Fourier analysis methods, nonstandard maximal regularity and applications to fluid mechanics

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Talk 1. Basic Fourier analysis and maximal regularity

Talk 2. Some applications to fluid mechanics

Talk 3. Partially parabolic or dissipative PDEs

Talk 4. Applications to the global existence issue and incompressible limit of the compressible Navier-Stokes equations

Fundamental fact: for spectrally localized functions over $\mathbb{R}^d$, derivatives act almost as homotheties. In effect:

$$\mathcal{F}(\nabla u)(\xi) = i\xi\mathcal{F}(u).$$

Hence, Parseval equality implies that if $\operatorname{Supp}\mathcal{F}(u) \subset \{\xi \in \mathbb{R}^d : r\lambda \leq |\xi| \leq R\lambda\}$ then

$$\|\nabla u\|_{L^2} \approx \lambda \|u\|_{L^2}.$$

- Direct Bernstein inequality: Let $R>0$. A constant C exists so that, for any $k\in\mathbb{N}$, any couple (p,q) in $[1,\infty]^2$ with $q\geq p\geq 1$ and any function u of L^p with $\operatorname{Supp}\widehat{u}\subset B(0,\lambda R)$ for some $\lambda>0$, we have

$$\|\nabla^k u\|_{L^q}\leq C^{k+1}\lambda^{k+d(\frac{1}{p}-\frac{1}{q})}\|u\|_{L^p}.$$

- Reverse Bernstein inequality: Let $0<r<R$. There exists a constant C so that for any $k\in\mathbb{N}$, $p\in[1,\infty]$ and any function u of L^p with $\operatorname{Supp}\widehat{u}\subset\{\xi\in\mathbb{R}^d\,/\,r\lambda\leq|\xi|\leq R\lambda\}$, we have

$$\lambda^k\|u\|_{L^p}\leq C^{k+1}\|\nabla^k u\|_{L^p}.$$

Proposition

Assume $\operatorname{Supp}\widehat{u} \subset \{\xi \in \mathbb{R}^d : r\lambda \leq |\xi| \leq R\lambda\}$. *Then for any* $\sigma \in \mathbb{R}$, *there exists* c *and* C *so that for all* $p \in [1, +\infty]$,

$$\|e^{-t|D|^\sigma} u\|_{L^p} \leq C e^{-ct\lambda^\sigma} \|u\|_{L^p}.$$

If $p = 2$ then this is an obvious consequence of the localization of $\mathcal{F}u$ and of Parseval equality, since

$$\mathcal{F}\big(e^{-t|D|^\sigma} u\big)(\xi) = e^{-t|\xi|^\sigma} \mathcal{F}u(\xi) \quad \text{and} \quad |\xi| \sim \lambda \ \text{ on } \ \operatorname{Supp}\mathcal{F}u.$$

Proposition

Assume $\operatorname{Supp}\widehat{u} \subset \{\xi \in \mathbb{R}^d : r\lambda \leq |\xi| \leq R\lambda\}$. *Then for any* $\sigma \in \mathbb{R}$, *there exists* c *and* C *so that for all* $p \in [1, +\infty]$,

$$\|e^{-t|D|^\sigma} u\|_{L^p} \leq Ce^{-ct\lambda^\sigma} \|u\|_{L^p}.$$

The proof for general p stems from the following lemma:

Lemma

Let $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ *supported in (say)* $\{\xi \in \mathbb{R}^d : r/2 \leq |\xi| \leq 2R\}$. *There exist two positive constants* c *and* C *such that for any* $\lambda > 0$, *and* $p \in [1, \infty]$ *we have*

$$\|e^{-t|D|^\sigma} \phi(\lambda^{-1}D)\|_{\mathcal{L}(L^p;L^p)} \leq Ce^{-ct\lambda^\sigma}.$$

Proof:

1. Change of scale reduces the proof to the case $\lambda = 1$;
2. $\mathcal{F}(e^{-t|D|^\sigma}\phi(D)u)(\xi) = \big(e^{-t|\xi|^\sigma}\phi(\xi)\big)\mathcal{F}(u)(\xi)$.
3. Young inequality reduces the proof to $\|\mathcal{F}^{-1}\big(e^{-t|\xi|^\sigma}\phi(\xi)\big)\|_{L^1} \leq Ce^{-ct}$. This follows from standard computations : integration by parts, ...

Let χ be a bump function with $\operatorname{Supp}\chi \subset B(0, \frac{4}{3})$ and $\chi \equiv 1$ on $B(0, \frac{3}{4})$. We set $\varphi(\xi) = \chi(\xi/2) - \chi(\xi)$ so that:

$$\chi(\xi) + \sum_{j\in\mathbb{N}} \varphi(2^{-j}\xi) = 1 \quad \text{and} \quad \sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1 \ \text{ if } \ \xi \neq 0.$$

The homogeneous dyadic blocks $\dot\Delta_j$ are defined by

$$\dot\Delta_j := \varphi(2^{-j}D) \quad \text{for} \quad j \in \mathbb{Z}.$$

The homogeneous Littlewood-Paley decomposition for u reads

$$u = \sum_j \dot\Delta_j u. \tag{1}$$

That equality holds true in the set $\mathcal{S}'$ of tempered distributions *modulo polynomials only*. A way to overcome this is to restrict to the set $\mathcal{S}'_h$ of tempered distributions u such that

$$\lim_{j\to-\infty} \|\dot S_j u\|_{L^\infty} = 0 \quad \text{with} \quad \dot S_j := \chi(2^{-j}D).$$

Equality (1) holds true in $\mathcal{S}'$ whenever u is in $\mathcal{S}'_h$.

Littlewood Paley decomposition allows to characterize some classical norms or semi-norms such as:

- homogeneous Sobolev semi-norm: $\|u\|_{\dot H^s}^2 \approx \sum_j (2^{js}\|\dot\Delta_j u\|_{L^2})^2$;
- homogeneous Hölder semi-norm: $\|u\|_{\dot C^{0,r}} \approx \sup_j 2^{jr}\|\dot\Delta_j u\|_{L^\infty}$.

This motivates the following definition of homogeneous Besov spaces:

Definition

For $s\in\mathbb{R}$ and $1\leq p,r\leq\infty$, we set

$$\|u\|_{\dot B^s_{p,r}} := \left(\sum_j 2^{rjs}\|\dot\Delta_j u\|_{L^p}^r\right)^{\frac1r} \quad \textit{if} \quad r<\infty \quad \textit{and} \quad \|u\|_{\dot B^s_{p,\infty}} := \sup_j 2^{js}\|\dot\Delta_j u\|_{L^p}.$$

The homogeneous Besov space $\dot B^s_{p,r}$ is the subset of $u\in\mathcal{S}'_h$ s.t. $\|u\|_{\dot B^s_{p,r}}<\infty$.

Besov spaces are *independent* of the Littlewood-Paley decomposition $(\dot\Delta_j)_{j\in\mathbb{Z}}$. This is a consequence of the following fundamental lemma:

Lemma

Let $0<r<R$. *Let* $s\in\mathbb{R}$ *and* $1\leq p,r\leq\infty$. *Let* $(u_j)_{j\in\mathbb{Z}}$ *be such that* $u:=\sum_{j\in\mathbb{Z}}u_j$ *converges in* $\mathcal{S}'_h$ *and* $\mathrm{Supp}\,\widehat{u}_j\subset 2^j\mathcal{C}(0,r,R)$ *for all* $j\in\mathbb{Z}$. *Then*

$$\Big\|2^{js}\|u_j\|_{L^p(\mathbb{R}^d)}\Big\|_{\ell^r(\mathbb{Z})}<\infty \quad\Longrightarrow\quad u:=\sum_{j\in\mathbb{Z}}u_j \text{ is in } \dot B^s_{p,r}(\mathbb{R}^d)$$

and we have $\|u\|_{\dot B^s_{p,r}}\approx\Big\|2^{js}\|u_j\|_{L^p(\mathbb{R}^d)}\Big\|_{\ell^r(\mathbb{Z})}$.

If $s>0$ *then the result is still true under the weaker assumption that* $\mathrm{Supp}\,\widehat{u}_j\subset B(0,2^jR)$.

In other words, for spectrally localized series, proving that the sum is in a Besov space amounts to getting suitable bounds for the L^p norm of each term.

Comparison with Lebesgue spaces:

$$\dot{B}^0_{p,\min(p,2)} \hookrightarrow L^p \hookrightarrow \dot{B}^0_{p,\max(p,2)} \quad \text{for any} \;\; p \in (1,\infty).$$

We also have

$$\dot{B}^0_{p,1} \hookrightarrow L^p \hookrightarrow \dot{B}^0_{p,\infty} \quad \text{if} \;\; p = 1, \infty.$$

Having u in $\dot{B}^s_{p,r}$ means that u has s fractional derivatives in L^p:

Proposition (Characterization by finite differences)

For $s \in]0,1[$ and finite p, r, we have

$$\|u\|_{\dot{B}^s_{p,r}} \approx \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\left(\frac{|u(y)-u(x)|}{|y-x|^s}\right)^p \frac{dy}{|y-x|^d}\right)^{\frac{r}{p}} dx\right)^{\frac{1}{r}}.$$

Similar result holds for p or r infinite.

A few classical properties of Besov spaces:

- With our definition, the spaces $\dot B^s_{p,r}(\mathbb{R}^d)$ are complete if and only if $s<d/p$, or $s=d/p$ and $r=1$.
- The following real interpolation property is fulfilled for all $\theta\in(0,1)$:

$$[\dot B^{s_1}_{p,r_1},\dot B^{s_2}_{p,r_2}]_{(\theta,r)}=\dot B^{\theta s_2+(1-\theta)s_1}_{p,r}\quad\text{if}\quad 1\leq p,r_1,r_2,r\leq\infty\quad\text{and}\quad s_1\neq s_2.$$

- Functional embedding: If $s\in\mathbb{R}$, $1\leq p_1\leq p_2\leq\infty$, $1\leq r_1\leq r_2\leq\infty$ then $\dot B^s_{p_1,r_1}(\mathbb{R}^d)\hookrightarrow \dot B^{s-d(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,r_2}(\mathbb{R}^d)$;
- Fatou property: if $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence of functions of $\dot B^s_{p,r}$ with $u_n\rightharpoonup u$ in $\mathcal{S}'_h$ then

$$u\in\dot B^s_{p,r}\quad\text{and}\quad \|u\|_{\dot B^s_{p,r}}\leq C\liminf\|u_n\|_{\dot B^s_{p,r}}.$$

- Action of Fourier multipliers: For any smooth homogeneous of degree m function F on $\mathbb{R}^d\setminus\{0\}$ the operator $F(D)$ maps $\dot B^s_{p,r}$ in $\dot B^{s-m}_{p,r}$. In particular $\nabla:\dot B^s_{p,r}\longrightarrow\dot B^{s-1}_{p,r}$.

Consider the heat equation

$$\partial_t u - \Delta u = f, \qquad u_{|t=0} = u_0$$

or, more generally,

$$\partial_t v + |D|^\sigma v = g, \qquad v_{|t=0} = v_0.$$

We want to establish estimates of the form

$$\|\partial_t u, D^2 u\|_{L^1(X)} \leq C\big(\|u_0\|_X + \|f\|_{L^1(X)}\big) \tag{2}$$

$$\|\partial_t v, |D|^\sigma v\|_{L^1(X)} \leq C\big(\|v_0\|_X + \|g\|_{L^1(X)}\big). \tag{3}$$

It is well known that if $r \in (1,\infty)$ and $X = L^q$ or $\dot{W}^{s,q}$ for some $s \in \mathbb{R}$ and $q \in (1,\infty)$ then, for the heat equation with $u_0 \equiv 0$,

$$\|\partial_t u, D^2 u\|_{L^r(X)} \leq C\|f\|_{L^r(X)}.$$

However the inequality fails for the endpoint case $r = 1$ for those spaces X.

Theorem

Inequality (2) *holds true for any* $p \in [1,\infty]$ *and* $s \in \mathbb{R}$, *if* $X = \dot{B}^s_{p,1}$.

Proof for the heat equation (for simplicity):

We start with $\partial_t\dot\Delta_j u-\Delta\dot\Delta_j u=\dot\Delta_j f$ for any $j\in\mathbb{Z}$. Hence, according to Duhamel's formula

$$\dot\Delta_j u(t)=e^{t\Delta}\dot\Delta_j u_0+\int_0^t e^{(t-\tau)\Delta}\dot\Delta_j f(\tau)\,d\tau.$$

Therefore, taking the L^p norm of both sides, we get

$$\|\dot\Delta_j u(t)\|_{L^p}\leq\|e^{t\Delta}\dot\Delta_j u_0\|_{L^p}+\int_0^t\|e^{(t-\tau)\Delta}\dot\Delta_j f(\tau)\|_{L^p}\,d\tau. \tag{4}$$

According to Bernstein inequality for the heat semi-group, we have

$$\|e^{\lambda\Delta}\dot\Delta_j z\|_{L^p}\leq Ce^{-c\lambda 2^{2j}}\|\dot\Delta_j z\|_{L^p}.$$

Therefore, applying this inequality and taking the L^1 or L^∞ norm of both sides of (4) on $[0,t]$,

$$\|\dot\Delta_j u\|_{L^\infty_t(L^p)}+2^{2j}\|\dot\Delta_j u\|_{L^1_t(L^p)}\leq C\Big(\|\dot\Delta_j u_0\|_{L^p}+\|\dot\Delta_j f\|_{L^1_t(L^p)}\Big).$$

Multiplying by 2^{js} and summing up over j yields

$$\underbrace{\sum_j 2^{js}\|\dot\Delta_j u\|_{L^\infty_t(L^p)}}_{\|u\|_{\widetilde L^\infty_t(\dot B^s_{p,1})}}+\|u\|_{L^1_t(\dot B^{s+2}_{p,1})}\lesssim\|u_0\|_{\dot B^s_{p,1}}+\|f\|_{L^1_t(\dot B^s_{p,1})}.\qquad\square$$

More maximal regularity estimates:

- Those results may be somewhat generalized to domains if restricting to indices (p,s) with $-1+1/p<s<1/p$ and $1<p<\infty$.
- In $\mathbb{R}^d$, taking the L^{ρ_1} norm of each $\|\dot\Delta_j u\|_{L^p}$ over the time interval $[0,t]$ yields:

$$\|u\|_{\widetilde{L}_t^{\rho_1}(\dot B^{s+\frac{2}{\rho_1}}_{p,r})} \lesssim \|u_0\|_{\dot B^s_{p,1}} + \|f\|_{\widetilde{L}_t^{\rho_2}(\dot B^{s-2+\frac{2}{\rho_2}}_{p,r})} \quad \text{for } \ 1\leq \rho_2\leq\rho_1\leq\infty$$

with $\|v\|_{\widetilde{L}_t^a(\dot B^\sigma_{b,c})} := \Big\|2^{j\sigma}\|\dot\Delta_j v\|_{L^a(0,t;L^b(\mathbb{R}^d))}\Big\|_{\ell^c(\mathbb{Z})}.$

Note that time integration has been performed *before* spectral summation.

- Lamé system: $\partial_t u-\mu\Delta u-\mu'\nabla\mathrm{div}\, u=f$ in $\mathbb{R}^d$ with $\mu>0$ and $\mu+\mu'>0$:

$$\|u\|_{\widetilde{L}_t^\infty(\dot B^s_{p,1})} + \min(\mu,\mu+\mu')\|\nabla^2 u\|_{L^1_t(\dot B^s_{p,1})} \lesssim \|u_0\|_{\dot B^s_{p,1}} + \|f\|_{L^1_t(\dot B^s_{p,1})}.$$

- Stokes system: $\partial_t u-\mu\Delta u+\nabla P=f$ and $\mathrm{div}\, u=0$ in $\mathbb{R}^d$:

$$\|u\|_{\widetilde{L}_t^\infty(\dot B^s_{p,1})} + \|(\partial_t u,\mu\nabla^2 u,\nabla P)\|_{L^1_t(\dot B^s_{p,1})} \lesssim \|u_0\|_{\dot B^s_{p,1}} + \|f\|_{L^1_t(\dot B^s_{p,1})}.$$

Consider u and v in two different Besov spaces:

- Does uv make sense ?
- If so, where does uv lie ?

Formally, we have

$$uv = T_u v + R(u,v) + T_v u \tag{5}$$

with

$$T_u v := \sum_j \dot{S}_{j-1} u \, \dot{\Delta}_j v \quad \text{and} \quad R(u,v) := \sum_j \sum_{|j'-j|\leq 1} \dot{\Delta}_j u \, \dot{\Delta}_{j'} v.$$

The above operator T is called paraproduct whereas R is called remainder. Relation (5) (the so called Bony's decomposition) has been introduced by J.-M. Bony in the early eighties.

Proposition

For any $(s,p,r)\in\mathbb{R}\times[1,\infty]^2$ *and* $t<0$ *we have*

$$\|T_u v\|_{\dot B^s_{p,r}}\lesssim\|u\|_{L^\infty}\|v\|_{\dot B^s_{p,r}}\quad\text{and}\quad\|T_u v\|_{\dot B^{s+t}_{p,r}}\lesssim\|u\|_{\dot B^t_{\infty,\infty}}\|v\|_{\dot B^s_{p,r}}.$$

For any (s_1,p_1,r_1) *and* (s_2,p_2,r_2) *in* $\mathbb{R}\times[1,\infty]^2$ *we have*

- *if* $s_1+s_2>0,\ 1/p:=1/p_1+1/p_2\leq 1$ *and* $1/r:=1/r_1+1/r_2\leq 1$ *then*

$$\|R(u,v)\|_{\dot B^{s_1+s_2}_{p,r}}\lesssim\|u\|_{\dot B^{s_1}_{p_1,r_1}}\|v\|_{\dot B^{s_2}_{p_2,r_2}};$$

- *if* $s_1+s_2=0,\ 1/p:=1/p_1+1/p_2\leq 1$ *and* $1/r_1+1/r_2\geq 1$ *then*

$$\|R(u,v)\|_{\dot B^0_{p,\infty}}\lesssim\|u\|_{\dot B^{s_1}_{p_1,r_1}}\|v\|_{\dot B^{s_2}_{p_2,r_2}}.$$

Idea of proof. The general term defining T_uv and $R(u,v)$ (namely $\dot S_{j-1}u\,\dot\Delta_j v$ and $\dot\Delta_j u\dot\Delta_j v$) is spectrally localized in $2^jC(0,r,R)$ and $2^jB(0,R)$, respectively. Hence, according to the "fundamental lemma", it suffices to establish a suitable L^p estimate for each term. □

Corollary

Let u and v be in $L^\infty \cap \dot{B}^s_{p,r}$ for some $s > 0$ and $(p,r) \in [1,\infty]^2$. Then there exists a constant C depending only on d, p and s and such that

$$\|uv\|_{\dot{B}^s_{p,r}} \leq C\big(\|u\|_{L^\infty}\|v\|_{\dot{B}^s_{p,r}} + \|v\|_{L^\infty}\|u\|_{\dot{B}^s_{p,r}}\big).$$

Proof:

1. Write Bony's decomposition $uv = T_u v + T_v u + R(u,v)$;
2. Use $T : L^\infty \times \dot{B}^s_{p,r} \to \dot{B}^s_{p,r}$;
3. Use $R : \dot{B}^0_{\infty,\infty} \times \dot{B}^s_{p,r} \to \dot{B}^s_{p,r}$ if $s > 0$;
4. Notice that $L^\infty \hookrightarrow \dot{B}^0_{\infty,\infty}$.

Corollary

If $p < \infty$ then $\dot{B}^{\frac{d}{p}}_{p,1}$ is a Banach algebra continuously embedded in the set of continuous functions decaying to 0 at infinity.

Basic question:

Let $u \in \dot{B}^s_{p,r}$ and $F : \mathbb{R} \times \mathbb{R}$ smooth. What can be said of $F(u)$?

Proposition

Let $F:\mathbb{R}\to\mathbb{R}$ be a smooth function with $F(0)=0$. Then for all $(p,r)\in[1,\infty]^2$ and all $s>0$, there exists a constant C such that for all $u\in\dot B^s_{p,r}\cap L^\infty$ we have $F(u)\in\dot B^s_{p,r}\cap L^\infty$ and

$$\|F(u)\|_{\dot B^s_{p,r}}\leq C\|u\|_{\dot B^s_{p,r}}$$

with C depending only on $\|u\|_{L^\infty}$, F, s, p and d.

Sketchy proof: We use Meyer's first linearization method:

$$F(u)=\sum_j F(\dot S_{j+1}u)-F(\dot S_j u)=\sum_j \underbrace{\dot\Delta_j u\int_0^1 F'(\dot S_j u+\tau\dot\Delta_j u)\,d\tau}_{u_j}.$$

We notice that

$$\|u_j\|_{L^p}\leq C\|\dot\Delta_j u\|_{L^p}.$$

Unfortunately, $\mathcal{F}u_j$ is not localized in a ball of size 2^j. However, we find out that

$$\|D^k u_j\|_{L^p}\leq C2^{jk}\|\dot\Delta_j u\|_{L^p}.$$

Hence everything happens as if the $\mathcal{F}u_j$ were well localized. This suffices to complete the proof.

Multiplier space $\mathcal{M}(X)$ for the Banach space $X =$ set of distributions f such that ψf is in X whenever ψ is in X, endowed with the norm

$$\|f\|_{\mathcal{M}(X)} := \sup \|\psi f\|_X$$

where the supremum is taken over all functions ψ in X with norm 1.

Proposition

Let Z be a bi-Lipschitz diffeomorphism of $\mathbb{R}^d$ and (s,p,q) with $1 \leq p < \infty$ and $-d/p' < s < d/p$ (or just $-d/p' < s \leq d/p$ if $q = 1$ and just $-d/p' \leq s < d/p$ if $q = \infty$).
Then $a \mapsto a \circ Z$ is a self-map over $\dot{B}^s_{p,q}$ in the following cases:

1. *$s \in (0,1)$ and $J_{Z^{-1}}$, DZ are bounded,*
2. *$s \in (-1,0]$, J_Z, DZ^{-1} are bounded and $J_{Z^{-1}}$ is in $\mathcal{M}(\dot{B}^{-s}_{p',q'})$.*

Proof.

Case $s \in (0,1)$ is based on characterization by finite differences and change of variables.

Case $s \in (-1,0)$ follows by duality. □

Remark : Higher order estimates are available under stronger condition over Z: use chain rule and induction.

Consider the following transport equation:

$$(T) \qquad \begin{cases} \partial_t a + v\cdot\nabla a = f \in L^1([0,T);X) \\ a_{|t=0} = a_0 \in X. \end{cases}$$

Roughly, if v is a Lipschitz time-dependent vector-field and if X is a "reasonable" Banach space then we expect (T) to have a unique solution $a\in\mathcal{C}([0,T);X)$ satisfying

$$\|a(t)\|_X \leq e^{CV(t)}\biggl(\|a_0\|_X + \int_0^t e^{-CV(\tau)}\|f(\tau)\|_X\,d\tau\biggr)$$
$$\text{with} \quad V(t) := \int_0^t \|\nabla v(\tau)\|_{L^\infty}\,d\tau. \tag{6}$$

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Basic example: Hölder space $C^{0,\varepsilon}$.

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$$\text{with } V(t) := \int_0^t \|\nabla v(\tau)\|_{L^\infty}\,d\tau. \tag{6}$$

Theorem

The above result holds true for $X=\dot B^s_{p,r}$ *with* $V(t)=\displaystyle\int_0^t \|\nabla v(\tau)\|_{\dot B^{\frac{d}{p_1}}_{p_1,1}}\,d\tau$ *whenever* $1\leq p\leq p_1\leq\infty,\quad 1\leq r\leq\infty,\quad -\min\Bigl(\frac{d}{p_1},\frac{d}{p'}\Bigr)\leq s\leq 1+\frac{d}{p_1}.$
If $r>1$ *then we need* $s<1+d/p_1$.

Sketch of the proof:

Applying $\dot\Delta_j$ to (T) gives

$$\partial_t\dot\Delta_j a+v\cdot\nabla\dot\Delta_j a=\dot\Delta_j f+\dot R_j\quad\text{with}\quad \dot R_j:=[v\cdot\nabla,\dot\Delta_j]a. \tag{7}$$

Under the above conditions over s,p, the remainder term $\dot R_j$ satisfies

$$\|\dot R_j(t)\|_{L^p}\leq Cc_j(t)2^{-js}\|\nabla v(t)\|_{\dot B^{\frac{d}{p_1}}_{p_1,1}}\|a(t)\|_{\dot B^s_{p,r}}\quad\text{with}\quad \|(c_j(t)\|_{\ell^r}=1. \tag{8}$$

Applying standard L^p estimates for the transport equation (7) yields

$$\|\dot\Delta_j a(t)\|_{L^p}\leq\|\dot\Delta_j a_0\|_{L^p}+\int_0^t\biggl(\|\dot\Delta_j f\|_{L^p}+\|\dot R_j\|_{L^p}+\frac{\|\mathrm{div}\,v\|_{L^\infty}}{p}\|\dot\Delta_j a\|_{L^p}\biggr)\,d\tau.$$

Multiplying by 2^{js} then summing up over j yields

$$\|a\|_{L^\infty_t(\dot B^s_{p,r})}\leq\|a\|_{\widetilde L^\infty_t(\dot B^s_{p,r})}\leq\|a_0\|_{\dot B^s_{p,r}}+\int_0^t\|f\|_{\dot B^s_{p,r}}\,d\tau+C\int_0^t V'\|a\|_{\dot B^s_{p,r}}\,d\tau$$

with $\|a\|_{\widetilde L^\infty_t(\dot B^s_{p,r})}:=\bigl\|2^{js}\|\dot\Delta_j a\|_{L^\infty_t(L^p)}\bigr\|_{\ell^r}$.

Then applying Gronwall's lemma yields the desired inequality for a. $\square$

The homogeneous incompressible Navier-Stokes equations read:

$$(NS) \qquad \begin{cases} \partial_t u + \mathrm{div}\,(u\otimes u) - \mu\Delta u + \nabla P = 0, \\ \mathrm{div}\, u = 0. \end{cases}$$

Here $u : [0,T[\times\mathbb{R}^d \to \mathbb{R}^d$ stands for the velocity field, and $P : [0,T[\times\mathbb{R}^d \to \mathbb{R}$, for the pressure. The viscosity μ is a given positive number.
If we want to solve the Cauchy problem for (NS) then we have to prescribe some initial divergence-free velocity field u_0.

Introducing the Leray projector over divergence-free vector fields:
$\mathcal{P} := \mathrm{Id} + \nabla(-\Delta)^{-1}\mathrm{div}$, System (NS) recasts in

$$\partial_t u + \mathcal{P}\mathrm{div}\,(u\otimes u) - \mu\Delta u = 0.$$

This equation enters in the class of *generalized Navier-Stokes equations*:

$$(GNS) \qquad \partial_t u + Q(u,u) - \mu\Delta u = 0$$

with $\mathcal{F}Q^j(u,v)(\xi) := \sum \alpha_{k,\ell}^{j,m,n,p} \dfrac{\xi_n\,\xi_p\,\xi_m}{|\xi|^2}\,\mathcal{F}(u^k\,v^\ell)(\xi).$

Scaling invariance for (GNS): for all $\lambda > 0$ it is clear that v is a solution if and only if $T_\lambda v$ is a solution with

$$T_\lambda v(t,x) := \lambda v(\lambda^2 t, \lambda x).$$

Examples of scaling invariance spaces for (GNS)**:**

- $\mathcal{C}(\mathbb{R}^+;\dot H^{\frac d2 -1}) \cap L^2(\mathbb{R}^+;\dot H^{\frac d2})$ (Fujita and Kato, 1964);
- $\mathcal{C}(\mathbb{R}^+;L^d)$ (Giga-Miyakawa, Kato, (1984) Furioli-Lemarié-Terraneo (1998));
- $\mathcal{C}(\mathbb{R}^+;\dot B^{\frac dp -1}_{p,1}) \cap L^1(\mathbb{R}^+;\dot B^{\frac dp +1}_{p,1})$ and more general Besov spaces (Cannone-Meyer-Planchon, Kozono-Yamazaki 1994).

Theorem (global existence for small data)

Let $u_0 \in \dot B^{\frac dp -1}_{p,r}$ *with* $\operatorname{div} u_0 = 0$. *Assume that p is finite. There exists $c > 0$ such that if*

$$\|u_0\|_{\dot B^{\frac dp -1}_{p,r}} \leq c\mu$$

then (GNS) *has a unique global solution* u *in the space*

$$X := \widetilde L^\infty(\mathbb{R}^+;\dot B^{\frac dp -1}_{p,r}) \cap \widetilde L^1(\mathbb{R}^+;\dot B^{\frac dp +1}_{p,r}).$$

Mild formulation of (GNS): finding $u:\mathbb{R}^+\times\mathbb{R}^d\to\mathbb{R}^d$ so that
$u(t)=u_L(t)+\mathcal{B}(u,u)(t)$ with

$$u_L(t):=e^{\mu t\Delta}u_0 \quad\text{and}\quad \mathcal{B}(u,v)(t):=-\int_0^t e^{\mu(t-\tau)\Delta}Q(u,v)\,d\tau.$$

Claim : if $\mu^{-1}\|u_0\|_{\dot B^{\frac{d}{p}-1}_{p,r}}$ is small enough then $\Phi: v\mapsto u_L+\mathcal{B}(v,v)$ possesses a unique fixed point in the closed ball $\bar B_X(0,2C\|u_0\|_{\dot B^{\frac{d}{p}-1}_{p,r}})$ of the Banach space X.

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1. Maximal regularity estimates for the heat equation imply that

$$\|u_L\|_X:=\|u_L\|_{\widetilde L^\infty(\dot B^{\frac{d}{p}-1}_{p,r})}+\mu\|u_L\|_{\widetilde L^1(\dot B^{\frac{d}{p}+1}_{p,r})}\leq C\|u_0\|_{\dot B^{\frac{d}{p}-1}_{p,r}}.$$

2. Continuity results for the paraproduct and remainder $\Longrightarrow$ $\mathcal{B}:X\times X\to X$ whenever $p<\infty$. Indeed, for some $C=C(d,p,Q)$, we have

$$\|Q(u,v)\|_{\widetilde L^1(\dot B^{\frac{d}{p}-1}_{p,r})}\leq C\mu^{-1}\|u\|_X\|v\|_X.$$

Hence

$$\|\mathcal{B}(u,v)\|_X\leq C\mu^{-1}\|u\|_X\|v\|_X.$$

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2. Continuity results for the paraproduct and remainder $\Longrightarrow$ $\mathcal{B} : X \times X \to X$ whenever $p < \infty$. Indeed, for some $C = C(d,p,Q)$, we have

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Hence

$$\|\mathcal{B}(u,v)\|_X \leq C\mu^{-1}\|u\|_X\|v\|_X.$$

Therefore $\|\Phi(v)\|_X \leq C\|u_0\|_{\dot{B}^{\frac{d}{p}-1}_{p,r}} + C\mu^{-1}\|v\|_X^2 \leq 2C\|u_0\|_{\dot{B}^{\frac{d}{p}-1}_{p,r}}$ if $\|v\|_X \leq 2C\|u_0\|_{\dot{B}^{\frac{d}{p}-1}_{p,r}}$ and $4C^2\|u_0\|_{\dot{B}^{\frac{d}{p}-1}_{p,r}} \leq \mu$, and, under the same conditions,

$$\|\Phi(v) - \Phi(w)\|_X \leq C\mu^{-1}(\|v\|_X + \|w\|_X)\|v-w\|_X \leq \frac{1}{2}\|v-w\|_X.$$

Hence Φ is a contraction on $\bar{B}_X(0, 2C\|u_0\|_{\dot{B}^{\frac{d}{p}-1}_{p,r}})$. $\square$

The system for incompressible nonhomogeneous viscous fluids reads:

$$(INS) \qquad \begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0 \\ \rho(\partial_t u + u \cdot \nabla u) - \mu \Delta u + \nabla P = 0 \\ \operatorname{div} u = 0. \end{cases}$$

For simplicity, we restrict ourselves to the case where the density ρ of the fluid goes to 1 at infinity. So we set $\rho = 1 + a.$

System (INS) is invariant by the rescaling

$$\rho(t,x) \to \rho(\lambda^2 t, \lambda x), \qquad u(t,x) \to \lambda u(\lambda^2 t, \lambda x).$$

In the Besov spaces scale, this induces to take data $(\rho_0 = 1 + a_0, u_0)$ with

$$a_0 \in \dot{B}^{\frac{d}{p_1}}_{p_1,r_1} \quad \text{and} \quad u_0 \in \dot{B}^{\frac{d}{p_2}-1}_{p_2,r_2}.$$

- To avoid vacuum (and loss of ellipticity), we need a to be bounded away from 0. Notice that $\dot{B}^{\frac{d}{p_1}}_{p_1,r_1} \hookrightarrow L^\infty$ iff $r_1 = 1$. Hence we take $r_1 = 1$.
- If $r_2 = 1$ then regularity properties of the heat equation give $u \in L^1_T(\dot{B}^{\frac{d}{p_2}+1}_{p_2,1})$. This is exactly what we need to transport the Besov regularity of a.
- We take $p_1 = p_2 = p$ for simplicity.

Theorem (Global existence for small data)

Let $a_0 \in \dot B^{\frac dp}_{p,1}$ *and* $u_0 \in \dot B^{\frac dp -1}_{p,1}$ *with* $\operatorname{div} u_0 = 0$ *and* $1 \le p < 2d$. *If in addition*

$$\|a_0\|_{\dot B^{\frac dp}_{p,1}} + \mu^{-1}\|u_0\|_{\dot B^{\frac dp -1}_{p,1}} \le c$$

for a small enough $c>0$ *then* (INS) *has a unique global solution* (a,u) *with*

$$a \in \mathcal{C}(\mathbb{R}^+; \dot B^{\frac dp}_{p,1}) \quad \text{and} \quad u \in \mathcal{C}(\mathbb{R}^+; \dot B^{\frac dp -1}_{p,1}) \cap L^1(\mathbb{R}^+; \dot B^{\frac dp +1}_{p,1}).$$

Owing to the **hyperbolic nature of the density equation**, one cannot use the contracting mapping argument in Banach spaces because there is a **loss of one derivative** in the stability estimates. Nevertheless, one may proceed as follows:

- 1) proving a priori estimates in high norm (that is in the space E of the statement) for a solution;
- 2) proving stability estimates in low norm (with one less derivative);
- 3) Use functional analysis (Fatou property) to justify that the constructed solution is in E.

As regards uniqueness, this approach works only for $1 \le p \le d$. For the full range $1 \le p < 2d$, one has to reformulate the system in Lagrangian coordinates.

Sketchy proof of existence in the Eulerian framework

Step 1 : A priori estimates in large norm. Estimate a in $\mathcal{C}_b(\mathbb{R}^+;\dot{B}^{d/p}_{p,1})$ and $(u,\nabla P)$ in

$$\mathcal{C}_b(\mathbb{R}^+;\dot{B}^{d/p-1}_{p,1})\times L^1(\mathbb{R}^+;\dot{B}^{d/p-1}_{p,1})\ \text{ with }\ \partial_t u,\nabla^2 u\in L^1(\mathbb{R}^+;\dot{B}^{d/p-1}_{p,1}).$$

Main ingredients:

1. Estimates in Besov space for the transport equation.
2. The previous maximal regularity estimates for the Stokes equation.
3. Product estimates : $\dot{B}^{d/p}_{p,1}$ is a Banach algebra and the product maps $\dot{B}^{d/p}_{p,1}\times\dot{B}^{d/p-1}_{p,1}\to\dot{B}^{d/p-1}_{p,1}$ if $1\leq p<2d.$

Step 2 : Stability estimates in small norm. The difference $\delta\rho:=\rho_2-\rho_1,$ $\delta u:=u_2-u_1$ and $\nabla\delta P:=\nabla P_2-\nabla P_1$ between two solutions satisfies

$$\left\{\begin{aligned}&\partial_t\delta\rho+u_2\cdot\nabla\delta\rho=-\delta u\cdot\nabla\rho_1\\&\rho_2(\partial_t\delta u+u_2\cdot\nabla\delta u)-\mu\Delta\delta u+\nabla\delta P=-\delta\rho(\partial_t u_1+(\rho_2u_2-\rho_1u_1)\cdot\nabla u_1)\end{aligned}\right.$$

$\Longrightarrow$ loss of one derivative in the stability estimates.
We need to use that the product maps

$$\dot{B}^{d/p-1}_{p,1}\times\dot{B}^{d/p-1}_{p,1}\to\dot{B}^{d/p-2}_{p,1}.$$

But this is true if and only if $1\leq p<d$ and $d>2.$

Lagrangian change of coordinates

Flow of $u = u(t,x)$:

$$X_u(t,y) = y + \int_0^t u(\tau, X_u(\tau,y))\,d\tau$$

Change of coordinates: $(t,x) \longrightarrow (t,y)$ with $x = X_u(t,y)$.

$$\begin{aligned} \bar{u}(t,y) &= u(t,x), \\ \bar{P}(t,y) &= P(t,x). \end{aligned}$$

Chain rule:

$$\nabla_y \bar{F} = \nabla_y X_u \cdot \nabla_x F.$$

Hence the divergence-free condition recasts in

$$\mathrm{div}_y \bar{u} = g := D_y \bar{u} : (\mathrm{Id} - A) \quad \text{with} \quad A := (D_y X_u)^{-1}.$$

Lagrangian change of coordinates

Flow of $u = u(t,x)$:

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In general, $\mathrm{div}\,\bar{u}$ need not be 0 for $t > 0$.

The generalized Stokes equations

The momentum equation now reads:

$$(S):\begin{cases}\partial_t u-\mu\Delta u+\nabla P=f\\ \operatorname{div} u=g.\end{cases}$$

Set $u=v+w$ with w s.t. $\operatorname{div} w=g$. One can take $w=-\nabla(-\Delta)^{-1}g$. Then v has to satisfy

$$\begin{cases}\partial_t v-\mu\Delta v+\nabla P=f-\nabla(-\Delta)^{-1}\partial_t g+\mu\nabla g\\ \operatorname{div} v=0.\end{cases}$$

Needed conditions for g:

- $\nabla g\in L^1(\mathbb{R}^+;\dot B^s_{p,1})$
- $\partial_t g=\operatorname{div} R$ with $R\in L^1(\mathbb{R}^+;\dot B^s_{p,1})$.

If so, then we get

$$\begin{aligned}\|(u,\nabla P)\|_{E^s_p}&:=\|u\|_{L^\infty(\dot B^s_{p,1})}+\|(\partial_t u,\mu\nabla^2 u,\nabla P)\|_{L^1(\dot B^s_{p,1})}\\ &\lesssim\|u_0\|_{\dot B^s_{p,1}}+\|f\|_{L^1(\dot B^s_{p,1})}+\mu\|\nabla g\|_{L^1(\dot B^s_{p,1})}+\|R\|_{L^1(\dot B^s_{p,1})}.\end{aligned}$$

We are interested in the case $s=d/p-1$.

Estimates for g

Recall that $g = D_y\bar u : (\mathrm{Id} - A)$ with $A = (D_yX_u)^{-1}$ and that

$$D_yX_u(t) - \mathrm{Id} = \int_0^t D\bar u(\tau)\,d\tau \in \dot B^{d/p}_{p,1}.$$

As $\dot B^{d/p}_{p,1}$ is a Banach algebra, if the red term is small enough then one may write

$$A = (\mathrm{Id} + (D_yX_u - \mathrm{Id})) = \sum_{k=0}^{+\infty}(-1)^k\biggl(\int_0^t D\bar u\,d\tau\biggr)^k\cdot$$

Hence

$$\|\mathrm{Id} - A(t)\|_{\dot B^{d/p}_{p,1}} \lesssim \|D\bar u\|_{L^1(0,t;\dot B^{d/p}_{p,1})},$$

whence

$$\boxed{\|g\|_{L^1(0,t;\dot B^{d/p}_{p,1})} \lesssim \|D\bar u\|^2_{L^1(0,t;\dot B^{d/p}_{p,1})}.}$$

Do we have $\partial_t g = \operatorname{div} R$ with $R \in L^1(\mathbb{R}^+;\dot B^{d/p-1}_{p,1})$?

Estimates for g (continued)

Magic identity: X_u measure preserving implies that

$$\mathrm{div}_x u = D_y\bar{u} : A = \mathrm{div}_y(A\bar{u}).$$

Hence

$$\partial_t g = \mathrm{div}\, R \quad \text{with} \quad R = -\partial_t A\,\bar{u} + (\mathrm{Id} - A)\,\partial_t\bar{u}.$$

Under the same smallness condition as in the previous slide, one can write

$$\partial_t A = D\bar{u}\sum_{k\geq 1} k(-1)^k\left(\int_0^t D\bar{u}\,d\tau\right)^{k-1}\cdot$$

So finally, if $1 \leq p < 2d$ then we get

$$\|R\|_{L^1(\dot{B}^{d/p-1}_{p,1})} \lesssim \|D\bar{u}\|_{L^1(\dot{B}^{d/p}_{p,1})}\big(\|\bar{u}\|_{L^\infty(\dot{B}^{d/p-1}_{p,1})} + \|\partial_t\bar{u}\|_{L^1(\dot{B}^{d/p-1}_{p,1})}\big).$$

A priori estimates for the Lagrangian INS equations

In Lagrangian coordinates ρ_0 is time-independent, hence no loss of derivatives in the stability estimates.

For the velocity, we have

$$\begin{cases} \rho_0\partial_t\bar u-\mu\,\mathrm{div}_y(A_u{}^TA_u\nabla_y\bar u)+{}^TA_u\nabla_y\bar P=0\\ \mathrm{div}_y(A_u\bar u)=0.\end{cases}\qquad \text{with}\quad A_u=(D_yX_u)^{-1}$$

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This equation rewrites

$$\begin{cases}\partial_t\bar u-\mu\Delta\bar u+\nabla_y\bar P=(1-\rho_0)\partial_t\bar u+\mu\,\mathrm{div}_y((A_u{}^TA_u-\mathrm{Id})\nabla_y\bar u)+(\mathrm{Id}-{}^TA_u)\nabla_y\bar P\\ \mathrm{div}_y\bar u=g:=\mathrm{div}_y((\mathrm{Id}-A_u)\bar u)=D\bar u:(\mathrm{Id}-A_u).\end{cases}$$

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From the above estimates for g, A_u and for the Stokes equations, we thus get

$$U(t)\lesssim\|u_0\|_{\dot B^{d/p-1}_{p,1}}+U^2(t)+\int_0^t\|(1-\rho_0)\partial_t\bar u\|_{\dot B^{d/p-1}_{p,1}}\,d\tau$$

with $U(t):=\|\bar u\|_{L^\infty(0,t;\dot B^{d/p-1}_{p,1})}+\|\partial_t\bar u,\mu D^2\bar u,\nabla\bar P\|_{L^1(0,t;\dot B^{d/p-1}_{p,1})}.$

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From the above estimates for $g,\ A_u$ and for the Stokes equations, we thus get

$$U(t)\lesssim\|u_0\|_{\dot B^{d/p-1}_{p,1}}+U^2(t)+\int_0^t\|(1-\rho_0)\partial_t\bar u\|_{\dot B^{d/p-1}_{p,1}}\,d\tau$$

with $U(t):=\|\bar u\|_{L^\infty(0,t;\dot B^{d/p-1}_{p,1})}+\|\partial_t\bar u,\mu D^2\bar u,\nabla\bar P\|_{L^1(0,t;\dot B^{d/p-1}_{p,1})}$.

Let $\mathcal M(\dot B^{d/p-1}_{p,1})$ be the multiplier space for $\dot B^{d/p-1}_{p,1}$. By definition,

$$\|(1-\rho_0)\partial_t\bar u\|_{\dot B^{d/p-1}_{p,1}}\leq\|(1-\rho_0)\|_{\mathcal M(\dot B^{d/p-1}_{p,1})}\|\partial_t\bar u\|_{\dot B^{d/p-1}_{p,1}}.$$

So we just need $\|(1-\rho_0)\|_{\mathcal M(\dot B^{d/p-1}_{p,1})}\ll1$ and $\|u_0\|_{\dot B^{d/p-1}_{p,1}}\ll\mu$ to close the estimates.

Implementing the fixed point argument

Let $\Phi : (\bar v, \nabla\bar Q) \mapsto (\bar u, \nabla\bar P)$ where $(\bar u, \nabla\bar P)$ stands for the solution to the *linear* system

$$\begin{cases} \rho_0\partial_t\bar u - \mu\,\mathrm{div}\,(A_v{}^TA_v\nabla\bar u) + {}^TA_v\nabla\bar P = 0\\ \mathrm{div}\,(A_v\bar u) = 0, \end{cases}$$

with $A_v := (DX_v)^{-1}$ and $X_v(t,y) := y + \int_0^t \bar v(\tau,y)\,d\tau.$

Step 1. Existence of Φ.

If $(\bar v, \nabla\bar Q)$ belongs to a small ball B_R of $E_p^{d/p-1}$ and X_v is measure preserving in the "original" Eulerian coordinates then the previous slide implies that the same holds for $(\bar u, \nabla\bar P)$.

Important: the corresponding set $\mathcal{E}_R$ is a closed subset of $E_p^{d/p-1}$.

Step 2. Contraction estimates for Φ.

One just has to write $\Phi(\bar v_2, \nabla\bar Q_2) - \Phi(\bar v_1, \nabla\bar Q_1)$ as a solution to the Stokes equation and slightly generalize the previous estimates. No loss of derivative here !

Applying the Banach fixed point theorem allows to conclude to the existence of a solution in $\mathcal{E}_R$.

Step 3. Uniqueness. This is a straightforward modification of Step 2.

Theorem (D. and P. Mucha, 2011)

Let $p \in [1, 2d)$ and $u_0 \in \dot{B}^{d/p-1}_{p,1}(\mathbb{R}^d)$ with $\operatorname{div} u_0 = 0$. Assume that $\rho_0 \in \mathcal{M}(\dot{B}^{d/p-1}_{p,1})$. There exists a constant $c = c(p, d)$ such that if

$$\|\rho_0 - 1\|_{\mathcal{M}(\dot{B}^{d/p-1}_{p,1})} + \mu^{-1}\|u_0\|_{\dot{B}^{d/p-1}_{p,1}} \leq c$$

then the Lagrangian (INS) system has a unique global solution $(\bar{u}, \nabla\bar{P})$ in $E^{d/p-1}_p$. Moreover, there exists $C = C(p, d)$ so that

$$\|\bar{u}\|_{L^\infty(\dot{B}^{d/p-1}_{p,1})} + \|\mu\nabla^2\bar{u}, \partial_t\bar{u}, \nabla\bar{P}\|_{L^1(\dot{B}^{d/p-1}_{p,1})} \leq C\|u_0\|_{\dot{B}^{d/p-1}_{p,1}}$$

and the flow map $(\rho_0, u_0) \longmapsto (\bar{u}, \nabla\bar{P})$ is Lipschitz continuous from $\mathcal{M}(\dot{B}^{d/p-1}_{p,1}) \times \dot{B}^{d/p-1}_{p,1}$ to $E^{d/p-1}_p$.

Remarks

- *Local-in-time statement if only $\rho_0 - 1$ is small.*
- *Propogation of interfaces: if $d/p - 1 < 1/p$ then one can take $\rho_0 = 1 + c1_D$ with c small enough, and D any C^1 domain.*
- *Corollary : same statement for the original system in Eulerian coordinates (except for the continuity of the flow map).*

The barotropic Navier-Stokes equations read :

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \mu \Delta u - \mu' \nabla \operatorname{div} u + \nabla P = 0. \end{cases} \tag{9}$$

- $\rho = \rho(t,x) \in \mathbb{R}^+$ (with $t \in \mathbb{R}^+$ and $x \in \mathbb{R}^d$) is the density.
- $u = u(t,x) \in \mathbb{R}^d$ is the velocity field.
- The pressure P is a given smooth function of ρ.
- The viscosity coefficients μ and μ' satisfy $\mu > 0$ and $\nu := \mu + \mu' > 0$ and are constant (for simplicity only).
- Boundary conditions: u decays to zero at infinity and ρ tends to some positive constant $\bar{\rho}$ at infinity. We take $\bar{\rho} = 1$ for simplicity.

Denoting $\rho = 1 + a$ and assuming that the density is positive everywhere the barotropic system rewrites

$$\begin{cases} \partial_t a + u \cdot \nabla a = -(1+a) \operatorname{div} u, \\ \partial_t u - \mathcal{A} u = -u \cdot \nabla u - J(a) \mathcal{A} u - \nabla G(a) \end{cases}$$

with $\mathcal{A} := \mu \Delta + \mu' \nabla \operatorname{div}$, $J(a) := a/(1+a)$ and $G'(a) = P'(1+a)/(1+a)$.

If neglecting the pressure term then the scaling invariance of the system still reads:

$$\rho(t,x) \to \rho(\lambda^2 t, \lambda x), \qquad u(t,x) \to \lambda u(\lambda^2 t, \lambda x).$$

As for the incompressible Navier-Stokes equations, in the Besov spaces scale, this induces to take data $(\rho_0 = 1 + a_0, u_0)$ with

$$a_0 \in \dot{B}^{\frac{d}{p}}_{p,1} \quad \text{and} \quad u_0 \in \dot{B}^{\frac{d}{p}-1}_{p,1}.$$

Goal:

Solving the compressible Navier-Stokes equations with $u_0 \in \dot{B}^{\frac{d}{p}-1}_{p,1}$ and $a_0 \in \dot{B}^{\frac{d}{p}}_{p,1}$ such that $1 + a_0 > 0$ (no vacuum assumption).

According to the preceding results on the transport equation and the Lamé system, we expect that

$$a \in \mathcal{C}([0,T]; \dot{B}^{\frac{d}{p}}_{p,1}) \quad \text{and} \quad u \in \mathcal{C}([0,T]; \dot{B}^{\frac{d}{p}-1}_{p,1}) \cap L^1([0,T]; \dot{B}^{\frac{d}{p}+1}_{p,1}).$$

Owing to the **hyperbolic nature of the density equation**, there is a **loss of one derivative** in the stability estimates. Hence it is tempting to use again Lagrangian coordinates.

Given some solution (ρ, u) to the compressible Navier-Stokes equations, we introduce X the flow associated to the vector-field u:

$$X(t,y) = y + \int_0^t u(\tau, X(\tau,y))\,d\tau. \tag{10}$$

Let $\bar\rho(t,y) := \rho(t, X(t,y)),\quad \bar u(t,y) = u(t, X(t,y)),\quad J := |\det DX|,$ and $A := (D_y X)^{-1}.$

- $J\bar\rho$ is time independent,
- As X need not preserve the Lebesgue measure, the "magic relation" becomes
$$\mathrm{div}_x H(x) = D_y \bar H(y)\cdot A(y) = J^{-1}\mathrm{div}_y(\mathrm{adj}\,(D_y X)\bar H)(y).$$
- Hence

$$J\partial_t(J\bar\rho\bar u) - \mu\,\mathrm{div}\left(\mathrm{adj}\,(DX)^T A\nabla_y \bar u\right) - \mu'\mathrm{div}\left(\mathrm{adj}\,(DX)^T A{:}\nabla\bar u\right) + \mathrm{div}\left(\mathrm{adj}\,(DX)P(\bar\rho)\right) = 0.$$

As before X may be directly computed from $\bar u$:

$$X(t,y) = y + \int_0^t \bar u(\tau, y)\,d\tau.$$

As $J\bar\rho = \rho_0,$ we just have to solve the following parabolic type equation for $\bar u$:

$$\begin{aligned} J\bar\rho_0\partial_t\bar u &- \mu\,\mathrm{div}\left(\mathrm{adj}\,(DX)^T A\nabla_y\bar u\right) \\ &-\mu'\mathrm{div}\left(\mathrm{adj}\,(DX)^T A : \nabla\bar u\right) + \mathrm{div}\left(\mathrm{adj}\,(DX)P(\bar\rho)\right) = 0. \end{aligned} \tag{11}$$

Theorem

Let $p \in [1, 2d)$ (with $d \geq 2$) and u_0 be a vector-field in $\dot{B}^{\frac{d}{p}-1}_{p,1}$. Assume that the initial density ρ_0 is positive and satisfies $a_0 := (\rho_0 - 1) \in \dot{B}^{\frac{d}{p}}_{p,1}$. Then the above equation has a unique local solution $(\bar{\rho}, \bar{u})$ with $\bar{a} \in \mathcal{C}([0,T]; \dot{B}^{\frac{d}{p}}_{p,1})$ and $\bar{u} \in \mathcal{C}([0,T]; \dot{B}^{\frac{d}{p}}_{p,1}) \cap L^1(0,T; \dot{B}^{\frac{d}{p}+1}_{p,1})$. Moreover, the flow map $(a_0, u_0) \longmapsto (\bar{a}, \bar{u})$ is Lipschitz continuous.

In Eulerian coordinates, this result recasts in:

Theorem

Under the above assumptions, the barotropic Navier-Stokes equations have a unique local solution (ρ, u) with the above regularity.

Remark

If working directly on the barotropic compressible Navier-Stokes equations in Eulerian coordinates, then uniqueness may be proved only under the stronger condition that $p \leq d$.

To simplify the presentation, we assume that $a_0 := \rho_0 - 1$ is small enough. The three ingredients are

- Regularity estimates for the Lamé system;
- Flow estimates in Besov spaces,
- The Banach fixed point theorem.

Let $E_p(T) := \{\bar{u} \in \mathcal{C}([0,T];\dot{B}^{\frac{d}{p}-1}_{p,1}) \,/\, \partial_t \bar{u}, \nabla^2 \bar{u} \in L^1(0,T;\dot{B}^{\frac{d}{p}-1}_{p,1}\}$. Define a map $\Phi : \bar{v} \mapsto \bar{u}$ on $E_p(T)$ where $\bar{u}$ stands for the solution to

$$\partial_t \bar{u} - \mu \Delta \bar{u} - (\lambda+\mu)\nabla \mathrm{div}\, \bar{u} = I_1(\bar{v},\bar{v}) + 2\mu \mathrm{div}\, I_2(\bar{v},\bar{v}) + \lambda \mathrm{div}\, I_3(\bar{v},\bar{v}) - \mathrm{div}\, I_4(\bar{v})$$

with

$$\begin{aligned} I_1(\bar{v},\bar{w}) &= (1-\rho_0 J_v)\partial_t \bar{w} & I_2(\bar{v},\bar{w}) &= \mathrm{adj}\,(DX_v) D_{A_v}(\bar{w}) - D(\bar{w}) \\ I_3(\bar{v},\bar{w}) &= \mathrm{div}_{A_v} \bar{w}\, \mathrm{adj}\,(DX_v) - \mathrm{div}\, \bar{w}\, \mathrm{Id} & I_4(\bar{v}) &= \mathrm{adj}\,(DX_v) P(J_v^{-1}\rho_0). \end{aligned}$$

Any fixed point of Φ is a solution in $E_p(T)$ to (11). We claim that the existence of such points is a consequence of the standard Banach fixed point theorem in a suitable closed ball of $E_p(T)$.

We first need to prove suitable a priori estimates for the Lamé system, that is, the linearized velocity equation (neglecting the pressure term). This system reads:

$$\partial_t u - \mu\Delta u - (\lambda+\mu)\nabla \mathrm{div}\, u = f, \qquad u_{|t=0} = u_0 \tag{12}$$

with $\mu > 0$ and $\nu := \lambda + 2\mu > 0$.

Apply the projector $\mathcal{P}$ over divergence-free vector-fields, or $\mathcal{Q}$ the projector over potential vector fields. We get

$$\partial_t \mathcal{P}u - \mu\Delta\mathcal{P}u = \mathcal{P}f \quad \text{and} \quad \partial_t \mathcal{Q}u - \nu\Delta\mathcal{Q}u = \mathcal{Q}f.$$

Hence applying the estimates for the heat equation yields in particular:

There exists a constant C depending only on μ/ν and λ/ν such that if $u_0 \in \dot{B}^s_{p,1}$ and $f \in L^1(\mathbb{R}^+;\dot{B}^s_{p,1})$ then (12) has a unique solution u in $\mathcal{C}(\mathbb{R}^+;\dot{B}^s_{p,1}) \cap L^1(\mathbb{R}^+;\dot{B}^s_{p,1})$ and

$$\|u\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})} + \|\partial_t u, \min(\mu,\nu)\nabla^2 u\|_{L^1_t(\dot{B}^{\frac{d}{p}-1}_{p,1})} \leq C\Big(\|u_0\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}} + \|f\|_{L^1_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}\Big).$$

Estimates for I_1, I_2, I_3 and I_4.

Throughout we assume that

$$\int_0^T \|Dv\|_{\dot B^{\frac{d}{p}}_{p,1}}\,dt <\!\!< 1. \tag{13}$$

In order to bound $I_1(v,w)$, we decompose it into

$$I_1(v,w)=(1-J_v)\partial_t w - a_0(1+(J_v-1))\partial_t w \quad\text{with}\quad a_0:=\rho_0-1.$$

Hence, product laws, definition of $\mathcal{M}(\dot B^{\frac{d}{p}-1}_{p,1})$ and flow estimates imply

$$\|I_1(v,w)\|_{L^1_T(\dot B^{\frac{d}{p}-1}_{p,1})}\leq C\Big(\|a_0\|_{\mathcal{M}(\dot B^{\frac{d}{p}-1}_{p,1})}+\|Dv\|_{L^1_T(\dot B^{\frac{d}{p}}_{p,1})}\Big)\|\partial_t w\|_{L^1_T(\dot B^{\frac{d}{p}-1}_{p,1})}.$$

Similarly, we have

$$\|I_2(v,w)\|_{L^1_T(\dot B^{\frac{d}{p}}_{p,1})}+\|I_3(v,w)\|_{L^1_T(\dot B^{\frac{d}{p}}_{p,1})}\leq C\|Dv\|_{L^1_T(\dot B^{\frac{d}{p}}_{p,1})}\|Dw\|_{L^1_T(\dot B^{\frac{d}{p}}_{p,1})}.$$

As regards the pressure term $I_4(v)$, we use the fact that under assumption (13), we have, by virtue of Proposition 1.4 and of flow estimates

$$\|I_4(v)\|_{L^\infty_T(\dot B^{\frac{d}{p}}_{p,1})}\leq C\Big(1+\|Dv\|_{L^1_T(\dot B^{\frac{d}{p}}_{p,1})}\Big)\Big(1+\|a_0\|_{\dot B^{\frac{d}{p}}_{p,1}}\Big).$$

Stability of a small enough ball by Φ.

We introduce u_L the solution to

$$\partial_t u_L - \mu\Delta u_L - (\lambda+\mu)\nabla\mathrm{div}\, u_L = 0, \qquad u_L|_{t=0} = u_0.$$

Claim: if R and T are small enough then

$$v \in \bar{B}_{E_p(T)}(u_L, R) \Longrightarrow u \in \bar{B}_{E_p(T)}(u_L, R).$$

Indeed $\widetilde{u} := u - u_L$ satisfies $\widetilde{u}(0) = 0$ and

$$\partial_t\widetilde{u} - \mu\Delta\widetilde{u} - (\lambda+\mu)\nabla\mathrm{div}\,\widetilde{u} = I_1(v,v) + 2\mu\mathrm{div}\, I_2(v,v) + \lambda\mathrm{div}\, I_3(v,v) - \mathrm{div}\, I_4(v).$$

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So regularity estimates for Lamé system imply that

$$\|\widetilde{u}\|_{E_p(T)} \lesssim \sum_{j=1}^{3} \|I_j(v,v)\|_{L^1_T(\dot{B}^{\frac{d}{p}-1}_{p,1})} + T\|I_4(v)\|_{L^\infty_T(\dot{B}^{\frac{d}{p}}_{p,1})}.$$

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Indeed $\widetilde{u} := u - u_L$ satisfies $\widetilde{u}(0) = 0$ and

$$\partial_t \widetilde{u} - \mu\Delta\widetilde{u} - (\lambda+\mu)\nabla \mathrm{div}\,\widetilde{u} = I_1(v,v) + 2\mu \mathrm{div}\, I_2(v,v) + \lambda \mathrm{div}\, I_3(v,v) - \mathrm{div}\, I_4(v).$$

Using the previous inequalities for $I_j(v,v)$ and that $v \in \bar{B}_{E_p(T)}(u_L, R)$, we get

$$\|\widetilde{u}\|_{E_p(T)} \leq C\Big(\big(\|a_0\|_{\mathcal{M}(\dot{B}^{\frac{d}{p}-1}_{p,1})} + \|Du_L\|_{L^1_T(\dot{B}^{\frac{d}{p}}_{p,1})} + R\big)\big(R + \|\partial_t u_L\|_{L^1_T(\dot{B}^{\frac{d}{p}-1}_{p,1})}\big) + \|Du_L\|^2_{L^1_T(\dot{B}^{\frac{d}{p}}_{p,1})} + R^2 + T(1 + \|a_0\|_{\dot{B}^{\frac{d}{p}}_{p,1}})\Big).$$

Hence there exists a small constant $\eta = \eta(d,p)$ such that if

$$\|a_0\|_{\mathcal{M}(\dot{B}^{\frac{d}{p}-1}_{p,1})} \leq \eta,$$

and if R, T have been chosen small enough then u is in $\bar{B}_{E_p(T)}(u_L, R)$.

Contraction properties for Φ on $\bar{B}_{E_p(T)}(u_L, R)$.

Let v_1 and v_2 in $\bar{B}_{E_p(T)}(u_L, R)$ and $u_1 := \Phi(v_1)$ and $u_2 := \Phi(v_2)$. The equation satisfied by $\delta u := u_2 - u_1$ reads

$$\partial_t \delta u - \mu\Delta\delta u - (\lambda+\mu)\nabla \mathrm{div}\,\delta u = \delta f_1 + \delta f_2 + \mathrm{div}\,\delta f_3 + 2\mu \mathrm{div}\,\delta f_4 + \lambda \mathrm{div}\,\delta f_5$$

with $\delta f_1 := (1-\rho_0 J_2)\partial_t \delta u, \quad \delta f_2 := -\rho_0(J_2 - J_1)\partial_t u_1,$

$$\delta f_3 := \mathrm{adj}\,(DX_1)P(\rho_0 J_1^{-1}) - \mathrm{adj}\,(DX_2)P(\rho_0 J_2^{-1}),$$
$$\delta f_4 := \mathrm{adj}\,(DX_2)D_{A_2}(u_2) - \mathrm{adj}\,(DX_1)D_{A_1}(u_1) - D(\delta u),$$
$$\delta f_5 := \mathrm{adj}\,(DX_2)^T A_2 : \nabla u_2 - \mathrm{adj}\,(DX_1)^T A_1 : \nabla u_1 - \mathrm{div}\,\delta u\,\mathrm{Id}\,.$$

Bounding δu stems from the maximal regularity estimates: we get

$$\|\delta u\|_{E_p(T)} \lesssim \sum_{i=1}^{2} \|\delta f_i\|_{L^1_T(\dot{B}^{\frac{d}{p}-1}_{p,1})} + T\|\delta f_3\|_{L^\infty_T(\dot{B}^{\frac{d}{p}}_{p,1})} + \sum_{i=4}^{5} \|\delta f_i\|_{L^1_T(\dot{B}^{\frac{d}{p}}_{p,1})}.$$

For instance we have

$$\|\delta f_3\|_{L^\infty_T(\dot{B}^{\frac{d}{p}-1}_{p,1})} \lesssim T(1+\|a_0\|_{\dot{B}^{\frac{d}{p}}_{p,1}})\|D\delta v\|_{L^1_T(\dot{B}^{\frac{d}{p}}_{p,1})}.$$

We eventually find that if η, R and T then

$$\|\delta u\|_{E_p(T)} \leq \frac{1}{2}\|\delta v\|_{E_p(T)}.$$

Banach theorem ensures that Φ has a unique fixed point in $\bar{B}_{E_p(T)}(u_L, R)$.

Aim: proving a global existence result for small data for the barotropic Navier-Stokes equation

$$(NSC) \qquad \begin{cases} \partial_t a + u\cdot\nabla a = -(1+a)\mathrm{div}\, u, \\ \partial_t u - \mathcal{A}u = -u\cdot\nabla u - J(a)\mathcal{A}u - \nabla G(a), \end{cases}$$

in the spirit of those for the incompressible Navier-Stokes equation.

Above we saw that just assuming that

$$\|a_0\|_{\dot B^{\frac dp}_{p,1}} + \|u_0\|_{\dot B^{\frac dp -1}_{p,1}} \ll 1$$

is not enough because the pressure term (which has not the right-scaling) entails a linear growth in time in the estimates. In effect, in order to bound the term $I_4(\bar v) := \mathrm{adj}\,(DX_v)P(J_v^{-1}\rho_0)$, we just wrote

$$\|I_4(v)\|_{L^1_T(\dot B^{\frac dp}_{p,1})} \leq T\|I_4(v)\|_{L^\infty_T(\dot B^{\frac dp}_{p,1})} \leq CT\Big(1+\|Dv\|_{L^1_T(\dot B^{\frac dp}_{p,1})}\Big)\Big(1+\|a_0\|_{\dot B^{\frac dp}_{p,1}}\Big).$$

We have to include the pressure term in the linearized system. This will be done in Eulerian coordinates, in the next section.

Part III. Partially dissipative or parabolic linear systems

We focus on linear systems of the type

$$\partial_t w + A(D)w + B(D)w = 0 \tag{14}$$

with $w : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^n$, and

- $A(D) = (A_{ij}(D))_{1\leq i,j\leq n}$ with $A_{ij}(D)$ homogeneous Fourier multiplier of degree α,
- $B(D) = (B_{ij}(D))_{1\leq i,j\leq n}$ with $B_{ij}(D)$ homogeneous Fourier multiplier of degree β.

We assume in addition that $A(D)$ is *antisymmetric*:

$$\mathrm{Re}\,((A(\xi)\eta)\cdot\eta) = 0 \quad \text{for all} \quad (\xi,\eta) \in \mathbb{R}^d \times \mathbb{C}^n,$$

and that $B(D)$ satisfies the following ellipticity property :

$$|\xi|^\beta\,\mathrm{Re}\,((B(\xi)\eta)\cdot\eta) \geq \kappa|B(\xi)\eta|^2 \quad \text{for all} \quad (\xi,\eta) \in \mathbb{R}^d \times \mathbb{C}^n$$

where κ is a positive real number.

Examples

- A partially dissipative symmetric linear one-dimensional system:

$$\begin{cases} \partial_t u + \partial_x v = 0 \\ \partial_t v + \partial_x u + \lambda v = 0; \end{cases} \qquad \lambda > 0.$$

The general conditions are fulfilled with $d = 1,\ n = 2,\ \alpha = 1,\ \beta = 0$ and $\kappa = \lambda^{-1}$,

- The linearized barotropic Navier-Stokes equations :

$$\begin{cases} \partial_t a + \operatorname{div} u = 0 \\ \partial_t u - \mu\Delta u - \mu'\nabla\operatorname{div} u + \nabla a = 0; \end{cases} \qquad \mu > 0 \ \text{ and } \ \mu + \mu' > 0.$$

The general conditions are fulfilled with $n = d + 1,\ \alpha = 1,\ \beta = 2$ and $\kappa = c\nu^{-1}$ (with c depending only on μ/ν).

The general solution formula

Set $A_\omega := \rho^{-\alpha} A(\xi)$ and $B_\omega := \kappa\rho^{-\beta} B(\xi)$ with $\rho := |\xi|$ and $\omega := \xi/|\xi|$.

Therefore

$$\partial_t \widehat{w}(t,\xi) + E(\xi)\widehat{w}(t,\xi) = 0 \quad \text{with} \quad E(\xi) := \rho^\alpha A_\omega + \kappa^{-1}\rho^\beta B_\omega.$$

Hence

$$\widehat{w}(t,\xi) = \widehat{w}_0(\xi)\exp\left(-\frac{t\rho^\beta}{\kappa}\left(\kappa\rho^{\alpha-\beta}A_\omega + B_\omega\right)\right),$$

Let $z_0 := \widehat{w}_0(\xi)$, $z(\tau) = \widehat{w}(t,\xi)$ with $\tau := (t\rho^\beta)/\kappa$, and $\varrho := \kappa\rho^{\alpha-\beta}$ We have

$$z(\tau) := z_0 \exp\Big(-\tau(\varrho A_\omega + B_\omega)\Big)$$

Hence one may restrict our attention to the case $\alpha = 1$, $\beta = 0$ and $\kappa = 1$ (that is first order antisymmetric terms and partial dissipation). Indeed

$$\mathrm{Re}\,((A_\omega\eta)\cdot\eta) = 0 \quad \text{and} \quad \mathrm{Re}\,((B_\omega\eta)\cdot\eta) \geq |B_\omega\eta|^2 \quad \text{for all} \quad (\omega,\eta) \in \mathbb{S}^{d-1}\times\mathbb{C}^n.$$

For $\varepsilon_1,\cdots,\varepsilon_{k-1}$ small positive parameters, define the *Lyapunov functional*:

$$L(\tau):=|z(\tau)|^2+\min(\varrho,\varrho^{-1})\sum_{k=1}^{n-1}\varepsilon_k\,\mathrm{Re}\,\big((B_\omega A_\omega^{k-1}z)\cdot(B_\omega A_\omega^k z)\big).$$

As $z'+(\varrho A_\omega+B_\omega)z=0,$ $\mathrm{Re}\,((A_\omega\eta)\cdot\eta)=0$ and $\mathrm{Re}\,((B_\omega\eta)\cdot\eta)\geq|B_\omega\eta|^2$ we get

$$L'(\tau)+2\,\mathrm{Re}\,\big((B_\omega z)\cdot z\big)+\min(1,\varrho^2)\sum_{k=1}^{n-1}\varepsilon_k|B_\omega A_\omega^k z|^2$$

$$=-\min(\varrho,\varrho^{-1})\Big\{\sum_{k=1}^{n-1}\varepsilon_k\Big(\mathrm{Re}\,\big((B_\omega A_\omega^{k-1}B_\omega z)\cdot(B_\omega A_\omega^k z)+(B_\omega A_\omega^{k-1}z)\cdot(B_\omega A_\omega^k B_\omega z)\big)\Big)\Big\}$$

$$-\min(1,\varrho^2)\sum_{k=1}^{n-1}\varepsilon_k\,\mathrm{Re}\,\big((B_\omega A_\omega^{k-1}B_\omega z)\cdot(B_\omega A_\omega^{k+1}z)\big).$$

One may take $\varepsilon_0,\cdots,\varepsilon_{k-1}$ so small as, for all $\omega\in\mathbb{S}^{d-1}$ and $\rho>0$,

$$L'(\tau)+\frac{\min(1,\varrho^2)}{2}\sum_{k=0}^{n-1}\varepsilon_k|B_\omega A_\omega^k z(\tau)|^2\leq 0\quad\text{and}\quad L\approx|z|^2.$$

Setting $N_\omega:=\min_{\eta\in\mathbb{S}^{n-1}}\sum_{k=0}^{n-1}\varepsilon_k|B_\omega A_\omega^k\eta|^2,$ we end up with

$$L(t)\leq e^{-\frac14\min(1,\varrho^2)N_\omega\tau}L(0).$$

Assumption: $\boxed{\min_{\omega\in\mathbb{S}^{d-1}} N_\omega > 0.}$

This entails the following decay inequality:

$$|\widehat{w}(t,\xi)| \leq 2|\widehat{w}_0(\xi)|\, e^{-c\kappa^{-1}\min(|\xi|^\beta,\kappa^2|\xi|^{2\alpha-\beta})t}. \tag{15}$$

The above assumption is equivalent to the *Kalman rank condition*:

$$\begin{pmatrix} B_\omega \\ B_\omega A_\omega \\ \cdots \\ BA_\omega^{n-1}\end{pmatrix} \quad \text{has rank} \quad n$$

or to the *Shizuta-Kawashima condition*:

$$\ker B_\omega \cap \{\text{eigenvectors of} \quad A_\omega\} = \{0\}.$$

From (15), using Parseval equality, we get

$$\|\dot\Delta_j w(t)\|_{L^2} \leq 2\|\dot\Delta_j w_0\|_{L^2} e^{-\min(2^{j\beta},\kappa^2 2^{(2\alpha-\beta)j})\kappa^{-1}t} \quad \text{for all} \quad j\in\mathbb{Z}. \tag{16}$$

Taking advantage of Duhamel's formula, we may afford to have a right-hand side f in the linear system: Inequality (16) implies that

$$\|\dot\Delta_j w\|_{L^\infty_t(L^2)} + \kappa^{-1}\min\Big(2^{j\beta},\kappa^2 2^{j(2\alpha-\beta)}\Big)\|\dot\Delta_j w\|_{L^1_t(L^2)} \lesssim \|\dot\Delta_j w_0\|_{L^2} + \|\dot\Delta_j f\|_{L^1_t(L^2)}.$$

This means that there is a gain of $\max(\beta, 2\alpha-\beta)$ (resp. $\min(\beta,2\alpha-\beta)$) in low frequencies (resp. high frequencies) when performing a L^1-in-time integration.

Application to a partially dissipative system

$$\begin{cases} \partial_t u + \partial_x v = 0 \\ \partial_t v + \partial_x u + \lambda v = 0; \end{cases} \qquad \lambda > 0$$

The corresponding matrices A_ω and B_ω read

$$A_\omega = \begin{pmatrix} 0 & i\,\mathrm{sgn}\,\omega \\ i\,\mathrm{sgn}\,\omega & 0 \end{pmatrix} \quad \text{and} \quad B_\omega = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Ellipticity condition is satisfied with $\kappa = \lambda^{-1}$ and $\beta = 0$. In addition,

$$B_\omega A_\omega = \begin{pmatrix} 0 & 0 \\ i\,\mathrm{sgn}\,\omega & 0 \end{pmatrix}.$$

Therefore the Kalman rank condition is satisfied.

The threshold between low and high frequencies is at λ. The corresponding Lyapunov functional reads (for small enough ε):

$$\|(u,v)\|_{L^2}^2 + \varepsilon\lambda^{-1} \int_{\mathbb{R}} v\partial_x u\,dx \qquad \text{in low frequencies}$$
$$\|(u,v)\|_{L^2}^2 + \varepsilon\lambda \int_{\mathbb{R}} v\,|D|^{-2}\partial_x u\,dx \qquad \text{in high frequencies.}$$

There is parabolic smoothing with diffusion λ^{-1} on the whole solution (u,v) in low frequency, and exponential decay with parameter λ for high frequencies.

For A and B two given functions, consider

$$\begin{cases} \partial_t u + A\partial_x u + \partial_x v = 0 \\ \partial_t v + B\partial_x v + \partial_x u + \lambda v = 0. \end{cases}$$

Applying $\dot\Delta_j$ to the system, we get

$$\begin{cases} \partial_t \dot\Delta_j u + \dot S_{j-1} A\partial_x \dot\Delta_j u + \partial_x \dot\Delta_j v = R_j(A,u) \\ \partial_t \dot\Delta_j v + \dot S_{j-1} B\partial_x \dot\Delta_j v + \partial_x \dot\Delta_j u + \lambda \dot\Delta_j v = R_j(B,v) \end{cases}$$

where the terms $R_j(A,u)$ and $R_j(B,v)$ may be estimated as follows:

$$\sum_j \|\dot R_j(C,w)\|_{L^2} \leq C2^{-js} \|\nabla C\|_{\dot B^{\frac{d}{2}}_{2,1}} \|w\|_{\dot B^s_{2,1}} \quad \text{if} \quad -d/2 < s \leq d/2.$$

For A and B two given functions, consider

$$\begin{cases} \partial_t u + A\partial_x u + \partial_x v = 0 \\ \partial_t v + B\partial_x v + \partial_x u + \lambda v = 0. \end{cases}$$

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$$\begin{cases} \partial_t \dot\Delta_j u + \dot S_{j-1} A\partial_x \dot\Delta_j u + \partial_x \dot\Delta_j v = R_j(A,u) \\ \partial_t \dot\Delta_j v + \dot S_{j-1} B\partial_x \dot\Delta_j v + \partial_x \dot\Delta_j u + \lambda \dot\Delta_j v = R_j(B,v) \end{cases}$$

where the terms $R_j(A,u)$ and $R_j(B,v)$ may be estimated as follows:

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For the localized system, the relevant Lyapunov functionals read:

$$\begin{aligned} &\|(\dot\Delta_j u, \dot\Delta_j v)\|_{L^2}^2 + \varepsilon\lambda^{-1} \int_{\mathbb{R}} \dot\Delta_j v\, \partial_x \dot\Delta_j u\, dx && \text{if } 2^j \leq \lambda \\ &\|(\dot\Delta_j u, \dot\Delta_j v)\|_{L^2}^2 + \varepsilon\lambda \int_{\mathbb{R}} \dot\Delta_j v\, |D|^{-2} \partial_x \dot\Delta_j u\, dx && \text{if } 2^j > \lambda. \end{aligned}$$

For A and B two given functions, consider

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Taking ε small enough (independently of A and B) yields

$$\begin{aligned} \|(\dot\Delta_j u, \dot\Delta_j v)(t)\|_{L^2} + \min(\lambda, \lambda^{-1}2^{2j})\|(\dot\Delta_j u, \dot\Delta_j v)\|_{L^1_t(L^2)} \leq C\biggl(\|(\dot\Delta_j u_0, \dot\Delta_j v_0)\|_{L^2} \\ + \int_0^t \|(R_j(A,u), R_j(B,v))\|_{L^2}\,d\tau + \int_0^t \|(\nabla A, \nabla B)\|_{L^\infty}\|(\dot\Delta_j u, \dot\Delta_j v)\|_{L^2}\,d\tau\biggr), \end{aligned}$$

For A and B two given functions, consider

$$\begin{cases} \partial_t u + A\partial_x u + \partial_x v = 0 \\ \partial_t v + B\partial_x v + \partial_x u + \lambda v = 0. \end{cases}$$

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$$\|(\dot\Delta_j u, \dot\Delta_j v)(t)\|_{L^2} + \min(\lambda, \lambda^{-1}2^{2j})\|(\dot\Delta_j u, \dot\Delta_j v)\|_{L^1_t(L^2)} \leq C\biggl(\|(\dot\Delta_j u_0, \dot\Delta_j v_0)\|_{L^2} + \int_0^t \|(R_j(A,u), R_j(B,v))\|_{L^2}\,d\tau + \int_0^t \|(\nabla A, \nabla B)\|_{L^\infty}\|(\dot\Delta_j u, \dot\Delta_j v)\|_{L^2}\,d\tau\biggr),$$

whence, for $-d/2 < s \leq d/2$,

$$\|(u,v)(t)\|_{\dot B^s_{2,1}} + \lambda\int_0^t \Bigl(\lambda^{-2}\|(u,v)\|^\ell_{\dot B^{s+2}_{2,1}} + \|(u,v)\|^h_{\dot B^s_{2,1}}\Bigr)\,d\tau \leq C\biggl(\|(u_0,v_0)\|_{\dot B^s_{2,1}} + \int_0^t \|\nabla(A,B)\|_{\dot B^{\frac d2}_{2,1}}\|(u,v)\|_{\dot B^s_{2,1}}\,d\tau\biggr).$$

Similar estimates may be proved in any Besov space $\dot B^\sigma_{2,r}$ with $|\sigma| < d/2$.

Part 4. Global existence results for the compressible NS equations

The linearized Navier-Stokes equations read:

$$\begin{cases} \partial_t a + \operatorname{div} u = 0 \\ \partial_t u - \mu\Delta u - \mu'\nabla\operatorname{div} u + \nabla a = 0; \end{cases} \qquad \mu > 0 \ \text{ and } \ \nu := \mu + \mu' > 0.$$

Part 4. Global existence results for the compressible NS equations

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We may apply the former results with $n = d+1,\ \alpha = 1,\ \beta = 2,\ \kappa = \nu^{-1}$.
Let $\widetilde{\mu} = \mu/\nu$ and $\widetilde{\mu}' = \mu'/\nu$. We have

$$A_\omega = \begin{pmatrix} 0 & i\,\mathrm{sgn}\,\vec{\omega} \\ i^T\mathrm{sgn}\,\vec{\omega} & 0 \end{pmatrix} \quad \text{and} \quad B_\omega = \begin{pmatrix} 0 & 0 \\ 0 & \widetilde{\mu} I_d + (\widetilde{\mu} + \widetilde{\mu}')\,\mathrm{sgn}\,\vec{\omega} \otimes \mathrm{sgn}\,\vec{\omega} \end{pmatrix}.$$

Hence

$$B_\omega A_\omega = \begin{pmatrix} 0 & 0 \\ i\,\mathrm{sgn}\,\vec{\omega} & 0 \end{pmatrix}$$

and the Kalman rank condition is satisfied.

Part 4. Global existence results for the compressible NS equations

The linearized Navier-Stokes equations read:

$$\begin{cases} \partial_t a + \operatorname{div} u = 0 \\ \partial_t u - \mu \Delta u - \mu' \nabla \operatorname{div} u + \nabla a = 0; \end{cases} \qquad \mu > 0 \ \text{ and } \ \nu := \mu + \mu' > 0.$$

We thus get

$$\|(\dot{\Delta}_j a, \dot{\Delta}_j u)(t)\|_{L^2} \leq e^{c\nu t \min(2^{2j}, \nu^{-2})} \|(\dot{\Delta}_j a_0, \dot{\Delta}_j u_0)\|_{L^2}.$$

In low frequencies $2^j \nu \leq 1$, we have parabolic smoothing (with diffusion ν) for a and u and the corresponding Lyapunov functional reads

$$\|(a,u)\|_{L^2}^2 + \varepsilon\nu \int_{\mathbb{R}^d} u \cdot \nabla a \, dx \quad \text{with } \ \varepsilon \ \text{ small enough.}$$

In high frequency, we get exponential decay. Parabolic smoothing may be recovered afterward by using the global L^1-in-time bound for ∇a, and estimates for the Lamé system: indeed

$$\partial_t u - \mu \Delta u - \mu' \nabla \operatorname{div} u = -\nabla a.$$

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$$\partial_t u - \mu\Delta u - \mu'\nabla \operatorname{div} u = -\nabla a.$$

As for the toy dissipative model, one may include a convection term in the analysis, which eventually leads to the following statement:

Theorem (R.D., 2000)

Assume that $P'(1)>0$, and that $a_0 \in \dot B^{\frac d2-1}_{2,1}\cap \dot B^{\frac d2}_{2,1}$ and $u_0\in \dot B^{\frac d2-1}_{2,1}$ are small enough. Then (9) has a unique global-in-time solution (a,u) with

$$a^\ell, u \in \mathcal{C}_b(\dot B^{\frac d2-1}_{2,1})\cap L^1(\dot B^{\frac d2+1}_{2,1}) \quad \text{and} \quad a^h \in \mathcal{C}_b(\dot B^{\frac d2}_{2,1})\cap L^1(\dot B^{\frac d2}_{2,1}).$$

Optimal decay estimates for high frequencies

We now present another approach leading to a global statement in Besov spaces related to L^p $(p\neq 2)$.

Equation for the *divergence-free part* $\mathcal{P}u$ of the velocity:

$$\partial_t\mathcal{P}u-\mu\Delta\mathcal{P}u=0.$$

Effective velocity : $w:=\mathcal{Q}u+\nu^{-1}(-\Delta)^{-1}\nabla a.$ We get

$$\begin{cases}\partial_t\nabla a+\nu^{-1}\nabla a=-\Delta w\\ \partial_t w-\nu\Delta w=\nu^{-1}w-\nu^{-2}(-\Delta)^{-1}\nabla a.\end{cases}$$

Therefore for any $j\in\mathbb{Z}$ and $p\in[1,+\infty]$,

$$\nu\|\dot\Delta_j\nabla a\|_{L^\infty_t(L^p)}+\|\nabla\dot\Delta_j a\|_{L^1_t(L^p)}\lesssim\nu\|\dot\Delta_j\nabla a_0\|_{L^p}+\nu 2^{2j}\|\dot\Delta_j w\|_{L^1_t(L^p)}$$
$$\|\dot\Delta_j w\|_{L^\infty_t(L^p)}+\nu 2^{2j}\|\dot\Delta_j w\|_{L^1_t(L^p)}\lesssim\|\dot\Delta_j w_0\|_{L^p}$$
$$+(\nu 2^j)^{-2}\big(\nu 2^{2j}\|\dot\Delta_j w\|_{L^1_t(L^p)}+\|\nabla\dot\Delta_j a\|_{L^1_t(L^p)}\big).$$

Hence, if $\nu 2^j$ is large enough,

$$\|\nu\dot\Delta_j\nabla a\|_{L^\infty_t(L^p)}+\|\nabla\dot\Delta_j a\|_{L^1_t(L^p)}\lesssim\nu\|\dot\Delta_j\nabla a_0\|_{L^p}+\|\dot\Delta_j w_0\|_{L^p}$$
$$\|\dot\Delta_j w\|_{L^\infty_t(L^p)}+\nu 2^{2j}\|\dot\Delta_j w\|_{L^1_t(L^p)}\lesssim\nu\|\dot\Delta_j\nabla a_0\|_{L^p}+\|\dot\Delta_j w_0\|_{L^p}.$$

Summary

In low frequency, the linearized equations tend to be hyperbolic (two eigenvalues with nonzero imaginary part). Hence it is hopeless to take a L^p framework with $p \neq 2$.

In high frequency, the fundamental observations are that, at the linear level:

- $\mathcal{P}u$ satisfies a heat equation;
- the effective velocity $w := \mathcal{Q}u + \nu^{-1}(-\Delta)^{-1}\nabla a$ has parabolic smoothing;
- a has exponential decay.

The only remaining difficulty is that we have to take care of the convection term $u \cdot \nabla a$ in the mass equation so as to avoid a loss of one derivative.

Step 1. Effective velocity

The effective velocity $w := \mathcal{Q}u + \nu^{-1}G'(0)(-\Delta)^{-1}\nabla a$ satisfies:

$$\partial_t w - \nu\Delta w = -\mathcal{Q}(u\cdot\nabla u) - \mathcal{Q}(J(a)\mathcal{A}u) \\ +(G'(0)-G'(a))\nabla a) - \nu^{-1}G'(0)(-\Delta)^{-1}\nabla((1+a)\mathrm{div}\, u).$$

The blue terms are quadratic hence small (if we start with small data). The red term has a linear part. So using regularity estimates for the heat equation yields

$$\|w\|_{L^\infty(\dot B^{\frac dp -1}_{p,1})} + \nu\|w\|_{L^1(\dot B^{\frac dp +1}_{p,1})} \lesssim \|w_0\|_{\dot B^{\frac dp -1}_{p,1}} + \nu^{-1}\|\mathcal{Q}u\|_{L^1(\dot B^{\frac dp -1}_{p,1})} + \text{quadratic}.$$

The red term has not the right scaling. It has two extra derivatives, hence it is good in high frequencies: if we put the threshold between low and high frequencies at j_0 s.t. $1 \ll 2^{j_0}\nu$ then

$$\nu^{-1}\|\mathcal{Q}u\|^h_{L^1(\dot B^{\frac dp -1}_{p,1})} \leq \nu^{-1}2^{-2j_0}\|\mathcal{Q}u\|^h_{L^1(\dot B^{\frac dp +1}_{p,1})} \ll \nu\|\mathcal{Q}u\|^h_{L^1(\dot B^{\frac dp +1}_{p,1})}.$$

Hence, because $\mathcal{Q}u = w - \nu^{-1}G'(0)(-\Delta)^{-1}\nabla a$,

$$\|w\|^h_{L^\infty(\dot B^{\frac dp -1}_{p,1})} + \nu\|w\|^h_{L^1(\dot B^{\frac dp +1}_{p,1})} \lesssim \|w_0\|^h_{\dot B^{\frac dp -1}_{p,1}} + \nu^{-2}G'(0)\|a\|^h_{L^1(\dot B^{\frac dp -2}_{p,1})} + \text{quadratic}.$$

The red term is very small compared to $\|a\|^h_{L^1(\dot B^{\frac dp}_{p,1})}$.

Step 2. Parabolic estimates for $\mathcal{P}u$.
Because

$$\partial_t \mathcal{P}u + \mathcal{P}(u\cdot\nabla u) - \mu\mathcal{P}u = -\mathcal{P}(J(a)\mathcal{A}u),$$

we readily have

$$\|\mathcal{P}u\|_{L^\infty(\dot B^{\frac{d}{p}-1}_{p,1})} + \mu\|\mathcal{P}u\|_{L^1(\dot B^{\frac{d}{p}+1}_{p,1})} \lesssim \|\mathcal{P}u_0\|_{\dot B^{\frac{d}{p}-1}_{p,1}} + \text{quadratic}.$$

Step 3. Decay estimates for a.
We notice that

$$\partial_t a + u\cdot\nabla a + \nu^{-1}G'(0)a = -a\,\mathrm{div}\,u - \mathrm{div}\,w.$$

As $G'(0)>0$, estimates for transport equation (with damping) imply if $\|u\|_{L^1(\dot B^{\frac dp+1}_{p,1})}$ is small enough, that

$$\|a\|^h_{L^\infty_t(\dot B^{\frac dp}_{p,1})} + \nu^{-1}\|a\|^h_{L^1_t(\dot B^{\frac dp}_{p,1})} \lesssim \|a_0\|^h_{\dot B^{\frac dp}_{p,1}} + \|\mathrm{div}\,w\|^h_{L^1(\dot B^{\frac dp}_{p,1})} + \text{quadratic}. \tag{17}$$

Recall that

$$\|w\|^h_{L^\infty(\dot B^{\frac dp-1}_{p,1})} + \nu\|w\|^h_{L^1(\dot B^{\frac dp+1}_{p,1})} \lesssim \|w_0\|^h_{\dot B^{\frac dp-1}_{p,1}} + (\nu 2^{j_0})^{-2}\nu^{-1}\|a\|^h_{L^1(\dot B^{\frac dp}_{p,1})} + \text{small}. \tag{18}$$

Hence plugging (17) in (18) and taking j_0 large enough, we deduce that

$$\|w\|^h_{L^\infty(\dot B^{\frac dp-1}_{p,1})} + \nu\|w\|^h_{L^1(\dot B^{\frac dp+1}_{p,1})} + \|a\|^h_{L^\infty_t(\dot B^{\frac dp}_{p,1})} + \nu^{-1}\|a\|^h_{L^1_t(\dot B^{\frac dp}_{p,1})} \lesssim \|w_0\|^h_{\dot B^{\frac dp-1}_{p,1}} + \|a_0\|^h_{\dot B^{\frac dp}_{p,1}} + \text{quadratic}.$$

As $\mathcal{Q}u = w - \nu^{-1}G'(0)(-\Delta)^{-1}\nabla a$, one may replace w by $\mathcal{Q}u$ in the above inequality.

Step 4. Low frequency estimates.

As explained before, we have to restrict to Besov spaces $\dot B^s_{2,1}$. By taking advantage of method for partially parabolic systems, we get:

$$\|(a,u)\|^\ell_{L^\infty_t(\dot B^{\frac d2-1}_{2,1})}+\|(a,u)\|^\ell_{L^1_t(\dot B^{\frac d2+1}_{2,1})}\lesssim \|(a_0,u_0)\|^\ell_{\dot B^{\frac d2-1}_{2,1}}+\text{quadratic.}$$

Step 5. Global estimate.

$$X(t):=\|(a,u)\|^\ell_{L^\infty_t(\dot B^{\frac d2-1}_{2,1})\cap L^1_t(\dot B^{\frac d2+1}_{2,1})}+\|a\|^h_{L^\infty_t\cap L^1_t(\dot B^{\frac dp}_{p,1})}+\|u\|^h_{L^\infty_t(\dot B^{\frac dp-1}_{p,1})\cap L^1_t(\dot B^{\frac dp+1}_{p,1})}.$$

All the nonlinear terms may be bounded by $X^2(t)$ (split them into low and high frequencies) provided $p<2d$ and $p\leq 4$. We eventually get

$$X\leq C\bigl(X(0)+X^2\bigr). \tag{19}$$

Now it is clear that as long as

$$2CX(t)\leq 1, \tag{20}$$

the above blue inequality ensures that

$$X(t)\leq 2CX(0). \tag{21}$$

Using a bootstrap argument, one may conclude that if $X(0)$ is small enough then (20) is satisfied as long as the solution exists. Hence the red inequality holds. □

Theorem

Let $p\in[2,2d)\cap[2,4]$. *Assume that* $P'(1)>0$, $a_0\in\dot B^{\frac dp}_{p,1}$ *and* $u_0\in\dot B^{\frac dp-1}_{p,1}$ *and that in addition* a_0^ℓ *and* u_0^ℓ *are in* $\dot B^{\frac d2-1}_{2,1}$. *There exist two constants* c *and* M *depending only on* d, *and on the parameters of the system such that if*

$$\|(a_0,u_0)\|^\ell_{\dot B^{\frac d2-1}_{2,1}}+\|a_0\|^h_{\dot B^{\frac dp}_{p,1}}+\|u_0\|^h_{\dot B^{\frac dp-1}_{p,1}}\leq c$$

then (9) *has a unique global-in-time solution* (a,u) *with*

$$(a,u)^\ell\in\mathcal{C}_b(\dot B^{\frac d2-1}_{2,1})\cap L^1(\dot B^{\frac d2+1}_{2,1}),\quad a^h\in\mathcal{C}_b(\dot B^{\frac dp}_{p,1})\cap L^1(\dot B^{\frac dp}_{p,1}),$$
$$u^h\in\mathcal{C}_b(\dot B^{\frac dp-1}_{p,1})\cap L^1(\dot B^{\frac dp+1}_{p,1}).$$

Theorem

Let $p \in [2, 2d) \cap [2, 4]$. Assume that $P'(1) > 0$, $a_0 \in \dot{B}^{\frac{d}{p}}_{p,1}$ and $u_0 \in \dot{B}^{\frac{d}{p}-1}_{p,1}$ and that in addition a_0^ℓ and u_0^ℓ are in $\dot{B}^{\frac{d}{2}-1}_{2,1}$. There exist two constants c and M depending only on d, and on the parameters of the system such that if

$$\|(a_0, u_0)\|^\ell_{\dot{B}^{\frac{d}{2}-1}_{2,1}} + \|a_0\|^h_{\dot{B}^{\frac{d}{p}}_{p,1}} + \|u_0\|^h_{\dot{B}^{\frac{d}{p}-1}_{p,1}} \leq c$$

then (9) has a unique global-in-time solution (a, u) with

$$(a, u)^\ell \in \mathcal{C}_b(\dot{B}^{\frac{d}{2}-1}_{2,1}) \cap L^1(\dot{B}^{\frac{d}{2}+1}_{2,1}), \quad a^h \in \mathcal{C}_b(\dot{B}^{\frac{d}{p}}_{p,1}) \cap L^1(\dot{B}^{\frac{d}{p}}_{p,1}),$$
$$u^h \in \mathcal{C}_b(\dot{B}^{\frac{d}{p}-1}_{p,1}) \cap L^1(\dot{B}^{\frac{d}{p}+1}_{p,1}).$$

- The first global existence result of strong solutions has been established by Matsumura and Nishida in 1980 (high Sobolev regularity).
- The above statement has been first proved independently in a joint work with Charve and by Chen, Miao and Z. Zhang, in 2009.
- Here we adopted Haspot's method (2010).

Theorem

Let $p\in[2,2d)\cap[2,4]$. *Assume that* $P'(1)>0$, $a_0\in\dot B^{\frac dp}_{p,1}$ *and* $u_0\in\dot B^{\frac dp-1}_{p,1}$ *and that in addition* a_0^ℓ *and* u_0^ℓ *are in* $\dot B^{\frac d2-1}_{2,1}$. *There exist two constants* c *and* M *depending only on* d, *and on the parameters of the system such that if*

$$\|(a_0,u_0)\|^\ell_{\dot B^{\frac d2-1}_{2,1}}+\|a_0\|^h_{\dot B^{\frac dp}_{p,1}}+\|u_0\|^h_{\dot B^{\frac dp-1}_{p,1}}\leq c$$

then (9) *has a unique global-in-time solution* (a,u) *with*

$$(a,u)^\ell\in\mathcal C_b(\dot B^{\frac d2-1}_{2,1})\cap L^1(\dot B^{\frac d2+1}_{2,1}),\quad a^h\in\mathcal C_b(\dot B^{\frac dp}_{p,1})\cap L^1(\dot B^{\frac dp}_{p,1}),$$
$$u^h\in\mathcal C_b(\dot B^{\frac dp-1}_{p,1})\cap L^1(\dot B^{\frac dp+1}_{p,1}).$$

- The smallness condition is satisfied for small densities and large highly oscillating velocities: take $u_0^\varepsilon: x\mapsto\phi(x)\sin(\varepsilon^{-1}x\cdot\omega)\,n$ with ω and n in $\mathbb S^{d-1}$ and $\phi\in\mathcal S(\mathbb R^d)$. Then

$$\|u_0^\varepsilon\|_{\dot B^{\frac dp-1}_{p,1}}\leq C\varepsilon^{1-\frac dp}\quad\text{if }\ p>d.$$

Hence such data with small enough ε generate global unique solutions.

The incompressible limit issue

We now want to study the convergence of the barotropic Navier-Stokes equations when the Mach number ε tends to 0.

The relevant time scale is $1/\varepsilon$, hence one makes the following rescaling

$$(\rho, u)(t, x) = (\rho^\varepsilon, \varepsilon u^\varepsilon)(\varepsilon t, x)$$

and the original system (9) becomes

$$\begin{cases} \partial_t \rho^\varepsilon + \operatorname{div}(\rho^\varepsilon u^\varepsilon) = 0, \\ \partial_t(\rho^\varepsilon u^\varepsilon) + \operatorname{div}(\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon) - \mu\Delta u^\varepsilon - \mu'\nabla \operatorname{div} u^\varepsilon + \dfrac{\nabla P^\varepsilon}{\varepsilon^2} = 0. \end{cases}$$

In the case of well-prepared data:

$$\rho_0^\varepsilon = 1 + \mathcal{O}(\varepsilon^2) \quad \text{and} \quad u_0^\varepsilon \ \text{ with } \ \operatorname{div} u_0^\varepsilon = \mathcal{O}(\varepsilon),$$

one may use asymptotic expansions to show that the solutions to the above system tend to the solution to the incompressible Navier-Stokes equations when ε goes to 0.

In the case of ill-prepared data, time derivatives are of order ε^{-1} and highly oscillating acoustic waves do have to be considered. Whether they may interact or not is the main problem from a mathematical viewpoint. This is the question that we want to address now in the whole space framework.

The incompressible limit issue

So we consider data

$$\rho_0^\varepsilon = 1 + \varepsilon b_0 \quad \text{and} \quad u_0$$

with (b_0, u_0) independent of ε (just to simplify). Note that it is not assumed that $\operatorname{div} u_0 = 0$. We still assume that $P'(1) = 1$.

Denoting $\rho^\varepsilon = 1 + \varepsilon b^\varepsilon$, it is found that $(b^\varepsilon, u^\varepsilon)$ satisfies

$$(NSC_\varepsilon) \qquad \begin{cases} \partial_t b^\varepsilon + \dfrac{\operatorname{div} u^\varepsilon}{\varepsilon} = -\operatorname{div}(b^\varepsilon u^\varepsilon), \\ \partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon - \dfrac{\mathcal{A} u^\varepsilon}{1 + \varepsilon b^\varepsilon} + (1 + k(\varepsilon b^\varepsilon)) \dfrac{\nabla b^\varepsilon}{\varepsilon} = 0, \\ (b^\varepsilon, u^\varepsilon)_{|t=0} = (b_0, u_0), \end{cases}$$

with $\mathcal{A} := \mu\Delta + \mu'\nabla\operatorname{div}$ and k a smooth function satisfying $k(0) = 0$.

According to the previous parts, System (NSC_ε) is locally well-posed. We want to study whether u^ε tends in some sense to the solution v of the incompressible Navier-Stokes equations:

$$(NS) \qquad \begin{cases} \partial_t v + \mathcal{P}(v \cdot \nabla v) - \mu\Delta v = 0, \\ v_{|t=0} = \mathcal{P} u_0. \end{cases}$$

For simplicity, we restrict to the case of small data (b_0, u_0). Hence all the results will be global-in-time.

Scheme of the proof

The proof of the convergence to the incompressible Navier-Stokes equations comprises four steps:

1. Global existence and uniform estimates for (NSC_ε);
2. Global existence for the corresponding limit system (NS);
3. Convergence to 0 for the "compressible part" of the solution, namely $(b^\varepsilon, \mathcal{Q}u^\varepsilon)$;
4. Convergence of $\mathcal{P}u^\varepsilon$ to the solution v of (NS).

Step 1. Global existence for (NSC_ε) and uniform estimates.

Making the change of functions

$$b(t,x) := \varepsilon b^\varepsilon(\varepsilon^2 t, \varepsilon x),\ \ u(t,x) := \varepsilon u^\varepsilon(\varepsilon^2 t, \varepsilon x)$$

we notice that $(b^\varepsilon, u^\varepsilon)$ solves (NSC_ε) if and only if (b,u) solves (NSC) with rescaled data $b_0 := \varepsilon b_0(\varepsilon\cdot)$, $u_0 := \varepsilon u_0(\varepsilon\cdot)$ and h^ε. Hence the global existence theorem for (NSC) with $p=2$ ensures the first part of the theorem. We get a global solution $(b^\varepsilon, u^\varepsilon)$ such that

$$\|b^\varepsilon\|^\ell_{L^\infty(\dot B^{\frac d2-1}_{2,1})\cap L^1(\dot B^{\frac d2+1}_{2,1})} + \varepsilon\|b^\varepsilon\|^h_{L^\infty(\dot B^{\frac d2}_{2,1})} + \varepsilon^{-1}\|b^\varepsilon\|^h_{L^1(\dot B^{\frac d2}_{2,1})}$$
$$+\|u^\varepsilon\|_{L^\infty(\dot B^{\frac d2-1}_{2,1})\cap L^1(\dot B^{\frac d2+1}_{2,1})} \leq M\big(\|b_0\|^\ell_{\dot B^{\frac d2-1}_{2,1}} + \varepsilon\|b_0\|^h_{\dot B^{\frac d2}_{2,1}} + \|u_0\|_{\dot B^{\frac d2-1}_{2,1}}\big).$$

Warning: here the threshold between low and high frequencies is at ε^{-1}.

Step 2. Global existence for (NS).

As u_0 is small, one may apply the global existence result for small data. We get a (small) solution $v \in \mathcal{C}_b(\mathbb{R}^+; \dot B^{\frac d2-1}_{2,1}) \cap L^1(\mathbb{R}^+; \dot B^{\frac d2+1}_{2,1})$.

Step 3. Convergence to zero for the compressible modes $(b^\varepsilon, \mathcal{Q}u^\varepsilon)$.

The functions b and u defined above satisfy

$$\begin{cases} \partial_t b + \operatorname{div} \mathcal{Q}u = F := -\operatorname{div}(bu), \\ \partial_t \mathcal{Q}u + \nabla b = G := -\mathcal{Q}\left(u\cdot\nabla u + \frac{1}{1+b}\mathcal{A}u + K(b)\nabla b\right). \end{cases} \tag{22}$$

The left-hand side is the acoustic wave equation, which has dispersive properties in the whole space $\mathbb{R}^d$ (with $d \geq 2$). This will be the key to our convergence result.

A short digression on dispersive equations and Strichartz estimates

Let $(U(t))_{t\in\mathbb{R}}$ be a group of unitary operators on $L^2(\mathbb{R}^d)$ satisfying the *dispersion inequality*:

$$\|U(t)f\|_{L^\infty} \leq \frac{C}{|t|^\sigma}\|f\|_{L^1} \quad \text{for some} \quad \sigma > 0.$$

Example: $\sigma = (d-1)/2$ for the wave equation on $\mathbb{R}^d$, and $\sigma = d/2$ for the Schrödinger equation.

Interpolating between $L^2 \mapsto L^2$ and $L^1 \mapsto L^\infty$, we deduce that

$$\|U(t)f\|_{L^r} \leq \left(\frac{C}{|t|^\sigma}\right)^{\frac{1}{r'}-\frac{1}{r}}\|f\|_{L^{r'}} \quad \text{for all} \quad 2 \leq r \leq \infty.$$

Definition: A couple $(q,r) \in [2,\infty]^2$ is admissible if $1/q + \sigma/r = \sigma/2$ and $(q,r,\sigma) \neq (2,\infty,1)$.

Theorem (Strichartz estimates)

1. *For any admissible couple* (q,r) *we have* $\|U(t)u_0\|_{L^q(L^r)} \leq C\|u_0\|_{L^2}$;
2. *For any admissible couples* (q_1,r_1) *and* (q_2,r_2) *we have*

$$\left\|\int_0^t U(t-\tau)f(\tau)\,d\tau\right\|_{L^{q_1}(L^{r_1})} \lesssim \|f\|_{L^{q_2'}(L^{r_2'})}.$$

Remark: compared to Sobolev embedding $H^{d(\frac{1}{2}-\frac{1}{r})} \hookrightarrow L^r$, Strichartz estimates provides a gain of $d(\frac{1}{2}-\frac{1}{r}) = \frac{d}{q\sigma}$ derivative.

The $TT^\star$ argument

Lemma ($TT^\star$ argument)

Let $T:\mathcal{H}\to B$ a bounded operator from the Hilbert space $\mathcal{H}$ to the Banach space B and $T^\star : B'\to\mathcal{H}$ the adjoint operator defined by

$$\forall (x,y)\in B'\times\mathcal{H},\ (T^\star x\mid y)_{\mathcal{H}}=\langle x,\overline{Ty}\rangle_{B',B}.$$

Then we have

$$\|TT^\star\|_{\mathcal{L}(B';B)}=\|T\|^2_{\mathcal{L}(\mathcal{H};B)}=\|T^\star\|^2_{\mathcal{L}(B';\mathcal{H})}.$$

We take

$$\mathcal{H}=L^2(\mathbb{R}^d),\quad B=L^q(\mathbb{R};L^r(\mathbb{R}^d)),\quad B'=L^{q'}(\mathbb{R};L^{r'}(\mathbb{R}^d))\ \text{ and }\ T:u_0\longmapsto U(t)u_0.$$

Hence

$$T^\star:\phi\longmapsto\int_{\mathbb{R}}U(-t')\phi(t')\,dt'\quad\text{and}\quad TT^\star:\phi\longmapsto\Big[t\mapsto\int_{\mathbb{R}}U(t-t')\phi(t')\,dt'\Big].$$

Proving the homogeneous Strichartz estimate

We want to prove $\|Tu_0\|_{L^q(L^r)} \leq C\|u_0\|_{L^2}$. According to the $TT^\star$ lemma, it is equivalent to

$$\|TT^\star\phi\|_{L^q(L^r)} \leq C\|\phi\|_{L^{q'}(L^{r'})}. \tag{23}$$

Now, we have

$$\|TT^\star\phi(t)\|_{L^r} \leq \int_{\mathbb{R}} \|U(t-t')\phi(t')\|_{L^r}\,dt.$$

So taking advantage of the dispersion inequality $L^{r'} \to L^r$ and of the relation $\sigma(\frac{1}{r'}-\frac{1}{r}) = \frac{2}{q}$, we get

$$\|TT^\star\phi(t)\|_{L^r} \leq \int_{\mathbb{R}} \frac{1}{|t-t'|^{\frac{2}{q}}}\|\phi(t')\|_{L^{r'}}\,dt.$$

Applying the Hardy-Littlewood-Sobolev inequality gives (23) if $2 < q < \infty$.

Remarks:

1. Endpoint $(q,r) = (\infty, 2)$ stems from the fact that $(U(t))_{t\in\mathbb{R}}$ is unitary on L^2. Endpoint $(q,r) = (2, 2\sigma/(\sigma-1))$ if $\sigma > 1$ is more involved (Keel & Tao).
2. The nonhomogeneous Strichartz inequality follows from similar arguments.
3. In the case of the linear wave or Schrödinger equation, using $(\dot{\Delta}_j)_{j\in\mathbb{Z}}$ allows to get Strichartz estimates involving Besov norms.

Application to the acoustic wave equations

The system

$$\begin{cases} \partial_t b + \operatorname{div} \mathcal{Q}u = F := -\operatorname{div}(bu), \\ \partial_t \mathcal{Q}u + \nabla b = G := -\mathcal{Q}\left(u \cdot \nabla u + \frac{1}{1+b}\mathcal{A}u + K(b)\nabla b\right). \end{cases} \tag{24}$$

is associated to a group $U(t)$ of *unitary operators on* $L^2(\mathbb{R}^d)$ which satisfies the dispersion inequality

$$\|U(t)(b_0, v_0)\|_{L^\infty} \leq Ct^{-\frac{d-1}{2}}\|(b_0, v_0)\|_{L^1}.$$

Hence Strichartz estimates are available for this system if $d \geq 2$.

Application to the acoustic wave equations

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$$\left\{\begin{array}{l}\partial_t b + \mathrm{div}\,\mathcal{Q}u = F := -\mathrm{div}\,(bu),\\ \partial_t \mathcal{Q}u + \nabla b = G := -\mathcal{Q}\left(u\cdot\nabla u + \frac{1}{1+b}\mathcal{A}u + K(b)\nabla b\right).\end{array}\right. \tag{24}$$

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$$\|U(t)(b_0,v_0)\|_{L^\infty} \leq Ct^{-\frac{d-1}{2}}\|(b_0,v_0)\|_{L^1}.$$

Hence Strichartz estimates are available for this system if $d\geq 2$.

Localizing (24) by means of $(\dot\Delta_j)_{j\in\mathbb{Z}}$, we get

$$\|(b,\mathcal{Q}u)\|_{\widetilde{L}^r_T(\dot B^{\frac{d}{p}-1+\frac{1}{r}}_{p,1})} \lesssim \|(b_0,\mathcal{Q}u_0)\|_{\dot B^{\frac{d}{2}-1}_{2,1}} + \|(F,G)\|_{L^1_T(\dot B^{\frac{d}{2}-1}_{2,1})}$$

whenever $2\leq p\leq 2\left(\dfrac{d-1}{d-3}\right)$, $\quad \dfrac{2}{r}=(d-1)\left(\dfrac12-\dfrac1p\right)$ and $\quad (r,p,d)\neq(2,\infty,3)$.

Combining product laws and the global a priori estimate for (b,u) gives

$$\|(F,G)\|_{L^1(\dot B^{\frac{d}{2}-1}_{2,1})} \leq CC_0.$$

Application to the acoustic wave equations

Localizing (24) by means of $(\dot\Delta_j)_{j\in\mathbb{Z}}$, we get

$$\|(b,\mathcal{Q}u)\|_{\widetilde L^r_T(\dot B^{\frac dp-1+\frac1r}_{p,1})}\lesssim \|(b_0,\mathcal{Q}u_0)\|_{\dot B^{\frac d2-1}_{2,1}}+\|(F,G)\|_{L^1_T(\dot B^{\frac d2-1}_{2,1})}$$

whenever $2\leq p\leq 2\Big(\dfrac{d-1}{d-3}\Big),\quad \dfrac2r=(d-1)\Big(\dfrac12-\dfrac1p\Big)$ and $(r,p,d)\neq(2,\infty,3).$

Combining product laws and the global a priori estimate for (b,u) gives

$$\|(F,G)\|_{L^1(\dot B^{\frac d2-1}_{2,1})}\leq CC_0.$$

Eventually, resuming to the initial variables, we end up with

$$\|(b^\varepsilon,\mathcal{Q}u^\varepsilon)\|_{\widetilde L^r(\dot B^{\frac dp-1+\frac1r}_{p,1})}\leq CC_0\varepsilon^{\frac1r},$$

where

- $p=2(d-1)/(d-3)$ and $r=2$ if $d\geq4$,
- $p\in[2,\infty)$ and $r=2p/(p-2)$ if $d=3$,
- $p\in[2,\infty]$ and $r=4p/(p-2)$ if $d=2$.

Step 4. Convergence of the incompressible part.

The vector-field $w^\varepsilon := \mathcal{P}u^\varepsilon - v$ satisfies

$$\partial_t w^\varepsilon - \mu\Delta w^\varepsilon = H^\varepsilon, \qquad w^\varepsilon_{|t=0} = 0, \tag{24}$$

with

$$H^\varepsilon := -\mathcal{P}(w^\varepsilon\cdot\nabla v) - \mathcal{P}(u^\varepsilon\cdot\nabla w^\varepsilon) - \mathcal{P}(\mathcal{Q}u^\varepsilon\cdot\nabla v) - \mathcal{P}(u^\varepsilon\cdot\nabla\mathcal{Q}u^\varepsilon) - \mathcal{P}\left(J(\varepsilon b^\varepsilon)\mathcal{A}u^\varepsilon\right).$$

There are three types of (quadratic) terms in H^ε:

- The blue terms are linear in w^ε, but small because u^ε and v are small.
- Owing to $\mathcal{Q}u^\varepsilon$, the red terms decay like some power of ε (previous step).
- The green term is small because $J(\varepsilon b^\varepsilon) \sim \varepsilon b^\varepsilon$.

One has to use appropriate norms, keeping in mind that $\nabla v, \nabla\mathcal{Q}u^\varepsilon$ are bounded in e.g. $\widetilde{L}^2(\dot{B}^{\frac{d}{2}}_{2,1})$. For instance, in the (nonphysical !) case $d \geq 4$, one has

$$\|(b^\varepsilon, \mathcal{Q}u^\varepsilon)\|_{\widetilde{L}^2(\dot{B}^{\frac{d}{p}-\frac12}_{p,1})} \leq CC_0\varepsilon^{\frac12} \quad \text{with} \quad p = 2(d-1)/(d-3).$$

Estimates for the heat equation ensure that

$$\|w^\varepsilon\|_{L^1(\dot{B}^{\frac{d}{p}+\frac12}_{p,1})} + \|w^\varepsilon\|_{L^\infty(\dot{B}^{\frac{d}{p}-\frac32}_{p,1})} \lesssim \|H^\varepsilon\|_{L^1(\dot{B}^{\frac{d}{p}-\frac32}_{p,1})} \tag{25}$$

and the above heuristics combined with product laws in Besov spaces leads to

$$\|w^\varepsilon\|_{L^1(\dot{B}^{\frac{d}{p}+\frac12}_{p,1})} + \|w^\varepsilon\|_{L^\infty(\dot{B}^{\frac{d}{p}-\frac32}_{p,1})} \leq CC_0\varepsilon^{\frac12}.$$

Theorem

There exist $\eta, M > 0$ depending only on d and G, such that if

$$C_0 := \|b_0\|_{\dot B^{\frac d2-1}_{2,1}\cap B^{\frac d2}_{2,1}} + \|u_0\|_{\dot B^{\frac d2}_{2,1}} \leq \eta \tag{26}$$

then the following results hold:

1. *System (NSC_ε) has a unique global solution $(b^\varepsilon, u^\varepsilon)$ with*

$$\|b^\varepsilon\|_{L^\infty(\dot B^{\frac d2-1}_{2,1})\cap L^2(\dot B^{\frac d2}_{2,1})} + \varepsilon\|b^\varepsilon\|_{L^\infty(\dot B^{\frac d2}_{2,1})} + \|u^\varepsilon\|_{L^\infty(\dot B^{\frac d2-1}_{2,1})\cap L^1(\dot B^{\frac d2+1}_{2,1})} \leq MC_0$$

2. *the incompressible Navier-Stokes equations (NS) with data $\mathcal{P}u_0$ have a unique solution v with*

$$\|v\|_{L^\infty(\dot B^{\frac d2-1}_{2,1})\cap L^1(\dot B^{\frac d2+1}_{2,1})} \leq MC_0$$

3. *for any $\alpha \in]0,1/2]$ if $d \geq 4$, $\alpha \in]0,1/2[$ if $d = 3$, $\alpha \in]0,1/6]$ if $d = 2$, $\mathcal{P}u^\varepsilon$ tends to v in $\mathcal{C}(\mathbb{R}^+; \dot B^{-1-\alpha}_{\infty,1})$ when ε goes to 0.*

4. *$(b^\varepsilon, \mathcal{Q}u^\varepsilon)$ tends to 0 in some space $L^r(\dot B^\sigma_{p,1})$ (the value of r and p depending on the dimension) with an explicit rate of decay.*

A FEW REFERENCES

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