(x) \leq 0$.

Effect of inhomogeneous term of (2.7)

- If v is positive, the inhomogeneous term makes z increase and w decrease in t.
- If v is negative, the inhomogeneous term makes z increase and w decrease in t.

Riemann Invariants

We define Riemann invariants by

$$z = v - \frac{\rho^\theta}{\theta}, \quad w = v + \frac{\rho^\theta}{\theta}, \quad \theta = \frac{\gamma - 1}{2}. \tag{2.3}$$

Properties of Riemann Invariants

- z is bonded from below and w is bounded from above. $\Longleftrightarrow$ ρ and v are bounded.
- If $v \geq 0$, $|w| \geq |z|$ and $w \geq 0$.
- If $v \leq 0$, $|w| \leq |z|$ and $z \leq 0$.
- $v = \dfrac{w + z}{2}$.

Diagonalization of (2.1)

If (2.1) has a smooth solution, we can diagonalize (2.1) into

$$\begin{aligned} z_t + \lambda_1 z_x &= -a(x)\rho^\theta v, \\ w_t + \lambda_2 w_x &= a(x)\rho^\theta v. \end{aligned} \tag{2.7}$$

For symmetry, we consider the only case where $x \geq 0$. Then, we notice that $a(x) \leq 0$.

Effect of inhomogeneous term of (2.7)

- If v is positive, inhomogeneous term makes z increase and w decrease in t.
- If v is negative, inhomogeneous term makes z increase and w decrease in t.

Most Difficult Point

The bounded estimate of z from below in the case where v is negative.

Transformation of (2.7)

We set $\tilde{z} = \{A(x)\}^{\frac{\gamma-1}{\gamma+1}} z$. From $(2.6)_1$, we find $-M \leq \tilde{z}_0(x)$.
Moreover, from $(2.7)_1$, we have

$$\tilde{z}_t + \lambda_1(u)\tilde{z}_x = -\frac{\gamma-1}{\gamma+1} a(x) \{A(x)\}^{\frac{\gamma-1}{\gamma+1}} \left((v)^2 + \frac{(\rho)^{2\theta}}{\theta} \right). \tag{2.8}$$

Remark 2

The inhomogeneous term of (2.8) is nonnegative.

Set $\tilde{w} = \{A(0)\}^{\frac{\gamma-1}{\gamma+1}} w$. From $(2.6)_2$, we find $\tilde{w}_0(x) \leq M$.
Moreover, from $(2.7)_2$, we have

$$\tilde{w}_t + \lambda_2 \tilde{w}_x = a(x) \{A(0)\}^{\frac{\gamma-1}{\gamma+1}} \rho^\theta v = a(x) \{A(0)\}^{\frac{\gamma-1}{\gamma+1}} \rho^\theta \frac{w+z}{2} \tag{2.9}$$

Outline of the Proof

We formally prove that Δ_x is an invariant region, that is, for any fixed $M > 0$, initial data satisfy

$$-M\{A_-(x)\}^{-\frac{\gamma-1}{\gamma+1}} \leq z_0(x), \quad w_0(x) \leq M\{A_+(x)\}^{-\frac{\gamma-1}{\gamma+1}},$$

solutions to the Cauchy problem (2.1)–(2.2)

$$-M\{A_-(x)\}^{-\frac{\gamma-1}{\gamma+1}} \leq z(x,t), \quad w(x,t) \leq M\{A_+(x)\}^{-\frac{\gamma-1}{\gamma+1}}, \tag{2.10}$$

where

$$A_-(x) = \begin{cases} A(x), & x \geq 0, \\ A(0), & x < 0, \end{cases} \qquad A_+(x) = \begin{cases} A(0), & x \geq 0, \\ A(x), & x < 0. \end{cases}$$

We set $\tilde{z} = \{A_-(x)\}^{\frac{\gamma-1}{\gamma+1}} z,\ \tilde{w} = \{A_+(x)\}^{\frac{\gamma-1}{\gamma+1}} w$. We prove the following. If

$$-M \leq \tilde{z}_0(x), \quad \tilde{w}_0(x) \leq M,$$

$$-M \leq \tilde{z}(x,t), \quad \tilde{w}(x,t) \leq M.$$

For any positive constant ε and T, we set

$$\check{z}(x,t) = \tilde{z}(x,t) + \frac{\varepsilon}{T-t}, \quad \check{w}(x,t) = \tilde{w}(x,t) - \frac{\varepsilon}{T-t}.$$

Then we find

$$-M < \check{z}_0(x), \quad \check{w}_0(x) < M.$$

Purpose

We assume $\check{z}(x,t)$ and $\check{w}(x,t)$ have compact supports on $0 \leq t < T$. Then, on $0 \leq t < T$, we shall prove

$$-M < \check{z}(x,t), \quad \check{w}(x,t) < M.$$

At a point $x = x_*, t = t_* \quad (t_* < T)$, if

$$-M = \check{z}(x_*,t_*), \quad -M < \check{z}(x,t), \quad \check{w}(x,t) < M, \quad 0 \leq t < t_* \quad (2.11)$$

or

$$\check{w}(x_*,t_*) = M, \quad -M < \check{z}(x,t), \quad \check{w}(x,t) < M, \quad 0 \leq t < t_* \quad (2.12)$$

hold, we shall deduce a contradiction.
We hereafter consider the only case where $x_* \geq 0$. We can similarly treat with the case where $x_* < 0$.

The Case where (2.11) Holds

Then, since $\check{z}$ attains the minimum at $x = x_*, t = t_*$, we find
$\check{z}_t \leq 0,\ \check{z}_x = 0.$
On the other hand, from

$$\tilde{z}_t + \lambda_1(u)\tilde{z}_x = -\frac{\gamma-1}{\gamma+1}a(x)\left\{A(x)\right\}^{\frac{\gamma-1}{\gamma+1}}\left((v)^2 + \frac{(\rho)^{2\theta}}{\theta}\right) \qquad (2.9)$$

and $\check{z}(x,t) = \tilde{z}(x,t) + \varepsilon/(T-t)$, at $x = x_*, t = t_*$ we have

$$\check{z}_t + \lambda_1(u)\check{z}_x = -\frac{\gamma-1}{\gamma+1}a(x)\left\{A(x)\right\}^{\frac{\gamma-1}{\gamma+1}}\left((v)^2 + \frac{(\rho)^{2\theta}}{\theta}\right) + \frac{\varepsilon}{(T-t)^2} > 0.$$

This is a contradiction.

The Case where (2.12) Holds

Then, $\check{w}$ attains the maximum at $x = x_*, t = t_*$, we find $\check{w}_t \geq 0,\ \check{w}_x = 0$. On the other hand, at $x = x_*, t = t_*$, we notice that

$$\begin{aligned} w + z &= \{A(0)\}^{-\frac{\gamma-1}{\gamma+1}}\, \tilde{w} + \{A(x_*)\}^{-\frac{\gamma-1}{\gamma+1}}\, \tilde{z} \\ &= \{A(0)\}^{-\frac{\gamma-1}{\gamma+1}} \left(\check{w} + \frac{\varepsilon}{T-t}\right) + \{A(x_*)\}^{-\frac{\gamma-1}{\gamma+1}} \left(\check{z} - \frac{\varepsilon}{T-t}\right) \\ &\geq \left(M + \frac{\varepsilon}{T-t}\right) \left[\{A(0)\}^{-\frac{\gamma-1}{\gamma+1}} - \{A(x_*)\}^{-\frac{\gamma-1}{\gamma+1}}\right] \geq 0. \end{aligned}$$

Then, from

$$\tilde{w}_t + \lambda_2 \tilde{w}_x = a(x)\left\{A(0)\right\}^{\frac{\gamma-1}{\gamma+1}} \rho^{\theta} \frac{w+z}{2} \tag{2.10}$$

and $\check{w}(x,t) = \tilde{w}(x,t) - \varepsilon/(T-t)$, we have

$$\check{w}_t + \lambda_2 \check{w}_x = a(x)\left\{A(0)\right\}^{\frac{\gamma-1}{\gamma+1}} \rho^{\theta} \frac{w+z}{2} - \frac{\varepsilon}{(T-t)^2} < 0.$$

This is a contradiction.

Since ε and T are arbitrary, we can prove (2.10).

Outline

In this section, we consider a general nozzle.
We choose positive constants M, σ and nonnegative function $b \in C^1(\mathbf{R})$ such that

$$\|b\|_{L^1} \ll M \ll 1. \tag{3.1}$$

Main Theorem 2

Theorem 4 (to appear in ARMA)

For M and b in (3.1), we assume that initial data $u_0 = (\rho_0, m_0) \in L^\infty(\mathbf{R})$ satisfy

$$
\begin{aligned}
&0 \le \rho_0(x), \quad -M - \int_x^\infty b(y)dy \le z(u_0(x)), \\
&w(u_0(x)) \le M + \int_{-\infty}^x b(y)dy.
\end{aligned} \tag{3.2}
$$

Then the Cauchy problem (2.1)–(2.2) has a weak entropy solution.

Remark 3

The solution of Theorem 4 satisfy the same inequality as (3.2).

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