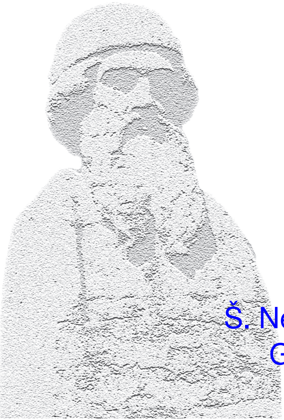
FSI with Application in Hemodynamics
Analysis and Simulation



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$\Longrightarrow$ importance of further detailed study e.g. using computational science

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Functions:

- transport oxygen and nutrients to all tissues
- remove waste products
- defend the body against infection through the action of antibodies

- leukocytes: (1%)
  play a vital role in fighting infection in the body

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  - deformability
  - tendency to align with the flow field at high shear rates

$\Longrightarrow$ effect non-Newtonian behaviour of blood

Motivation

Hemodynamic factors:
flow separation, flow recirculation, and oscillatory wall shear stress:
important role in the localization and development of vascular diseases $\Longrightarrow$

- healthy patient: high shear rate, typical vessel lengths: no time to build microstructures

Newtonian models reasonable approximation

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- disease states: diseases in which the arterial geometry has altered (aneurysms), aggregates get more stable

non-Newtonian models more relevant

Non-Newtonian nature of blood

Blood: plasma + cells (45% volume concentration)

vessel	radius	RE
aorta	1.25	3400
arteries	0.2	500
arterioles	$1.5.10^{-3}$	0.7
veins	0.25	140
vena cava	1.5	3300

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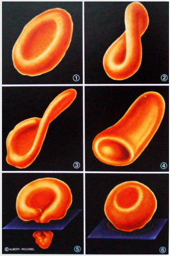

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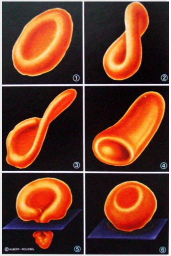

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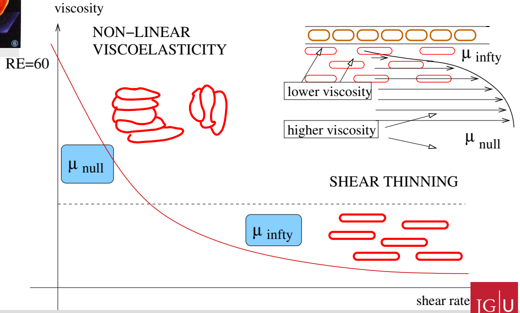
- shear thinning
- viscoelasticity
- yield stress



Characteristic numbers

$$Re = \frac{\rho\, U\, L}{\mu} \qquad \alpha = \frac{L}{2}\sqrt{\frac{\rho\,\omega}{\mu}}$$

Reynolds and Wormesley number

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$\rho \ldots$ density; 1.06×10^{-3} kg $\cdot$ m^{-3},
$U \ldots$ characteristic velocity,
$\boldsymbol{u} = (u_1, \ldots, u_d) \ldots$ fluid velocity
$L \ldots$ characteristic length
$\mu \ldots$ characteristic viscosity; $3 - 5.5$ mPa $\cdot$ s
$\omega \ldots$ characteristic angular frequency
$Re \in (0.0015, 6100) \qquad \alpha \in (0.003, 30)$

Outline

Fluid equations

- shear-thinning properties

$$
\begin{aligned}
\rho\partial_t \boldsymbol{u} + \rho\,(\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u} - \mathrm{div}\,[2\mu(|D(\boldsymbol{u})|)D(\boldsymbol{u})] + \nabla\pi &= 0\\
\mathrm{div}\,\boldsymbol{u} &= 0
\end{aligned}
$$

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Carreau-Yasuda model

$$\mu = \mu(D(\boldsymbol{u})) = \mu_\infty + (\mu_0 - \mu_\infty)(1 + \gamma|D(\boldsymbol{u})|^2)^{\frac{p-2}{2}}$$

Yeleswarapu model

$$\mu = \mu(D(\boldsymbol{u})) = \mu_\infty + (\mu_0 - \mu_\infty)\frac{\log(1 + \gamma|D(\boldsymbol{u})|) + 1}{(1 + \gamma|D(\boldsymbol{u})|)}$$

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- blood:
  - ability to store and release energy (from its branched 3D structures)
  - challenge: developing nonlinear viscoelastic constitutive models for blood
  - the simplest model that captures the shear rate dependence and elasticity the rate-type shear thinning model (Oldroyd-B type model)

$$\tau = -\pi \boldsymbol{I} + \tau^{v} + \tau^{e}$$

$$\lambda \left(\frac{\partial \tau^e}{\partial t} + \boldsymbol{u} \cdot \nabla \tau^e - \tau^e \cdot (\nabla \boldsymbol{u})^T - \nabla \boldsymbol{u} \cdot \tau^e \right) + \tau^e = 2 \left(\frac{\mu(D(\boldsymbol{u}))}{\mu_0} - (1-\alpha) \right) D(\boldsymbol{u})$$

viscoelastic shear-dependent fluids

$$div(\mathbf{u}) = 0$$

$$Re\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla\pi + (1-\alpha)\Delta\mathbf{u} + div(\tau^e) + \mathbf{f}$$

$$We\left(\frac{\partial \tau^e}{\partial t} + \mathbf{u}\cdot\nabla\tau^e - \tau^e\nabla\mathbf{u}^T - \nabla\mathbf{u}\,(\tau^e)\right) + \tau^e =$$
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$\alpha = \frac{\mu_e}{\mu_0}$ ... elastic part of the total viscosity $\mu = \mu(|D(\boldsymbol{u})|)$

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- nonlinear coupled parabolic-hyperbolic system
- singular behaviour for large Reynolds, Weissenberg

Outline

Structure equation

- assume: circular symmetry, deformation only in the radial direction
- radial wall displacement $\eta(x_1, t) = R(x_1, t) - R_0(x_1)$
  $\ldots$ constant w.r.t θ, $\theta \in [0, 2\pi)$
- linear elasticity, i.e. $\partial\eta/\partial_{x_1} \ll 1$

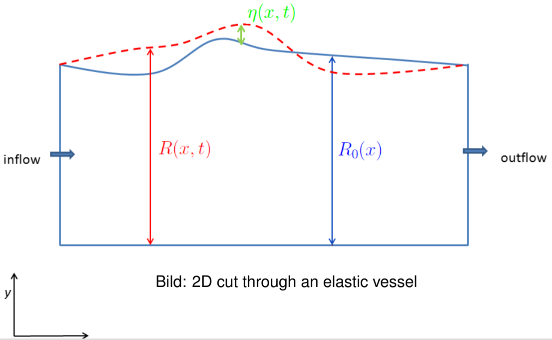

Bild: 2D cut through an elastic vessel

- Newton law: Force = mass $\times$ acceleration

$$F = \rho_w h \; dc \, d\ell \times \frac{\partial^2 \eta}{\partial t^2}$$

- ρ_W wall density
- h wall thickness
- dc infinitesimal length of arc $dc = R d\theta$
- $d\ell$ infinitesimal length in the x_1-direction

- Forces: external and internal

1. External forces

$$f_{ext} = \boldsymbol{f}_{ext} \cdot \boldsymbol{e}_r \approx (-\mathbf{T}_f - P_{ext}\mathbf{I})\boldsymbol{n} \cdot \boldsymbol{e}_r \, dc \, d\ell$$

$\mathbf{T_f}$ ... fluid Cauchy stress tensor on $\Gamma_W := \Gamma_{W(t)}$

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- $\Gamma_{W(t)} := \{(x_1, x_2); x_2 = R_0(x_1) + \eta(x_1)\}$

$$\tilde{\boldsymbol{\eta}}(X_1, X_2) = (0, \tilde{\eta}(X_1, X_2))^T = (0, \eta(x_1))^T$$

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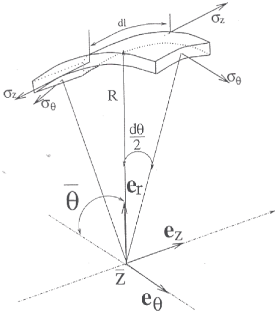
$$
\begin{aligned}
\boldsymbol{f}_{ext}\Big|_{\Gamma_{W_0}} &= \boldsymbol{f}_{ext} \cdot \boldsymbol{e}_r \, dc \, d\ell \\
&= -(\mathbf{T}_f + P_w \mathbf{I})\boldsymbol{n} \cdot \boldsymbol{e}_r \frac{R}{R_0} \frac{\sqrt{1 + (\partial_{x_1} R)^2}}{\sqrt{1 + (\partial_{x_1} R_0)^2}} \, dc_0 \, d\ell_0 \\
&\approx -(\mathbf{T}_f + P_w \mathbf{I})\boldsymbol{n} \cdot \boldsymbol{e}_r \frac{R}{R_0} \, dc_0 \, d\ell_0
\end{aligned}
$$

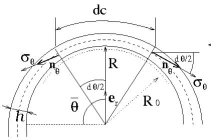
2. Internal forces:

$f_{int} = (\boldsymbol{f}_\theta + \boldsymbol{f}_{x_1}) \cdot \boldsymbol{e}_r$

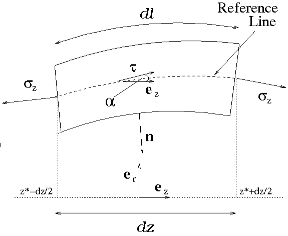
$\boldsymbol{f}_\theta$ ... internal forces due to circumferential stress

$\boldsymbol{f}_{x_1}$ ... internal forces due to longitudal stress





Transversal Section



Longitudinal Section

$$\boldsymbol{f}_\theta \cdot \boldsymbol{e}_r = -E\frac{h\eta}{R_0 R} dc_0 d\ell_0$$

$$\boldsymbol{f}_{x_1} \cdot \boldsymbol{e}_r = |\sigma_{x_1}| h \frac{\partial^2 R}{\partial x_1^2} \left[1 + \left(\frac{\partial R_0}{\partial x_1}\right)^2\right]^{-3/2} \boldsymbol{n} \cdot \boldsymbol{e}_r dc_0 d\ell_0$$

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Assuming:

$$\boldsymbol{n} \cdot \boldsymbol{e}_r \approx \left[1 + (\partial_{x_1} R_0^2)\right]^{-1/2}, \quad \eta/R \approx \eta/R_0$$

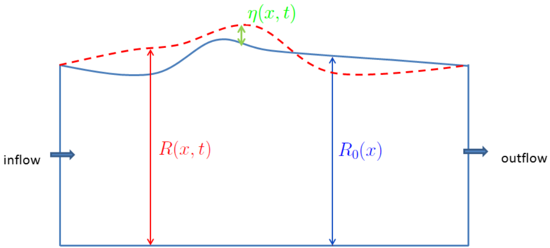
for $|\eta| < |R_0|$
We can rewrite:

$$f_{int} = \left\{ -E\frac{h\eta}{R_0 R} + |\sigma_{x_1}| h \frac{\partial^2 R}{\partial x_1^2} \left[1 + \left(\frac{\partial R_0}{\partial x_1}\right)^2\right]^{-2} \right\} dc_0 d\ell_0$$

Structure equation

$$\frac{\partial^2\eta}{\partial t^2} - \frac{|\sigma_{x_1}|}{\rho_w} \frac{\left(\frac{\partial^2\eta}{\partial x_1^2} + \frac{\partial^2 R_0}{\partial x_1^2}\right)}{\left[1 + \left(\frac{\partial R_0}{\partial x_1}\right)^2\right]^2} + \frac{E\eta}{\rho_w(R_0+\eta)R_0} - c\frac{\partial^\alpha \eta}{\partial t \partial x_1^2}$$

$$= -\frac{(\mathbf{T}_f + P_w \mathbf{I})}{\rho_w h}\boldsymbol{n}\cdot\boldsymbol{e}_r \frac{R_0+\eta}{R_0}\frac{\sqrt{1+(\partial_{x_1}R)^2}}{\sqrt{1+(\partial_{x_1}R_0)^2}}, \qquad \alpha = 3 \text{ or } 5$$



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Structure equation

$$\frac{\partial^2 \eta}{\partial t^2} - \frac{|\sigma_{x_1}|}{\rho_w} \frac{\left(\frac{\partial^2 \eta}{\partial x_1^2} + \frac{\partial^2 R_0}{\partial x_1^2}\right)}{\left[1 + \left(\frac{\partial R_0}{\partial x_1}\right)^2\right]^2} + \frac{E\eta}{\rho_w (R_0 + \eta) R_0} - c \frac{\partial^\alpha \eta}{\partial t \partial x_1^2}$$

$$= -g(R, R_0) \frac{(\mathbf{T}_f \boldsymbol{n} \cdot \boldsymbol{e}_r + P_w)}{\rho_w h}$$

- ρ_W wall density
- h wall thickness
- E the Young modulus of elasticity
- $|\sigma_{x_1}| = E/3$ shear modulus
- c viscoelastic constant
- $g(R, R_0) = \frac{R_0+\eta}{R_0} \frac{\sqrt{1+(\partial_{x_1} R)^2}}{\sqrt{1+(\partial_{x_1} R_0)^2}}$

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Mathematical model

- Fluid equations
- Structure equation
- Fluid-structure interaction

Theoretical result

- Analysis

Fluid-domain interaction, decoupling

- fluid and geometry are coupled

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1. Dirichlet boundary condition

$$\boldsymbol{u}(x_1, R_0 + \eta, t) = \frac{\partial \eta}{\partial t}\boldsymbol{n} \qquad \text{on } \Gamma^w, \tag{1}$$

$\boldsymbol{n}$ is unit outward normal to the domain boundary

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2. $$\mathbf{T}_S \boldsymbol{n} = g(R, R_0)\mathbf{T}_f \boldsymbol{n}$$

   the equation for domain deformation (1).

Further B.C.

- $\Gamma_{in} \ldots$ inflow conditions

$$\begin{aligned} u_2(0, x_2, t) &= 0, \\ \left(2\mu(|D(\boldsymbol{u})|)\frac{\partial u_1}{\partial x_1} - p + P_{in} - \frac{\rho}{2}|u_1|^2\right)(0, x_2, t) &= 0 \end{aligned}$$

$0 < x_2 < R_0(0)$, $0 < t < T$; $P_{in} = P_{in}(x_2, t)$ given;
Γ_{out} analogous

- $\Gamma_c \ldots$ flow symmetry conditions

$$u_2(x_1, 0, t) = 0 \,, \qquad \mu(|D(\boldsymbol{u})|)\frac{\partial u_1}{\partial x_2}(x_1, 0, t) = 0$$

$0 < x_1 < L$, $0 < t < T$

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conditions for structure

$$\eta(0,t) = \eta(L,t) = 0$$
$$\eta_{x_1}(0,t) = \eta_{x_1}(L,t) = 0$$
$$\eta(x_1, 0) = \frac{\partial \eta}{\partial t}(x_1, 0) = 0$$

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I. Analysis

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1. local existence of the strong solutions:
   B. da Veiga ('04)
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4. uniqueness and continuous dependence of the weak solution: Guidoboni, Guidorzi, Padula ('10)
   coupling of the 2D Navier-Stokes eqs. & elastic or viscoelastic plate (friction $\alpha\eta_{txx}$)

5 Čanić, Guidoboni et al. ('06) reduced structural model: viscoelastic Koiter model

$$\rho_w h \frac{\partial^2 \eta}{\partial t^2} - C_1 \frac{\partial^2 \eta}{\partial x^2} + C_2 \frac{\partial^4 \eta}{\partial x^4} + C_0 \eta + D_0 \frac{\partial \eta}{\partial t} - D_1 \frac{\partial^3 \eta}{\partial t \partial x^2} + D_2 \frac{\partial^5 \eta}{\partial t \partial x^4} = f$$

-multiscale analysis of the FSI problem

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-multiscale analysis of the FSI problem

- all of these results are for Newtonian fluid rheology

Non-Newtonian rheology

- We assume existence of $\mathcal{U} \in C^2(\mathbf{R}^{2\times 2})$

$$\begin{aligned} \frac{\partial \mathcal{U}(\eta)}{\partial \eta_{ij}} &= \tau_{ij}(\eta) \qquad \mathcal{U}(0) = \frac{\partial \mathcal{U}(0)}{\partial \eta_{ij}} = 0 \\ \frac{\partial^2 \mathcal{U}(\eta)}{\partial \eta_{mn} \partial \eta_{rs}} \xi_{mn} \xi_{rs} &\geq C_1 (1 + |\eta|)^{p-2} |\xi|^2 \\ \left| \frac{\partial^2 \mathcal{U}(\eta)}{\partial \eta_{ij} \partial \eta_{kl}} \right| &\leq C_2 (1 + |\eta|)^{p-2}. \end{aligned}$$

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$$
\begin{aligned}
\frac{\partial \mathcal{U}(\eta)}{\partial \eta_{ij}} &= \tau_{ij}(\eta) \qquad \mathcal{U}(0) = \frac{\partial \mathcal{U}(0)}{\partial \eta_{ij}} = 0 \\
\frac{\partial^2 \mathcal{U}(\eta)}{\partial \eta_{mn} \partial \eta_{rs}} \xi_{mn} \xi_{rs} &\geq C_1 (1 + |\eta|)^{p-2} |\xi|^2 \\
\left| \frac{\partial^2 \mathcal{U}(\eta)}{\partial \eta_{ij} \partial \eta_{kl}} \right| &\leq C_2 (1 + |\eta|)^{p-2}.
\end{aligned}
$$

Non-linear viscous term: Carreau model, power law model

$$
((\boldsymbol{u}, \psi)) = \int_{\Omega(\eta(t))} \tau_{ij}(D(\boldsymbol{u})) D(\psi), \quad \tau_{ij}(D(\boldsymbol{u})) = \mu(|D(\boldsymbol{u})|) D_{ij}(\boldsymbol{u})
$$

- $((\cdot,\cdot))$ defines a monotone and coercive operator
  - *p*- structure

$$((\boldsymbol{u},\cdot)) : L^p(0,T;W^{1,p}(\Omega)) \to L^{p'}\left(0,T;(W^{-1,p'}(\Omega))\right)$$

- $((\boldsymbol{u},\cdot))$ interplays with the convective term $\boldsymbol{u}\cdot\nabla\boldsymbol{u}$

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$$\int_0^T\int_\Omega \boldsymbol{u}\cdot\nabla\boldsymbol{u}\,\varphi \le C \qquad \Longleftrightarrow \qquad p\ge 11/5(3D);\ p\ge 3/2(2D)$$

first energy estimates:

$$\max_{t\in(0,T]}\|\boldsymbol{u}^n\|^2_{L^2(\Omega)} + \int_0^T\int_\Omega \|D(\boldsymbol{u}^n)\|^p \le C$$

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- theory of monotone operators: Ladyzhenskaja, Lions (existence of weak solutions) $p \geq \frac{11}{5}$ (3D), $p \geq 2$ (2D) (Dirichlet B.C.)

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- -testing by $\Delta \boldsymbol{u}$ & Vitali theorem Nečas, Málek, Ržička, et al. ('93-01)

- existence of global weak solutions for $p > 9/5$ (3D unsteady, per. BC)
- existence of local strong solutions for $p \geq 5/3$ (3D unsteady, per. BC)
- existence of global weak solutions for $p \in [2, 3)$ (3D unsteady, Dir. BC)

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- Diening, Ružička et al. ('05,'07):
  existence of local strong solutions for $p > 7/3$ (3D unsteady, per.BC)

Existence results ... continue

- local pressure method Wolf ('07)

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Existence results . . . continue

- local pressure method Wolf ('07)
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- Lipschitz-truncation & local pressure method
- Diening, Ružička , Wolf ('10):
existence of global weak solutions $p > 6/5$ (3D unsteady, Dir. BC), $p > 1$ (2D)

FSI for non-Newtonian fluids

Theorem (M.L., Hundertmark, Nečasová ('11))

Let $p \geq 2$. Assume $P|_{in,out,w} \in L^{p'}(0, T; L^2(\partial\Omega))$. Then $\exists(\boldsymbol{u}, \eta)$ a weak solution of the nonlinear FSI problem, s.t.

$$\boldsymbol{u} \in L^p(0, T; W^{1,p}(\Omega(\eta(t)))) \cap L^\infty(0, T; L^2(\Omega(\eta(t))))$$
$$\eta \in W^{1,\infty}(0, T; L^2(0, L)) \cap H^1(0, T; H_0^2(0, L)).$$

$$\gamma_{\eta(t)}(\boldsymbol{u}) = (0, \eta_t) \text{ on } \Gamma_w$$

$div(\boldsymbol{u}) = 0$ a.e. on $\Omega(\eta(t))$ and the corresponding integral identity of the weak formulation holds.

Weak formulation - problem $\mathcal{P}$

$$\int_0^T \int_{\Omega(\eta(t))} \left\{ -\rho \boldsymbol{u} \cdot \frac{\partial \boldsymbol{\varphi}}{\partial t} + ((\boldsymbol{u}, \varphi)) + \rho \sum_{i,j=1}^{2} u_i \frac{\partial u_j}{\partial x_i} \varphi_j \right\}$$

$$\pm \int_0^T \int_{\Gamma_{in,out}} \left(P_{in,out} - \frac{\rho}{2} |u_1|^2 \right) \varphi_1 + \int_0^T \int_{\Gamma_w} P_w \varphi_2$$

$$+ \int_0^T \int_0^L \left(-\frac{\partial \eta}{\partial t} \frac{\partial \xi}{\partial t} + c \frac{\partial^3 \eta}{\partial x_1^2 \partial t} \frac{\partial^2 \xi}{\partial x_1^2} + a \frac{\partial \eta}{\partial x_1} \frac{\partial \xi}{\partial x_1} + b \eta \, \xi \right) = 0$$

for every test functions

$$\begin{aligned}
\boldsymbol{\varphi}(x_1, x_2, t) \quad &\in \quad H^1(0, T; W^{1,p}(\Omega(\eta(t)))) \\
\text{such that} \quad & \quad \operatorname{div} \boldsymbol{\varphi} = 0 \quad \text{a.e on } \Omega(\eta(t)), \\
& \quad \gamma_{\eta(t)} \varphi_2 \in H^1(0, T; H_0^2(\Gamma_w)) \\
& \quad \xi(x_1, t) = \boldsymbol{\varphi}_2(x_1, R_0(x_1) + \eta(x_1, t), t)
\end{aligned}$$

Analysis

- Idea: approximate problem $\mathcal{P}(\kappa, \varepsilon, k)$

1. *Global iteration*: given domain deformation
$h(x_1, t) \equiv h^{(k)}(x_1, t) = R_0(x_1) + \eta^{(k-1)}(x_1, t)$

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  1. *Global iteration*: given domain deformation $h(x_1, t) \equiv h^{(k)}(x_1, t) = R_0(x_1) + \eta^{(k-1)}(x_1, t)$
  2. Transformation of $\Omega(h)$ to a domain $D := (0, L) \times (0, 1)$

$$\boldsymbol{v}(y_1, y_2, t) \equiv \boldsymbol{u}(y_1, h(y_1, t)y_2, t);$$
$$q(y_1, y_2, t) \equiv \rho^{-1}\pi(y_1, h(y_1, t)y_2, t);$$
$$\sigma(y_1, t) \equiv \frac{\partial \eta(y_1, t)}{\partial t}$$

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3. Split the structure equation into two steps:

$$-\mathbf{T}_f \boldsymbol{n} \cdot \boldsymbol{e}_r - P_w - \frac{\rho}{2} u_2 \left(u_2 - \frac{\partial \eta^{(k-1)}}{\partial t} \right) = \kappa \rho \left(\frac{\partial \eta}{\partial t} - u_2 \right)$$

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Reason:

by $\kappa \longrightarrow \infty$ we obtain $\partial_t \eta = u_2$, which is original Dirichlet condition

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4. Approximation of the incompressibility condition by pseudo-compressibility

$$\varepsilon \frac{\partial p}{\partial t} - \varepsilon \Delta \pi + \operatorname{div} \boldsymbol{u}_\varepsilon = 0.$$

Idea of the proof

1. Construct Galerkin approximations to the approximate problem $\mathcal{P}(\kappa, \varepsilon, k)$
2. Apriori estimates implies (independent on ε, κ) for a fixed $h(x_1, t)$
3. Limiting process in the Galerkin approximations: Lions-Aubin & Minty-Browder for convergence in nonlinear terms
4. Limiting process $\varepsilon \to 0, \kappa \to \infty$ for fixed $h(x_1, t)$ - problem $\mathcal{P}(k)$
5. Limiting in $h^{(k)} = R_0 + \eta^{(k-1)}$ for $k \to \infty$ by the Schauder fixed point arguments original problem $\mathcal{P}$ on $\Omega(\eta(t))$