Some recent results on regularity criteria for weak solutions of the Navier-Stokes equations

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1. Weak solution to the Navier–Stokes equations

Ω ... a domain in $\mathbb{R}^3$ with a Lipschitz–continuous boundary

$T > 0$, $Q_T := \Omega \times (0, T)$

We deal with **the Navier–Stokes initial–boundary value problem** for viscous incompressible fluid

$$\begin{aligned} \partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} &= -\nabla p + \nu \Delta \mathbf{v} + \mathbf{f} && \text{in } Q_T, && (1.1) \\ \operatorname{div} \mathbf{v} &= 0 && \text{in } Q_T, && (1.2) \\ \mathbf{v} &= \mathbf{0} && \text{on } \partial\Omega \times (0, T), && (1.3) \\ \mathbf{v} &= \mathbf{v}_0 && \text{in } \Omega \times \{0\}. && (1.4) \end{aligned}$$

(**H. Navier** 1824, **G. Stokes** 1845)

First qualitative results on the existence of solutions: **J. Leray** in the 30–ties of the 20th century.

Leray introduced the notion of the **weak solution** of the boundary–value problem (1.1)–(1.4). (In fact, Leray studied the case $\Omega = \mathbb{R}^3$. The case of a bounded domain Ω was treated by **E. Hopf** in 1951.)

A weak solution to problem (1.1)–(1.4). Let $\mathbf{v}_0 \in \mathbf{L}^2_\sigma(\Omega)$ and $\mathbf{f} \in \mathbf{L}^2(Q_T)$. A vector function $\mathbf{v} \in L^2(0,T;\, \mathbf{W}^{1,2}_0(\Omega)) \cap L^\infty(0,T;\, \mathbf{L}^2_\sigma(\Omega))$ is said to be a *weak solution* of the problem (1.1)–(1.4) if for all $\boldsymbol{\phi} \in \mathbf{C}^\infty_{0,\sigma}\big(\Omega \times [0,T)\big)$:

$$\int_0^T \int_\Omega \big[\mathbf{v} \cdot \partial_t \boldsymbol{\phi} - \nu \nabla \mathbf{v} : \nabla \boldsymbol{\phi} - \mathbf{v} \cdot \nabla \mathbf{v} \cdot \boldsymbol{\phi}\big] \, \mathrm{d}\mathbf{x} \, \mathrm{d}t$$
$$= -\int_0^T \int_\Omega \mathbf{f} \cdot \boldsymbol{\phi} \, \mathrm{d}\mathbf{x} \, \mathrm{d}t - \int_\Omega \mathbf{v}_0 \cdot \boldsymbol{\phi}(\,.\,, 0) \, \mathrm{d}\mathbf{x}. \tag{1.5}$$

Remark: function space $\mathbf{L}^q_\sigma(\Omega)$. Let $\mathbf{C}^\infty_{0,\sigma}(\Omega)$ be the linear space of all infinitely differentiable divergence–free vector functions in Ω with a compact support in Ω.

For $1 \leq q \leq \infty$, we denote by $\mathbf{L}^q_\sigma(\Omega)$ the closure of $\mathbf{C}^\infty_{0,\sigma}(\Omega)$ in $\mathbf{L}^q(\Omega)$.

Remark: characterization of the space $\mathbf{L}^q_\sigma(\Omega)$. Assume that Ω has a locally Lipschitzian boundary and $1 < q < \infty$. Let $\mathbf{L}^q_{\mathrm{div}}(\Omega)$ be the space of functions $\mathbf{v} \in \mathbf{L}^q(\Omega)$ such that $\operatorname{div}\mathbf{v} \in L^q(\Omega)$. One can prove that

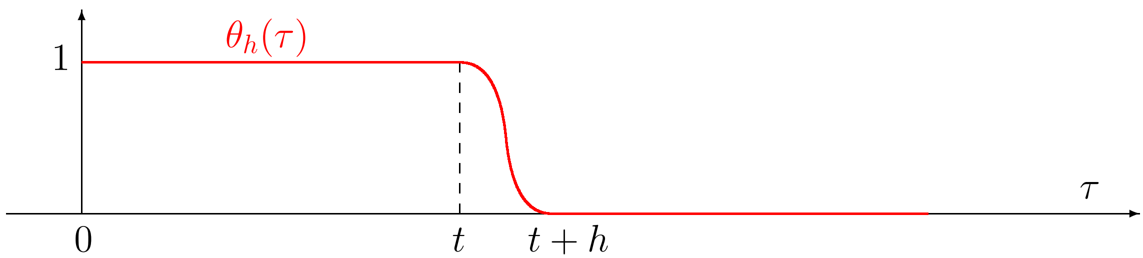
1) The space $\mathbf{C}^\infty(\overline{\Omega})$ is dense in $\mathbf{L}^q_{\mathrm{div}}(\Omega)$.
2) The mapping $\gamma_{\mathbf{n}} : \mathbf{v} \mapsto \mathbf{v}\cdot\mathbf{n}$ defined on $\mathbf{C}^\infty(\overline{\Omega})$ can be extended to a continuous linear mapping from $\mathbf{L}^q_{\mathrm{div}}(\Omega)$ to $W^{-1/q,q}(\partial\Omega)$.

The space $\mathbf{L}^q_\sigma(\Omega)$ can now be characterized as a space of functions from $\mathbf{L}^q_{\mathrm{div}}(\Omega)$, whose divergence equals zero in Ω (in the sense of distributions) and such that $\gamma_{\mathbf{n}}\mathbf{v} = 0$ (the zero element of $W^{-1/q,q}(\partial\Omega)$).

Lemma 1 (Hopf 1951, Prodi 1959, Serrin 1963). *The weak solution* $\mathbf{v}$ *to problem (1.1)–(1.4) can be redefined on a set of zero Lebesgue measure so that* $\mathbf{v}(\,.\,,t) \in \mathbf{L}^2(\Omega)$ *for all* $t \in [0,T)$ *and for all* $\boldsymbol{\phi} \in \mathbf{C}^\infty_{0,\sigma}\big(\Omega \times [0,T)\big)$*:*

$$\int_0^t \int_\Omega \big[\mathbf{v}\cdot\partial_\tau\boldsymbol{\phi} - \nu\nabla\mathbf{v} : \nabla\boldsymbol{\phi} - \mathbf{v}\cdot\nabla\mathbf{v}\cdot\boldsymbol{\phi}\big]\,\mathrm{d}\mathbf{x}\,\mathrm{d}\tau$$
$$= -\int_0^t \int_\Omega \mathbf{f}\cdot\boldsymbol{\phi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}\tau + \int_\Omega \mathbf{v}(\,.\,,t)\cdot\boldsymbol{\phi}(\,.\,,t)\,\mathrm{d}\mathbf{x} - \int_\Omega \mathbf{v}_0\cdot\boldsymbol{\phi}(\,.\,,0)\,\mathrm{d}\mathbf{x}. \qquad (1.6)$$

Principle of the proof: We use a C^1 function θ_h as on the figure. We use (1.6) with $\boldsymbol{\phi}(\mathbf{x},\tau)\,\theta_h(\tau)$ instead of $\boldsymbol{\phi}(\mathbf{x},\tau)$, and we consider the limit for $h \longrightarrow 0$.



Theorem 1 (existence of a weak solution – Leray 1934, Hopf 1951, et al). *Let* Ω *be a domain in* $\mathbb{R}^3$, $T > 0$, $\mathbf{v}_0 \in \mathbf{L}^2_\sigma(\Omega)$ *and* $\mathbf{f} \in \mathbf{L}^2(Q_T)$. *Then there exists at least one weak solution* $\mathbf{v}$ *to problem (1.1)–(1.4). The solution satisfies*

- *the energy inequality (EI)*

$$\|\mathbf{v}(\,.\,,t)\|_2^2 + 2\nu \int_0^t \|\nabla\mathbf{v}(\,.\,,\tau)\|_2^2 \,\mathrm{d}\tau \leq \|\mathbf{v}_0\|_2^2 + 2\int_0^t \big(\mathbf{v}(\,.\,,\tau), \mathbf{f}(\,.\,,\tau)\big)_2 \,\mathrm{d}\tau \tag{1.7}$$

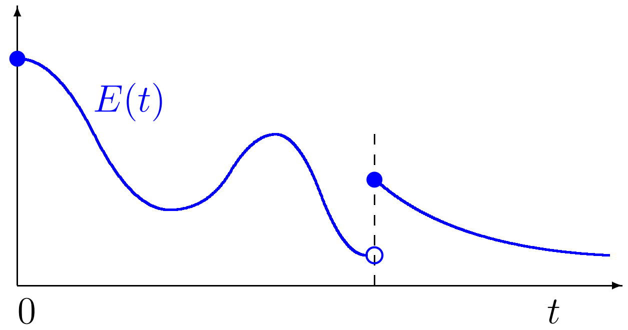
  for all $t \in [t, T)$,

- $\lim_{t\to 0+} \|\mathbf{v}(\,.\,,t) - \mathbf{v}_0\|_2 \;=\; 0.$

Open questions:

- Does each weak solution satisfy (EI), or even the energy equality (EE)?
- Is the weak solution unique?
- Is the weak solution regular provided that $\mathbf{v}_0$ and $\mathbf{f}$ are regular?

(EI) does not exclude e.g. this behaviour of the kinetic energy $E(t) := \|\mathbf{v}(\,.\,,t)\|_2$:



The inequality, which excludes the growth of $E(t)$, is the so called *strong energy inequality* (SEI):

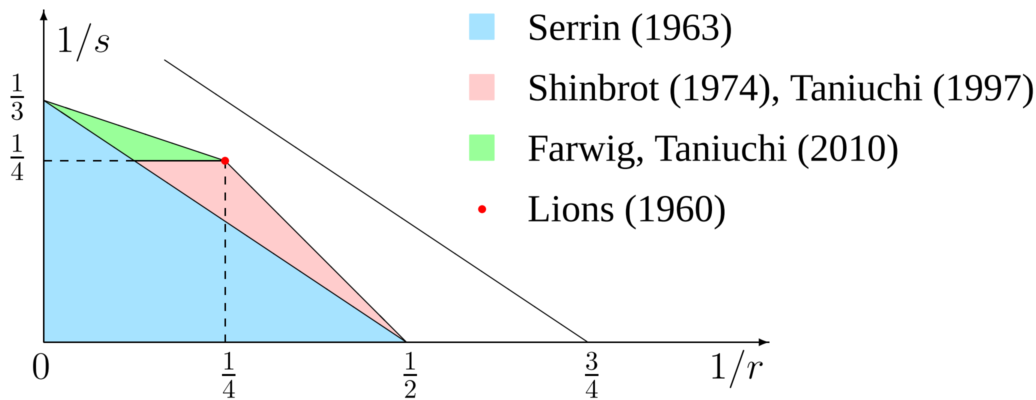
$$\|\mathbf{v}(\,.\,,t)\|_2^2 + 2\nu \int_s^t \|\nabla\mathbf{v}(\,.\,,\tau)\|_2^2 \,\mathrm{d}\tau \leq \|\mathbf{v}(\,.\,,s)\|_2^2 + 2\int_s^t \big(\mathbf{v}(\,.\,,\tau), \mathbf{f}(\,.\,,\tau)\big)_2 \,\mathrm{d}\tau \tag{1.8}$$

for a.a. $s \in [0,T)$ and all $t \in [s,T)$.

Question: Does the solution, provided by Theorem 1, satisfy (SEI)?

Partial answers regarding (EE): **Serrin** (1963): If $\mathbf{v} \in L^r(0,T;\,\mathbf{L}^s(\Omega))$, where $2/r + 3/s \le 1,\ 3 \le s \le \infty,\ 2 \le r \le \infty$ then $\mathbf{v}$ satisfies (EE).

- **Shinbrot** (1974), **Taniuchi** (1997): If $\mathbf{v} \in L^r(0,T;\,\mathbf{L}^s(\Omega))$, where $2/r + 3/s \le 1 + 1/s,\ 4 \le s \le \infty$, then $\mathbf{v}$ satisfies (EE).
- **Farwig and Taniuchi** (2010): observed that if $\mathbf{v} \in L^r(0,T;\,\mathbf{L}^s(\Omega))$, where $2/r + 3/s \le 1 + 1/r,\ 4 \le r \le \infty$, then $\mathbf{v}$ satisfies (EE).
- Further improvements: **Cheskidov, Friedlander and Shvydkoy** (2010), **Farwig and Taniuchi** (2010).



As to (SEI), Leray (1934), **Galdi and Maremonti** (1986), **Miyakawa and Sohr** (1988), **Farwig, Kozono and Sohr** (2005) proved: *Weak solution* $\mathbf{v}$ *can be constructed so that it satisfies not only (EI), but also (SEI).*

Partial answer to the question of uniqueness:

Theorem 2 (Prodi 1959, Lions and Prodi 1959, et al). *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be two weak solutions of the problem (1.1)–(1.4), with the same data* $\mathbf{v}_0$ *and* $\mathbf{f}$. *Assume that*

1) $\mathbf{u}$ *satisfies (EI),*

2) $\mathbf{v}$ *satisfied at least one of the conditions*

   (a) $\mathbf{v} \in L^r(0,T;\, \mathbf{L}^s(\Omega))$ *for some* r, s *satisfying* $2/r + 3/s = 1$, $3 < s \leq \infty$

   (b) $\mathbf{v} \in L^\infty(0,T;\, \mathbf{L}^3(\Omega))$ *and* $\mathbf{v}(\,.\,,t)$ *is right–continuous in the norm of* $\mathbf{L}^3(\Omega)$ *in dependence on* t *for* $0 \leq t < T$.

Then $\mathbf{u} = \mathbf{v}$ *a.e. in* Q_T.

Kozono and Sohr (1996): if Ω is a domain with a „smooth" bounded boundary, then the condition of right–continuity in condition (ii) can be omitted.

Interior regularity of a weak solution under Serrin's condition

Assume, for simplicity, that $\mathbf{f} \equiv \mathbf{0}$.

Theorem 3 (interior regularity – Serrin 1963). *Let* $\mathbf{v}$ *be a weak solution to (1.1)–(1.4) with* $\mathbf{f} \equiv \mathbf{0}$. *Assume, in addition, that there exists a sub–domain* Ω' *of* Ω *and* $0 \leq t_1 < t_2 \leq T$ *so that*

$$\text{(a)} \quad \mathbf{v} \in L^r(t_1, t_2; \mathbf{L}^s(\Omega')) \quad \textit{for some } r, s \textit{ satisfying } \frac{2}{r} + \frac{3}{s} = 1, \;\; 3 < s \leq \infty.$$

Then, given any bounded domain $\Omega'' \subset \overline{\Omega''} \subset \Omega'$ *and* $0 < \delta < (t_2 - t_1)/2$, *each space derivative of* $\mathbf{v}$ *is bounded in* $\overline{\Omega''} \times [t_1 + \delta, t_2 - \delta]$.

If, in addition,

$$\text{(b)} \quad \partial_t \mathbf{v} \in L^2(t_1, t_2; \mathbf{L}^q(\Omega')) \quad \textit{for some } q \geq 1$$

then each space derivative of $\mathbf{v}$ *is absolutely continuous function of* t.

Remark. The analogous result in the case $\Omega = \mathbb{R}^3$ and $s = 3$ follows from the work of **Escauriaza, Seregin, Šverák** (2003).

Remark: interior regularity of $\partial_t\mathbf{v}$ and p. Condition (a) implies that $\partial_t\mathbf{v}$ and p have all spatial derivatives in $L^\alpha(t_1+\delta, t_2-\delta;\, L^\infty(\Omega''))$ for each $\alpha \in [1,2)$, (see **J. N., Penel** 2001, **Kučera, Skalák** 2003.)

Principle of the proof.

Let $\zeta > 0$ be so small that $U_{4\zeta}(\Omega'') \subset \Omega'$,

Denote by ψ an infinitely differentiable cut–off function defined in $\mathbb{R}^3$ such that $0 \leq \psi \leq 1$ and

$$\psi \ = \ \begin{cases} 1 & \text{on } U_\zeta(\Omega''), \\ 0 & \text{on } \mathbb{R}^3 \smallsetminus U_{3\zeta}(\Omega''). \end{cases}$$

The product $\psi\, p$ satisfies

$$\psi(\mathbf{x})\, p(\mathbf{x},t) \ = \ -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|\mathbf{x}-\mathbf{y}|} \left[\, \Delta(\psi p)\, \right](\mathbf{y},t)\; \mathrm{d}\mathbf{y}.$$

If we use the integration by parts and the equation

$$\Delta p \ = \ -\partial_i\partial_j\, (v_i v_j),$$

we get (for $\mathbf{x} \in \Omega''$)

$$\psi(\mathbf{x})\, p(\mathbf{x},t) \;=\; p^I(\mathbf{x},t) + p^{II}(\mathbf{x},t)$$

where

$$p^I(\mathbf{x},t) \;=\; -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1}{|\mathbf{x}-\mathbf{y}|}\, \big[\partial_i\partial_j(\psi v_i v_j)\big](\mathbf{y},t)\; \mathrm{d}\mathbf{y},$$

$$p^{II}(\mathbf{x},t) \;=\; -\frac{1}{2\pi}\int_{\mathbb{R}^3} \frac{x_i-y_i}{|\mathbf{x}-\mathbf{y}|^3}\, \big[(\partial_i\psi) v_i v_j\big](\mathbf{y},t)\; \mathrm{d}\mathbf{y}$$

$$-\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1}{|\mathbf{x}-\mathbf{y}|}\, \big[(\partial_i\partial_j\psi) v_i v_j\big](\mathbf{y},t)\; \mathrm{d}\mathbf{y} + \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{x_i-y_i}{|\mathbf{x}-\mathbf{y}|^3}\, \big[(\partial_i\psi) p\big](\mathbf{y},t)\; \mathrm{d}\mathbf{y}$$

$$+\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1}{|x-y|}\, \big[\,\Delta\psi\, p\,\big](\mathbf{y},t)\; \mathrm{d}\mathbf{y}.$$

Using the boundedness of $\mathbf{v}$ and its spatial derivatives on $supp\psi \times (t_1,t_2)$, we obtain

$$\big|\nabla^k p^I(\mathbf{x},t) \;\le\; C(k).$$

The integrals in p^{II} can be considered only for $\mathbf{y} \in U_{3\zeta}(\Omega'') \smallsetminus U_\zeta(\Omega'')$ where $\mathbf{v}$ and its spatial derivatives are bounded and $|\mathbf{x}-\mathbf{y}| \geq \zeta$. Thus,

$$|\nabla^k p^{II}(\mathbf{x},t)| \ \leq \ C(k) \int_{\mathrm{supp}\,\nabla\psi} |p(\mathbf{y},t)| \, \mathrm{d}\mathbf{y} + C(k).$$

Hence

$$\begin{aligned} \int_{t_1}^{t_2} \Big[\max_{\mathbf{x}\in\overline{\Omega'}} |\nabla^k p^{II}(\mathbf{x},t)| \Big]^\alpha \, \mathrm{d}t \ &\leq \ C(k) \int_{t_1}^{t_2} \bigg(\int_{\mathrm{supp}\,\nabla\psi} |p(\mathbf{y},t)| \, \mathrm{d}\mathbf{y} \bigg)^\alpha \, \mathrm{d}t + C(k) \\ &\leq \ C(k) \int_{t_1}^{t_2} \bigg(\int_{U_\zeta(\overline{\Omega''})} |p(\mathbf{y},t)|^\beta \, \mathrm{d}\mathbf{y} \bigg)^{\alpha/\beta} \, \mathrm{d}t + C(k) \end{aligned}$$

where β is chosen so that $2/\alpha + 3/\beta = 3$. Due to the results of Taniuchi (1997) and Kozono (1998), $p \in L^\alpha(t_1,t_2;\, L^\beta(U_\zeta(\Omega')))$ for $1 < \alpha < 2$, $\frac{3}{2} < \beta < 3$ such that $2/\alpha + 3/\beta = 3$. Hence the last integral is finite. ■

Remark. If $\Omega = \mathbb{R}^3$ then one can use a little different technique to show that $\partial_t \mathbf{v}$ and p have all spatial derivatives in $L^\infty(t_1+\delta, t_2-\delta;\, L^\infty(\Omega''))$.

Corollary. If $(\mathbf{x}_1, t_1)$, $(\mathbf{x}_2, t_2) \in \Omega'' \times (t_1 + \delta, t_2 - \delta)$ then

$$\begin{aligned}
|\mathbf{v}(\mathbf{x}_1, t_1) - \mathbf{v}_2, t_2)| \; &\leq \; |\mathbf{v}(\mathbf{x}_1, t_1) - \mathbf{v}(\mathbf{x}_2, t_1)| + |\mathbf{v}(\mathbf{x}_2, t_1) - \mathbf{v}(\mathbf{x}_2, t_2)| \\
&\leq C\,|\mathbf{x}_1 - \mathbf{x}_2| + \int_{t_2}^{t_1} \partial_t \mathbf{v}(\mathbf{x}_2, t)\;\mathrm{d}t\Bigg| \\
&\leq \; C\,|\mathbf{x}_1 - \mathbf{x}_2| + \int_{t_2}^{t_1} \|\partial_t \mathbf{v}(\,.\,, t)\|_{\infty;\,\Omega''}\;\mathrm{d}t \\
&\leq \; C\,|\mathbf{x}_1 - \mathbf{x}_2| + \left(\int_{t_1}^{t_2} \|\partial_t \mathbf{v}(\,.\,, t)\|_{\infty;\,\Omega''}^{\alpha}\;\mathrm{d}t \right)^{1/\alpha} |t_1 - t_2|^{(\alpha-1)/\alpha} \\
&\leq \; C\,|\mathbf{x}_1 - \mathbf{x}_2| + C\,|t_1 - t_2|^{(\alpha-1)/\alpha}.
\end{aligned}$$

This implies the Hölder–continuity of $\mathbf{v}$ in $\Omega'' \times (t_1 + \delta, t_2 - \delta)$.

2. A suitable weak solution of the problem (1.2)–(1.4)

L. Caffarelli, R. Kohn and L. Nirenberg (1983) called a weak solution $\mathbf{v}$ of (1.1)–(1.4) a **suitable weak solution** if an associated pressure p belongs to $L^{5/4}(Q_T)$ and the pair $(\mathbf{v}, p)$ satisfies the so called **generalized energy inequality** (GEI)

$$2\nu \int_0^T \int_\Omega |\nabla \mathbf{v}|^2 \varphi \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \le \int_0^T \int_\Omega \big[\, |\mathbf{v}|^2 \left(\partial_t \varphi + \nu \Delta \varphi \right) + \left(|\mathbf{v}|^2 + 2p \right) \mathbf{v} \cdot \nabla \varphi \big] \, \mathrm{d}\mathbf{x} \, \mathrm{d}t$$
$$+ \int_0^T \int_\Omega 2\mathbf{v} \cdot \mathbf{f} \, \varphi \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \tag{2.1}$$

for every non–negative function φ from $C_0^\infty(Q_T)$.

C-K-N proved the existence of a suitable weak solution in the case when Ω is either $\mathbb{R}^3$ or a "smooth" bounded domain in $\mathbb{R}^3$. (The proof is based on the applications of the so called "retarded mollifications" in the nonlinear term ... $\Psi_\delta(\mathbf{v}) \cdot \nabla \mathbf{v}$.)

(See also **V. Scheffer** 1977 for the proof in the case $\mathbf{f} = \mathbf{0}$.)

C-K-N defined a **regular point** of a weak solution $\mathbf{v}$ as a point in Q_T such that there exists a neighbourhood U of this point, where $\mathbf{v}$ is essentially bounded.

A point in Q_T that is not regular is called **singular**.

$\mathcal{S}(\mathbf{v})$ ... the set of all singular points of solution $\mathbf{v}$ in Q_T

Clearly, since the set of regular points is open in Q_T, **the set $\mathcal{S}(\mathbf{v})$ of singular points is closed in Q_T.**

Put $Q_r^*(\mathbf{x},t) := B_r(\mathbf{x}) \times (t - \frac{7}{8}r^2, t + \frac{1}{8}r^2)$

Lemma 2 (C-K-N 1983). *Let* $\mathbf{v}$ *be a suitable weak solution of the problem (1.1)–(1.4). There exists a constant* $\epsilon > 0$ *such that if*

$$\limsup_{r\to 0+} \frac{1}{r} \iint_{Q_r^*(\mathbf{x}_0,t_0)} |\nabla \mathbf{v}|^2 \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \ \leq \ \epsilon \tag{2.2}$$

then $(\mathbf{x}_0, t_0) \notin \mathcal{S}(\mathbf{v})$.

Theorem 4 (C-K-N 1983). *Let* $\mathbf{v}$ *be a suitable weak solution of the problem (1.1)–(1.4). Then the* 1*–dimensional Hausdorff measure of se* $\mathcal{S}(\mathbf{v})$ *is zero.*

Principle of the proof.

$$(\mathbf{x}_0, t_0) \in \mathcal{S}(\mathbf{v}) \implies \limsup_{r\to 0+} \frac{1}{r} \iint_{Q_r^*(\mathbf{x}_0,t_0)} |\nabla \mathbf{v}|^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \;>\; \epsilon$$

Let U be a neighbourhood of $\mathcal{S}(\mathbf{v})$ in Q_T and $\delta > 0$.

To each $(\mathbf{x}_0, t_0) \in \mathcal{S}(\mathbf{v})$ choose $Q_r^*(\mathbf{x}, t) \subset U$ (with $r < \delta$) such that

$$\frac{1}{r} \iint_{Q_r^*(\mathbf{x}_0,t_0)} |\nabla \mathbf{v}|^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \;>\; \epsilon.$$

Let us denote by $\mathcal{J}$ the family of all such cylinders. Due to Vitali's covering lemma, there exists an at most countable sub–family $\mathcal{J}' = \{Q_{r_i}^*(\mathbf{x}_i, t_i)\}$ of $\mathcal{J}$ such that

$$Q_{r_i}^*(\mathbf{x}_i, t_i) \cap Q_{r_j}^*(\mathbf{x}_j, t_j) = \emptyset \quad \text{for } i \neq j,$$

$$\forall\, Q_r^*(\mathbf{x}, t) \in \mathcal{J} \quad \exists\, Q_{r_i}^*(\mathbf{x}_i, t_i) \in \mathcal{J}' \;:\; Q_r^*(\mathbf{x}, t) \subset Q_{5r_i}^*(\mathbf{x}_i, t_i).$$

Consequently: $\mathcal{S}(\mathbf{v}) \subset \bigcup_i Q_{5r_i}^*(\mathbf{x}_i, t_i)$. Moreover,

$$\sum_i 5r_i \quad \le \frac{5}{\epsilon}\sum_i \iint_{Q^*_{r_i}(\mathbf{x}_i,t_i)} |\nabla\mathbf{v}|^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \le \frac{5}{\epsilon}\iint_U |\nabla\mathbf{v}|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t,$$

$$\sum_i (5r_i)^3 \le 125\delta^2 \sum_i 5r_1 \le \frac{125\delta^2}{\epsilon}\iint_U |\nabla\mathbf{v}|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t.$$

Since $\delta > 0$ can be arbitrarily small, we deduce that the 3D Lebesgue measure of $\mathcal{S}_{\mathrm{CKN}}(\mathbf{v})$ is zero. Thus, neighbourhood U can be chosen so that its 3D Lebesgue measure is arbitrarily small. Hence the right hand side can be arbitrarily small. This implies that $\mathcal{P}^1(\mathcal{S}(\mathbf{v})) = 0$. Consequently, $\mathcal{H}^1(\mathcal{S}(\mathbf{v})) = 0$. ■

What is a real regularity of a suitable weak solution in the neighbourhood of C-K-N's regular point (x_0, t_0)?

Answer: there exists $R > 0$ and $\delta > 0$ such that

a) $\mathbf{v}$ and its all spatial derivatives are in $L^\infty\big(B_R(\mathbf{x}_0) \times (t_0-\delta, t_0+\delta)\big)$,

b) $\partial_t\mathbf{v}$ and p have all spatial derivatives in $L^\alpha\big(t_0-\delta, t_0+\delta;\ L^\infty(B_R(\mathbf{x}_0))\big)$ for each $\alpha \in [1,2)$ if Ω is bounded and for $\alpha = \infty$ if $\Omega = \mathbb{R}^3$.

c) $\mathbf{v}$ is Hölder–continuous in $B_R(\mathbf{x}_0) \times (t_0-\delta, t_0+\delta)$.

Later improvements or modifications:

F. Lin (1996) – used the same assumptions on domain Ω as in C–K–N, the definition of a suitable weak solution requires the pressure to be in $L^{3/2}(Q_T)$. Considered the case $\mathbf{f} = \mathbf{0}$.

Theorem 5 (Lin 1996). *Let $(\mathbf{v}, p)$ be a suitable weak solution of the problem (1.1)–(1.4). There exists a constant $\epsilon > 0$ such that if*

$$\frac{1}{\delta^2} \int_{t_0-\delta^2}^{t_0} \int_{B_\delta(\mathbf{x}_0)} \big(|\mathbf{v}|^3 + |p|^{3/2} \big) \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \ \leq \ \epsilon \tag{2.3}$$

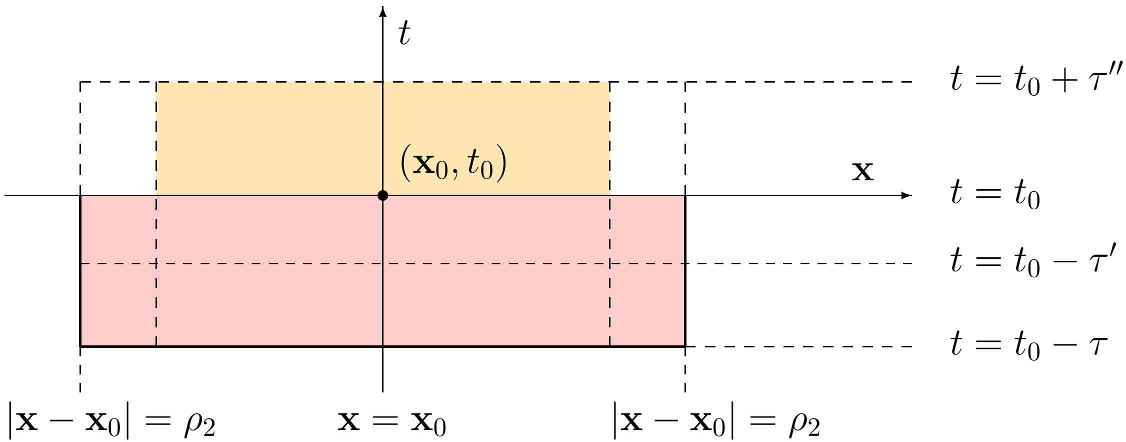
for some $\delta > 0$ then $\mathbf{v} \in C^\alpha\big(B_{\delta/2}(\mathbf{x}_0) \times (t_0, t_0 - \frac{1}{4}\delta^2)\big)$ for some $\alpha > 0$.

Corollary. (2.3) implies that $(\mathbf{x}_0, t_0)$ is a regular point in the sense of C-K-N.

Principle of the proof. There exist numbers $\tau > 0$ and $0 < \rho_1 < \rho_2$ so that $\mathbf{v}$ is bounded with all its spatial derivatives in $[B_{\rho_2}(\mathbf{x}_0) \smallsetminus B_{\rho_1}(\mathbf{x}_0)] \times (t_0 - \tau, t_0 + \tau)$.

We use a cut–off function $\eta \in C^\infty(\mathbb{R}^3)$ such that $0 \le \eta \le 1$, $\eta = 1$ in $B_{\rho_1}(\mathbf{x}_0)$ and $\eta = 0$ outside $B_{\rho_2}(\mathbf{x}_0)$.

We put $\mathbf{u} = \eta\mathbf{v} - \mathbf{V}$, where $\mathbf{V}$ is the correction such that $\operatorname{div}\mathbf{u} = 0$. (Function $\mathbf{V}$ satisfies $\operatorname{div}\mathbf{V} = \nabla\eta\cdot\mathbf{v}$; it can be constructed so that its support is in $[B_{\rho_2}(\mathbf{x}_0) \setminus B_{\rho_1}(\mathbf{x}_0)] \times (t_0-\tau, t_0+\tau)$.



Functions $\mathbf{u}$, ηp satisfy the Navier–Stokes equation with the right hand side $\mathbf{h} \in L^\alpha(t_0-\tau, t_0+\tau;\ \mathbf{W}^{k,\infty}(B_{\rho_2}(\mathbf{x}_0))$.

Function $\mathbf{u}$ satisfies the boundary condition $\mathbf{u} = \mathbf{0}$ on $\partial B_{\rho_2}(\mathbf{x}_0) \times (t_0 - \tau, t_0 + \tau)$.

Using the results on the local existence of strong solutions, one can show that there exist $\tau, \tau'' \in (0, \tau)$ so that the same Navier–Stokes problem as the one satisfied by functions $\mathbf{u}$, ηp, has a "smooth" solution $\mathbf{w}$, q on the time interval $(t_0 - \tau', t_0 + \tau'')$, satisfying the boundary condition $\mathbf{w} = \mathbf{0}$ on $\partial B_{\rho_2}(\mathbf{x}_0) \times (t_0 - \tau, t_0 + \tau'')$.'Moreover, $\mathbf{w} = \mathbf{u}$ at time $t = t_0 - \tau'$.

In fact, one obtains $\mathbf{u} \in L^\infty(t_0 - \tau', t_0 + \tau'';\ \mathbf{W}^{1,2}(B_{\rho_2}(\mathbf{x}_0))$.

Since $\mathbf{v}$ is a suitable weak solution, one can verify that solution $\mathbf{u}$ satisfies (SEI).

Consequently, due to theorems on uniqueness, one can identify solutions $\mathbf{u}$ and $\mathbf{w}$ in $B_{\rho_2}(\mathbf{x}_0) \times (t_0 - \tau', t_0 + \tau'')$.

Consequently, $\mathbf{v} \in L^\infty(t_0 - \tau', t_0 + \tau'';\ \mathbf{W}^{1,2}(B_{\rho_1}(\mathbf{x}_0))$, which means that $\mathbf{v}$ satisfies Serrin's regularity condition in $B_{\rho_1}(\mathbf{x}_0) \times (t_0 - \tau', t_0 + \tau'')$. $\blacksquare$

Remark. Lin also proved that (2.2) $\Longrightarrow \big[$ (2.3) holds for some $\delta > 0 \big]$.

Y. Taniuchi (1997) considered domain Ω in $\mathbb{R}^3$ that is either smooth bounded, or smooth exterior, or a half–space, or the whole space $\mathbb{R}^3$.

Provided $\mathbf{v}_0 \in \mathbf{L}^2_\sigma(\Omega)$ and $\mathbf{f} \in \mathbf{L}^2(Q_T)$, Taniuchi proved the existence of a suitable weak solution.

Lemma 3 (Taniuchi 1997). *If* $\mathbf{v}$ *is a weak solution and* K *is a bounded sub–domain of* Ω, $\mathbf{v} \in L^r(\epsilon, T;\, \mathbf{L}^s(K))$ *with*

$$1 < r, s < \infty, \qquad \frac{2}{r} + \frac{2}{s} \leq 1, \qquad \frac{1}{r} + \frac{3}{s} \leq 1 \tag{2.4}$$

then $\mathbf{v}$ *satisfies (GEE) in* $K \times (\epsilon, T)$*. Moreover, if* $\mathbf{v} \in L^r(0, T;\, \mathbf{L}^r(\Omega))$ *with* r, s *satisfying (2.4) then* $\mathbf{v}$ *satisfies both (SEE) and (GEE).*

Lemma 4 (Taniuchi 1997). *If* $\mathbf{v}$ *is a weak solution,* D *is a bounded sub–domain of* Ω *and* $0 < \epsilon < T$ *then* p *and* $\partial_t \mathbf{v}$ *can be taken so that*

$$p \in L^r(\epsilon, T;\, L^s(D)) \qquad \text{for } \frac{2}{r} + \frac{3}{s} = 3,\ 1 < r \leq 2,\ 1 < s < 3,$$

$$\partial_t \mathbf{v} \in L^r(\epsilon, T;\, \mathbf{L}^s(D)) \qquad \text{for } \frac{2}{r} + \frac{3}{s} = 4,\ 1 < r \leq 2,\ 1 < s \leq \frac{3}{2}.$$

O. A. Ladyzhenskaya and G. Seregin (1999) consider a bounded domain $\Omega \in \mathbb{R}^3$, they also consider $\mathbf{f} \neq \mathbf{0}$, $\mathbf{f} \in {}^M\mathbf{L}_{2,\gamma}(Q_T)$ (for some $\gamma > 0$; ${}^M\mathbf{L}_{2,\gamma}$ denotes the Morrey space), and they use the same definition of a suitable weak solution as Lin.

Ladyzhenskaya and Seregin define a **regular point** of a suitable weak solution $(\mathbf{v}, p)$ to be such a point in Q_T that there exists its neighbourhood U in Q_T, where $\mathbf{v}$ is Hölder–continuous.

Theorem 6 (Ladyzhenskaya, Seregin 1999). *Let $(\mathbf{v}, p)$ be a suitable weak solution of the problem (1.1)–(1.4). There exists a constant $\epsilon > 0$ such that if*

$$\limsup_{r \to 0+} \frac{1}{r} \int_{t_0 - r^2}^{t_0} \int_{B_r(\mathbf{x}_0)} |\nabla \mathbf{v}|^2 \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \;\leq\; \epsilon \tag{2.5}$$

then $(\mathbf{x}_0, t_0))$ is a (SL)–regular point of solution $(\mathbf{v}, p)$.

The proof is based on the estimates of solution $\mathbf{v}$ in parabolic Campanato spaces. (See e.g. W. Schlag, *Comm. PDE* 21, 1996, 1141-1175.)

L-S do not prove the existence of a suitable weak solution.

Remark: Basuc information on the Morrey and Campanato spaces.
(See e.g. Kufner et al: *Function Spaces*.)

Let Ω be a bounded domain in R^N. For $\lambda \geq 0$ and $1 \leq p < \infty$, we set

$$\begin{aligned}
{}^M L_{p,\lambda}(\Omega) &:= \Big\{ u \in L^p(\Omega);\ {}^M\|u\|_{p,\lambda} := \Big(\sup_{\mathbf{x}\in\Omega,\ r>0} \frac{1}{r^\lambda} \int_{B_r(\mathbf{x})\cap\Omega} |u(\mathbf{y})|^p \, \mathrm{d}\mathbf{y} \Big)^{1/p} < \infty \Big\}, \\
{}^C L_{p,\lambda}(\Omega) &:= \Big\{ u \in L^p(\Omega);\ [u]_{p,\lambda} < \infty \Big\}, \\
[u]_{p,\lambda} &:= \Big(\sup_{\mathbf{x}\in\Omega,\ r>0} \frac{1}{r^\lambda} \int_{B_r(\mathbf{x})\cap\Omega} |u(\mathbf{y}) - \overline{u}_r(\mathbf{x})|^p \, \mathrm{d}\mathbf{y} \Big)^{1/p} \\
{}^C\|u\|_{p,\lambda} &:= \|u\|_p + [u]_{p,\lambda}
\end{aligned}$$

- ${}^M L_{p,0}(\Omega) \rightleftarrows L^p(\Omega), \quad {}^M L_{p,N}(\Omega) \rightleftarrows L^\infty(\Omega)$
- If $1 \leq p \leq q < \infty,\ \lambda \geq 0,\ \mu \geq 0,\ \dfrac{\lambda - N}{p} \leq \dfrac{\mu - N}{q}$ then ${}^M L_{q,\mu}(\Omega) \hookrightarrow {}^M L_{p,\lambda}(\Omega)$.

- ${}^{M}L_{p,\lambda}(\Omega) = \{0\}$ for $\lambda > N$

- If $1 \leq p \leq q < \infty,\ \lambda \geq 0,\ \mu \geq 0,\ \dfrac{\lambda - N}{p} \leq \dfrac{\mu - N}{q}$ then ${}^{C}L_{q,\mu}(\Omega) \hookrightarrow {}^{C}L_{p,\lambda}(\Omega)$.

- ${}^{C}L_{p,\lambda}(\Omega \rightleftarrows {}^{M}L_{p,\lambda}(\Omega)$ for $0 \leq \lambda \leq N$

- ${}^{C}L_{p,\lambda}(\Omega \rightleftarrows C^{0,\alpha}(\overline{\Omega})$ with $\alpha = \dfrac{\lambda - N}{p}$ provided $\lambda \in (N, N + p)$.

R. Farwig, H. Kozono and H. Sohr (2005) considered an arbitrary domain Ω in $\mathbb{R}^3$ with a uniformly C^2–boundary, $0 < T \leq \infty$, $\mathbf{v}_0 \in \mathbf{L}^2_\sigma(\Omega)$, $\mathbf{f} \in L^{5/4}(0,T;\, \mathbf{L}^2(\Omega))$. **They proved the existence of a suitable weak solution** $(\mathbf{v}, p)$ with $\mathbf{v}$, $\partial_t\mathbf{v}$, $\nabla\mathbf{v}$, $\nabla^2\mathbf{v}$, ∇p in $L^{5/4}(\epsilon, T';\, \mathbf{L}^2(\Omega) + \mathbf{L}^{5/4}(\Omega))$, where $0 < \epsilon < T' < T$. **The solution satisfies (GEI)** in the form

$$\|\varphi\mathbf{v}(\,.\,,t)\|^2_{2;\,\Omega} + 2\nu \int_s^t \|\varphi\nabla\mathbf{v}(\,.\,,\tau)\|^2_{2;\,\Omega}\,\mathrm{d}\tau \;\leq\; \|\varphi\mathbf{v}(\,.\,,s)\|^2_{2;\,\Omega} + \int_s^t \big(\varphi\mathbf{f}, \varphi\mathbf{v}\big)_2\,\mathrm{d}\tau$$
$$-\nu\int_s^t\int_\Omega \nabla|\mathbf{v}|^2\cdot\nabla\varphi^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}\tau + \int_s^t\int_\Omega\big(|\mathbf{v}|^2 + 2p\big)\,(\mathbf{v}\cdot\nabla\varphi^2)\,\mathrm{d}\mathbf{x}\,\mathrm{d}\tau$$

for a.a. $s \in (0,T)$, all $t \in [s,T)$ and all $\varphi \in C_0^\infty(\mathbb{R}^3)$ **and (SEI)** in the form

$$\|\mathbf{v}(\,.\,,t)\|_2^2 + 2\nu\int_s^t \|\nabla\mathbf{v}(\,.\,,\tau)\|_2^2\,\mathrm{d}\tau \;\leq\; \|\mathbf{v}(\,.\,,s)\|_2^2 + 2\int_s^t \big(\mathbf{v}, \mathbf{f}\big)_2\,\mathrm{d}\tau$$

for a.a. $s \in [0,T)$ (including $s = 0$) and all $t \in [s,T)$.

Note that ϵ can be considered to be $= 0$ if $\mathbf{v}_0 \in D(\tilde{A}^{5/4})$.

T' can be considered to be $= T$ if $T < \infty$.

J. Wolf (2007) considered a general domain $\Omega \subset \mathbb{R}^3$, $\mathbf{f} = \mathbf{0}$.

Using the pressure representation $p = p_0 + \partial_t \tilde{p}_h$ where $p_0 \in L^{4/3}(0,\infty;\, L^2(\Omega))$ and $\tilde{p}_h \in C(Q)$ being harmonic, **Wolf proved the existence of of the so called generalized suitable weak solution** in $Q := \Omega \times (0,\infty)$, which is defined to be a weak solution in Q such that the function $\mathbf{V} := \mathbf{v} + \nabla \tilde{p}_h$ satisfies the identity

$$\int_0^\infty \int_\Omega \big[-\mathbf{V}\, \partial_t \boldsymbol{\phi} + \mathbf{v} \cdot \nabla \mathbf{v} \cdot \boldsymbol{\phi} + \nu\, \nabla \mathbf{V} : \nabla \boldsymbol{\phi} \big]\, \mathrm{d}\mathbf{x}\, \mathrm{d}t = \int_0^\infty \int_\Omega p_0\, \mathrm{div}\, \boldsymbol{\phi}\, \mathrm{d}\mathbf{x}\, \mathrm{d}t \tag{2.6}$$

for all $\boldsymbol{\phi} \in \mathbf{C}_0^\infty(Q)$.

Remark. Integral equation (2.6) formally follows from the Navier–Stokes equation if we use the representation $p = p_0 + \partial_t \tilde{p}_h$, the identities

$$\big\langle \nabla p, \boldsymbol{\phi} \big\rangle \;=\; -\big\langle p, \mathrm{div}\, \boldsymbol{\phi} \big\rangle \;=\; -\int_0^T \int_\Omega p_0\, \mathrm{div}\, \boldsymbol{\phi}\, \mathrm{d}\mathbf{x}\, \mathrm{d}t + \int_0^T \int_\Omega \tilde{p}_h\, \mathrm{div}\, \partial_t \boldsymbol{\phi}\, \mathrm{d}\mathbf{x}\, \mathrm{d}t$$

(where $\big\langle \nabla p, \boldsymbol{\phi} \big\rangle$ denotes the distribution ∇p, applied to function $\boldsymbol{\phi}$), and the fact that $\tilde{p}_h$ is harmonic.

Furthermore, Wolf proved that his solution satisfies

$$
\begin{aligned}
\int_\Omega |\mathbf{V}(t)|^2\, \varphi(t)\ \mathrm{d}\mathbf{x} + 2\nu \int_0^t \int_\Omega |\nabla \mathbf{V}|^2\, \varphi\ \mathrm{d}\mathbf{x}\,\mathrm{d}s \\
\leq\ \int_0^t \int_\Omega \big[\, |\mathbf{V}|^2\, \big(\partial_t\varphi + \nu\Delta\varphi\big) + \big(|\mathbf{v}|^2 + 2p_0\big)\, \mathbf{V}\cdot\nabla\varphi\big]\ \mathrm{d}\mathbf{x}\,\mathrm{d}s \\
+\, 2 \int_0^t \int_\Omega (\nabla\tilde{p}_h \times \mathbf{curl}\ \mathbf{V})\cdot\mathbf{V}\, \varphi\ \mathrm{d}\mathbf{x}\,\mathrm{d}s
\end{aligned}
\tag{2.7}
$$

for every non–negative function φ from $C_0^\infty(Q_T)$.

Inequality (2.7) formally follows from (2.6) if we choose $\boldsymbol{\phi} = \mathbf{V}\,\varphi\,\theta_{t,\delta}$, where θ_δ is an appropriate smooth cut–off function of one variable (equal to 1 on the interval $[0,t]$ and equal to zero on the interval $[t+\delta,\infty)$), and pass to zero with δ.

Theorem 7 (Wolf 2007). *Let* $\mathbf{v}$ *be a generalized suitable weak solution of the problem (1.1)–(1.4). There exists a constant* $\epsilon > 0$ *such that if*

$$
\limsup_{r\to 0+} \frac{1}{r} \int_{t_0-r^2}^{t_0} \int_{B_r(\mathbf{x}_0)} \left| \mathbf{curl}\ \mathbf{v} \times \frac{\mathbf{v}}{|\mathbf{v}|}\right|^2 \mathrm{d}\mathbf{x}\,\mathrm{d}t\ \leq\ \epsilon \tag{2.8}
$$

then $(\mathbf{x}_0, t_0)$ *is a (Lin)–regular point of solution* $\mathbf{v}$.

3. A brief survey of further criteria for regularity at a point $(\mathbf{x}_0, t_0)$ for weak or suitable weak solutions

- **S. Takahashi** (1990) proved that if the norm of solution $\mathbf{v}$ in $L^r_w(t_0-\rho^2, t_0;$ $\mathbf{L}^s(B_\rho(\mathbf{x}_0))$ (where $2/r+3/s \le 1,\ 3<s\le\infty$) is less than or equal to ϵ then $(\mathbf{x}_0, t_0)$ is a regular point of solution $\mathbf{v}$.

- **H. Kozono** (1998) proved that if $\mathbf{v}$ is a weak solution satisfying

$$\sup_{t_0-\sigma<t<t_0+\sigma} \|\mathbf{v}(\,.\,,t)\|_{L^3_w(B_\rho(\mathbf{x}_0))} \;\le\; \epsilon \tag{3.1}$$

  for some $\sigma>0$ and $\rho>0$ then $\partial_t\mathbf{v}$ and $\nabla^k\mathbf{v}$ ($k=0,1,2$) are bounded in some neighbourhood of point $(\mathbf{x}_0, t_0)$.

- **J. Nečas, J.N.** (2002) have shown that if $\mathbf{v}$ is a suitable weak solution then the condition

$$\lim_{\delta\to 0+} \frac{1}{\delta^2} \int_{t_0-\delta^2}^{t_0} \int_{B_\delta(\mathbf{x}_0)} |\mathbf{v}|^3 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \;=\; 0 \tag{3.2}$$

  implies that $(\mathbf{x}_0, t_0)$ is a regular point.

- **G. Seregin, V. Šverák** (2005) have shown that if $\mathbf{v}$ is a suitable weak solution satisfying

$$\frac{1}{\delta^2} \int_{t_0-\delta^2}^{t_0} \int_{B_\delta(\mathbf{x}_0)} |\mathbf{v}|^3 \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \ \leq \ \epsilon, \tag{3.3}$$

  for all $\delta > 0$ "sufficiently small" then $(\mathbf{x}_0, t_0)$ is a regular point.

- An improvement of the last criterion has been obtained by **J. Wolf** (2010), requiring the validity of

$$\frac{1}{\delta^2} \int_{t_0-\delta^2}^{t_0} \int_{B_\delta(\mathbf{x}_0)} |\mathbf{v}|^3 \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \ \leq \ \epsilon \tag{3.4}$$

  for at least one $\delta > 0$.

- Further generalizations and modifications of the aforementioned regularity criteria can be found in the paper by **G. Seregin, V. Šverák** (2005).

- Another generalization of the CKN–condition for suitable weak solutions has been proven by **A. Mahalov, B. Nicoalenko, G. Seregin** (2008), where the authors replace $\nabla\mathbf{v}$ by a quantity $d(\mathbf{v})$, which is in some sense related to $\nabla\mathbf{v}$.

- **S. Gustafson, K. Kang and T.-P. Tsai** (2006): $\mathbf{v}$ is a suitable weak solution, $\mathbf{f} \in {}^M\mathbf{L}_{2,\gamma}(Q_T)$ for some $\gamma > 0$. If

$$\limsup_{\rho \to 0+} \rho^{1-\left(\frac{2}{r}+\frac{3}{s}\right)} \, \|\mathbf{v} - \overline{\mathbf{v}}_\rho\|_{L^r(t_0-\rho^2,t_0;\,\mathbf{L}^s(B_\rho(\mathbf{x}_0)))} \;\leq\; \epsilon,$$

for some $r, s \in [1, \infty]$, satisfying $\quad 1 \;\leq\; \dfrac{2}{r} + \dfrac{3}{s} \;\leq\; 2,$

where $\quad \overline{\mathbf{v}}_\rho := \dfrac{1}{|B_\rho(\mathbf{x}_0)|} \displaystyle\int_{B_\rho(\mathbf{x}_0)} \mathbf{v} \, \mathrm{d}\mathbf{x},$

then $(\mathbf{x}_0, t_0)$ is a regular point.

- **S. Gustafson, K. Kang and T.-P. Tsai** (2006) also formulated a criterion in terms of vorticity:

$$\limsup_{\rho \to 0+} \rho^{2-\left(\frac{2}{r}+\frac{3}{s}\right)} \, \|\mathbf{curl}\,\mathbf{v}\|_{L^r(t_0-\rho^2,t_0;\,\mathbf{L}^s(B_\rho(\mathbf{x}_0)))} \;\leq\; \epsilon$$

for some $r, s \in [1, \infty]$, satisfying $\quad 2 \;\leq\; \dfrac{2}{r} + \dfrac{3}{s} \;\leq\; 3.$

Remark. Put

$$\mathbf{x}-\mathbf{x}_0=\rho\mathbf{x}', \quad t_0-t=t_0-\rho^2 t', \quad \mathbf{v}(\mathbf{x},t)=\frac{1}{\rho}\mathbf{v}'(\mathbf{x}',t'), \quad p(\mathbf{x},t)=\frac{1}{\rho^2}p'(\mathbf{x}',t').$$

Then $\mathbf{v}'$, p' satisfy the Navier–Stokes equation and the equation of continuity in $B_1(\mathbf{0})\times(t_0-1,t_0)$.

Furthermore,

$$\begin{aligned}\frac{1}{\rho}\int_{t_0-\rho^2}^{t_0}\int_{B_\rho(\mathbf{x}_0)}|\nabla\mathbf{v}|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t &= \int_{t_0-1}^{t_0}\int_{B_1(\mathbf{0})}|\nabla'\mathbf{v}'|^2\,\mathrm{d}\mathbf{x}'\,\mathrm{d}t',\\ \frac{1}{\rho^2}\int_{t_0-\rho^2}^{t_0}\int_{B_\rho(\mathbf{x}_0)}|\mathbf{v}|^3\,\mathrm{d}\mathbf{x}\,\mathrm{d}t &= \int_{t_0-1}^{t_0}\int_{B_1(\mathbf{0})}|\mathbf{v}'|^3\,\mathrm{d}\mathbf{x}'\,\mathrm{d}t'\end{aligned}$$

Similarly, one obtains

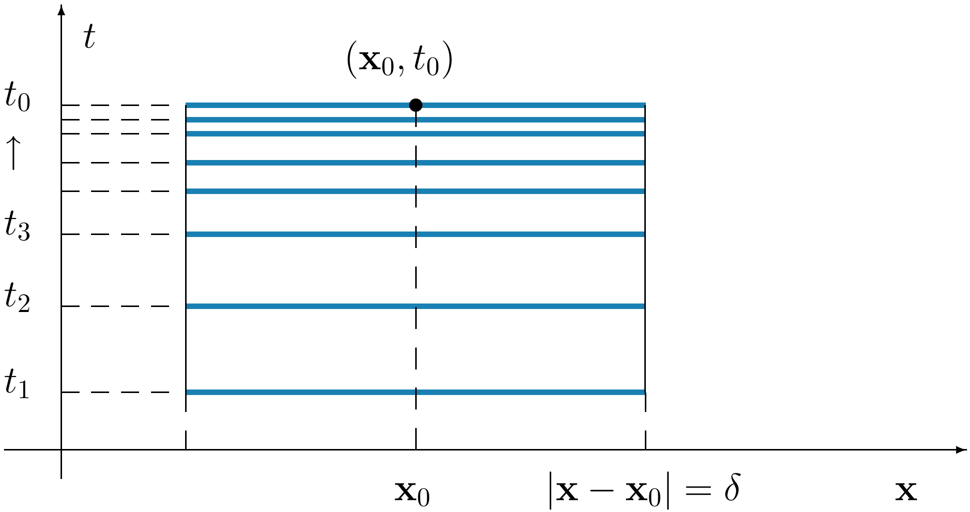
$$\begin{aligned}\rho^{1-\left(\frac{2}{r}+\frac{3}{s}\right)}\,\|\mathbf{v}-\overline{\mathbf{v}}_\rho\|_{L^r(t_0-\rho^2,t_0;\,\mathbf{L}^s(B_\rho(\mathbf{x}_0)))} &= \|\mathbf{v}'-\overline{\mathbf{v}'}_1\|_{L^r(t_0-1,t_0;\,\mathbf{L}^s(B_1(\mathbf{0})))},\\ \rho^{2-\left(\frac{2}{r}+\frac{3}{s}\right)}\,\|\mathbf{curl}\,\mathbf{v}\|_{L^r(t_0-\rho^2,t_0;\,\mathbf{L}^s(B_\rho(\mathbf{x}_0)))} &= \|\mathbf{curl}'\mathbf{v}'\|_{L^r(t_0-1,t_0;\,\mathbf{L}^s(B_1(\mathbf{0})))}.\end{aligned}$$

These formulas show that the mentioned criteria are “scale invariant”.

Theorem 8 (J.N. 2011). *If* $\mathbf{v}$ *is a suitable weak solution, satisfying the condition*

$$\liminf_{t\to t_0-} \|\mathbf{v}(\,.\,,t)\|_{3;\,B_\delta(\mathbf{x}_0)} \;\leq\; \epsilon, \tag{3.5}$$

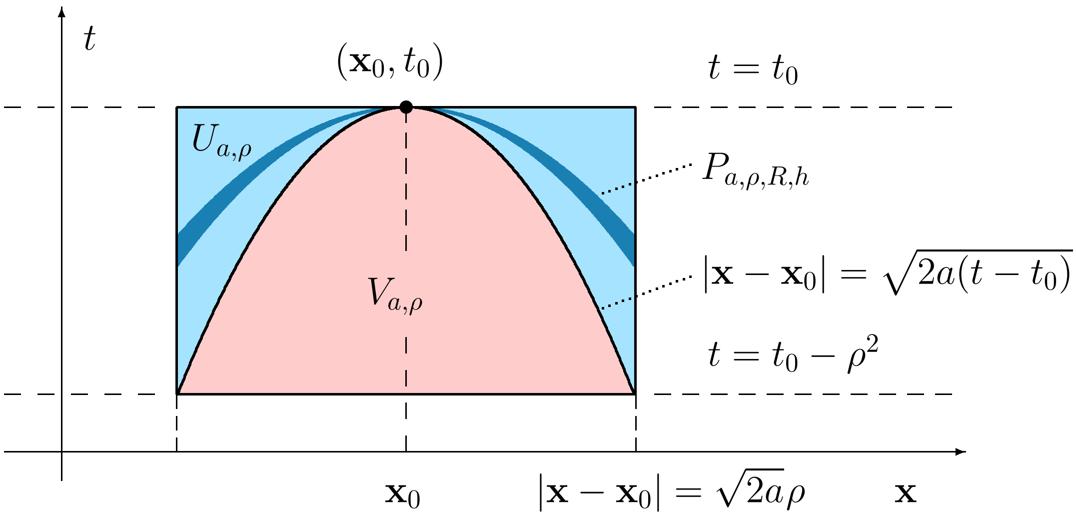
for some $\delta>0$ *then* $(\mathbf{x}_0,t_0)\notin\mathcal{S}(\mathbf{v})$.



J.N. (2012): $a > 0,\ 0 < \rho < \sqrt{t_0},\ R > 1,\ 0 < h < R - 1$; we denote

$$U_{a,\rho} := \left\{ (\mathbf{x}, t);\ t_0 - \rho^2 < t < t_0 \text{ and } \sqrt{2a(t_0 - t)} < |\mathbf{x} - \mathbf{x}_0| < \sqrt{2a}\,\rho \right\},$$

$$P_{a,\rho,R,h} := \left\{ (\mathbf{x}, t);\ t_0 - \rho^2/R^2 < t < t_0 \text{ and } (R - h)\sqrt{2a(t_0 - t)} < |\mathbf{x} - \mathbf{x}_0| < R\sqrt{2a(t_0 - t)} \right\},$$



Theorem 9 (J.N. 2012). *Let* $\mathbf{v}$ *be a suitable weak solution of (1.1)–(1.3),* p *be an associated pressure,* $(\mathbf{x}_0, t_0) \in \mathbb{R}^3 \times (0, T)$*,* $0 < a < 2\nu$ *and* $0 < \rho < \sqrt{t_0}$*. Suppose that*

(i) *function* $\mathbf{v}$ *satisfies the integrability condition in set* $U_{a,\rho}$*:*

$$\int_{t_0-\rho^2}^{t_0} \biggl(\int_{\sqrt{2a(t_0-t)}<|\mathbf{x}-\mathbf{x}_0|<\sqrt{2a}\,\rho} |\mathbf{v}(\mathbf{x},t)|^s \, \mathrm{d}\mathbf{x} \biggr)^{r/s} \mathrm{d}t \;<\; \infty \tag{3.6}$$

for some r, s, *satisfying* $\quad 3 \le r < \infty, \quad 3 < s < \infty, \quad \dfrac{2}{r} + \dfrac{3}{s} < 1,$

(ii) *there exist real numbers* $R > 1$, $0 < h < R - 1$, *such that function* p *satisfies the integrability condition in set* $P_{a,\rho,R,h}$*:*

$$\int_{t_0-\rho^2/R^2}^{t_0} \biggl(\int_{(R-h)\sqrt{2a(t_0-t)}<|\mathbf{x}-\mathbf{x}_0|<R\sqrt{2a(t_0-t)}} |p(\mathbf{x},t)|^\beta \, \mathrm{d}\mathbf{x} \biggr)^{\alpha/\beta} \mathrm{d}t \;<\; \infty \tag{3.7}$$

for some α, β, *satisfying* $\quad \dfrac{r}{r-1} \le \alpha < \infty, \quad \dfrac{3}{2} < \beta < \infty, \quad \dfrac{2}{\alpha} + \dfrac{3}{\beta} < 2.$

Then $(\mathbf{x}_0, t_0)$ *is a regular point of solution* $\mathbf{v}$*.*

4. Principle of the proof of Theorem 8

We call $\mathcal{A} \subset \mathcal{S}_{t_0}(\mathbf{v})$ a *separated subset of* $\mathcal{S}_{t_0}(\mathbf{v})$ if $\overline{\mathcal{A}} \cap \overline{\mathcal{S}_{t_0}(\mathbf{v}) \smallsetminus \mathcal{A}} = \emptyset$.

A separated subset of $\mathcal{S}_{t_0}(\mathbf{v})$ is a closed set in $\mathbb{R}^3$.

B_1 denotes a ball in $\mathbb{R}^3$ with radius 1.

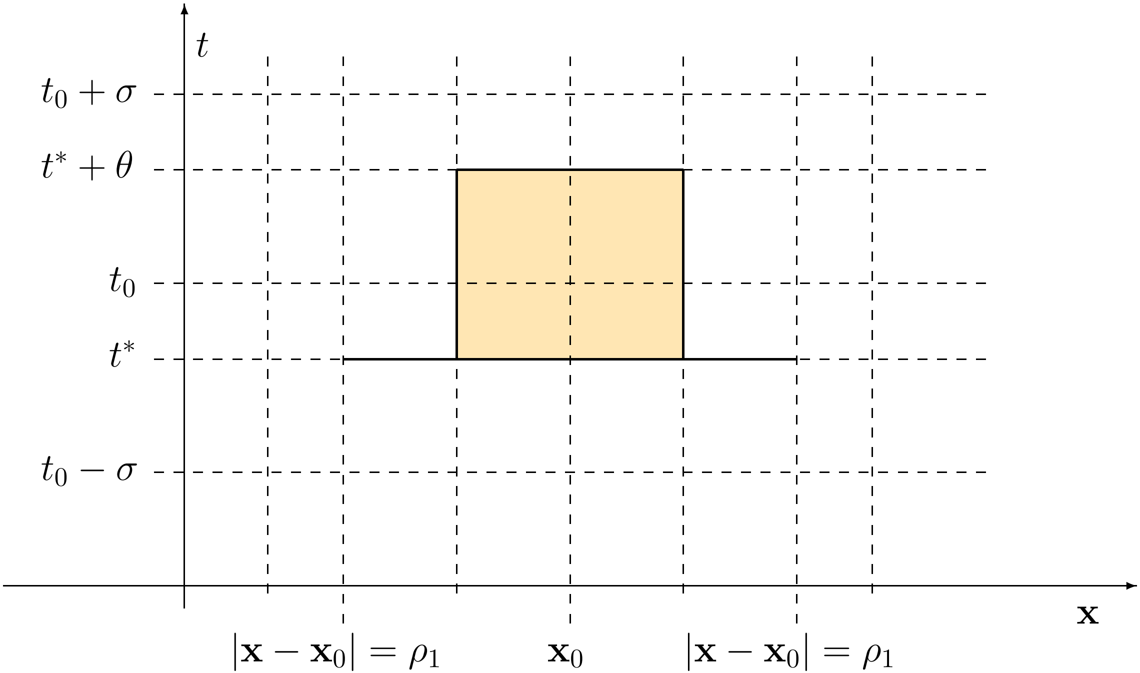
Lemma 5. *There exist $\sigma > 0$, $c_1 > 0$ and $\delta_1 > 0$ such that if $\mathcal{A}$ is a nonempty separated subset of $\mathcal{S}_{t_0}(\mathbf{v})$ such that $U_{1/2}(\mathcal{A}) \subset B_1$, and $\epsilon_1 > 0$, $\rho_1 \in (0, \frac{1}{2})$ are given numbers, then there exists $\theta > 0$ such that the inequality*

$$\|\mathbf{v}(\,.\,, t^*)\|_{3;\, U_{\rho_1}(\mathcal{A})} \; < \; \delta_1 \tag{4.1}$$

for some $t^* \in [t_0 - \sigma, t_0)$ implies that

$$\|\mathbf{v}(\,.\,, t)\|_{3;\, U_{\rho_1/2}(\mathcal{A})} \; \leq \; c_1\, \|\mathbf{v}(\,.\,, t^*)\|_{3;\, U_{\rho_1}(\mathcal{A})} + \epsilon_1 \tag{4.2}$$

for all $t \in (t^*, t^* + \theta) \cap (t_0 - \sigma, t_0 + \sigma)$.



Note that θ is independent of t^*.

Now, we prove Theorem 8 by contradiction: Suppose that to each $\epsilon > 0$ there exists a singular point $(\mathbf{x}_0, t_0) \in \mathbb{R}^3 \times (0, T)$ of solution $\mathbf{v}$ such that (3.5) holds.

Inequality (3.5) implies that there exists $\rho_2 > 0$ such that

$$\liminf_{t \to t_0-} \; \|\mathbf{v}(\,.\,, t)\|_{3;\, B_\rho(\mathbf{x}_0)} \; < \; \epsilon$$

holds for all $\rho \in (0, \rho_2)$. It means that there exists a sequence $t_n \nearrow t_0$ such that

$$\|\mathbf{v}(\,.\,, t_n)\|^3_{3;\, B_\rho(\mathbf{x}_0)} \; < \; \epsilon. \tag{4.3}$$

There exists a separated subset $\mathcal{A}$ of $\mathcal{S}_{t_0}(\mathbf{v})$ such that $\mathbf{x}_0 \in \mathcal{A} \subset B_{\rho/2}(\mathbf{x}_0)$.

We can assume without loss of generality that $t_n \in (t_0 - \sigma, t_0)$. Inequality (4.3) implies that

$$\|\mathbf{v}(\,.\,, t_n)\|^3_{3;\, U_{\rho/2}(\mathcal{A})} \; \leq \; \epsilon. \tag{4.4}$$

If ϵ is chosen so small that $\epsilon \leq \delta_1$ (where δ_1 is the number from Lemma 5) and if ϵ_1 is a positive number then Lemma 5 provides the existence of $\theta > 0$ such that

$$\|\mathbf{v}(\,.\,,t)\|_{3;\,U_{\rho/4}(\mathcal{A})} \;\leq\; c_1\,\|\mathbf{v}(\,.\,,t_n)\|_{3;\,U_{\rho/2}(\mathcal{A})} + \epsilon_1 \;\leq\; c_1\,\epsilon + \epsilon_1$$

for all $t \in (t_n, t_n + \theta) \cap (t_0 - \sigma, t_0 + \sigma)$. However, if ϵ_1 and ϵ are chosen so small that $c_1\,\epsilon + \epsilon_1 < \epsilon_3$, where ϵ_3 is the number on the right hand side of (3.1), then

$$\|\mathbf{v}(\,.\,,t)\|^3_{3;\,B_{\rho/4}(\mathbf{x}_0)} \;\leq\; \|\mathbf{v}(\,.\,,t)\|^3_{3;\,U_{\rho/4}(\mathcal{A})} \;\leq\; \epsilon_3^3$$

for all $t \in (t_0 - \sigma_0, t_0 + \sigma_0)$ for some $\sigma_0 > 0$. Since the L^3–norm dominates the weak L^3–norm, inequality (3.1) is fulfilled. Consequently, $(\mathbf{x}_0, t_0)$ cannot be a singular point of solution $\mathbf{v}$.

This is the contradiction.

The proof is completed. $\blacksquare$

5. Principle of the proof of Theorem 9

The used regularity criterion. We will prove that solution $\mathbf{v}$ satisfies the regularity criterion

$$\lim_{\delta\to 0+} \frac{1}{\delta^2} \int_{t_0-\delta^2}^{t_0} \int_{B_\delta(\mathbf{x}_0)} |\mathbf{v}|^3 \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \;=\; 0. \tag{3.1}$$

We denote

$$\theta(t) := \sqrt{2a(t_0-t)},$$

and derive an estimate of

$$\lim_{\delta\to 0+} \frac{1}{\delta^2} \int_{t_0-\delta^2}^{t_0} \int_{B_\delta(\mathbf{x}_0)} |\mathbf{v}|^3 \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \;=\; \lim_{\delta\to 0+} \big[A_\delta^I + A_\delta^{II} \big], \tag{5.1}$$

where

$$A_\delta^I \;:=\; \frac{1}{\delta^2} \int_{t_0-\delta^2}^{t_0} \int_{\theta(t)<|\mathbf{x}-\mathbf{x}_0|<\delta} |\mathbf{v}|^3 \, \mathrm{d}\mathbf{x} \, \mathrm{d}t,$$

$$A_\delta^{II} \;:=\; \frac{1}{\delta^2} \int_{t_0-\delta^2}^{t_0} \int_{|\mathbf{x}-\mathbf{x}_0|<\theta(t)} |\mathbf{v}|^3 \, \mathrm{d}\mathbf{x} \, \mathrm{d}t.$$

An estimate of A_δ^I. The integral in A_δ^I is an integral over a subset of $U_{a,\rho}$. Hence A_δ^I can be estimated as follows:

$$\begin{aligned} A_\delta^I &= \frac{1}{\delta^2} \int_{t_0-\delta^2}^{t_0} \int_{\theta(t)<|\mathbf{x}-\mathbf{x}_0|<\delta} |\mathbf{v}|^3 \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \\ &\leq \frac{1}{\delta^2} \int_{t_0-\delta^2}^{t_0} \biggl(\int_{\theta(t)<|\mathbf{x}-\mathbf{x}_0|<\delta} |\mathbf{v}|^s \, \mathrm{d}\mathbf{x} \biggr)^{\frac{3}{s}} \biggl(\frac{4\pi\delta^3}{3} \biggr)^{1-\frac{3}{s}} \mathrm{d}t \\ &\leq \biggl(\frac{4\pi}{3} \biggr)^{1-\frac{3}{s}} \delta^{3\left(1-\frac{2}{r}-\frac{3}{s}\right)} \biggl[\int_{t_0-\delta^2}^{t_0} \biggl(\int_{\theta(t)<|\mathbf{x}-\mathbf{x}_0|<\delta} |\mathbf{v}|^s \, \mathrm{d}\mathbf{x} \biggr)^{\frac{r}{s}} \mathrm{d}t \biggr]^{\frac{3}{r}}. \end{aligned}$$

The limit of the right hand side, for $\delta \to 0+$, equals zero due to condition (2.2) and the assumptions on r and s. Hence we have

$$\lim_{\delta \to 0+} A_\delta^I = 0. \tag{5.2}$$

Transformation to the new coordinates $\mathbf{x}', t'$. We use coordinates $\mathbf{x}'$ and t', which are related to $\mathbf{x}$ and t through the formulas

$$\mathbf{x}' = \frac{\mathbf{x}-\mathbf{x}_0}{\theta(t)}, \qquad t' = \int_{t_0-\rho^2}^{t} \frac{\mathrm{d}s}{\theta^2(s)} = \frac{1}{2a}\,\ln\frac{\rho^2}{t_0-t}. \tag{5.3}$$

Then

$$t = t_0 - \rho^2\,\mathrm{e}^{-2at'} \qquad \text{and} \qquad \theta(t) = \sqrt{2a}\,\rho\,\mathrm{e}^{-at'}.$$

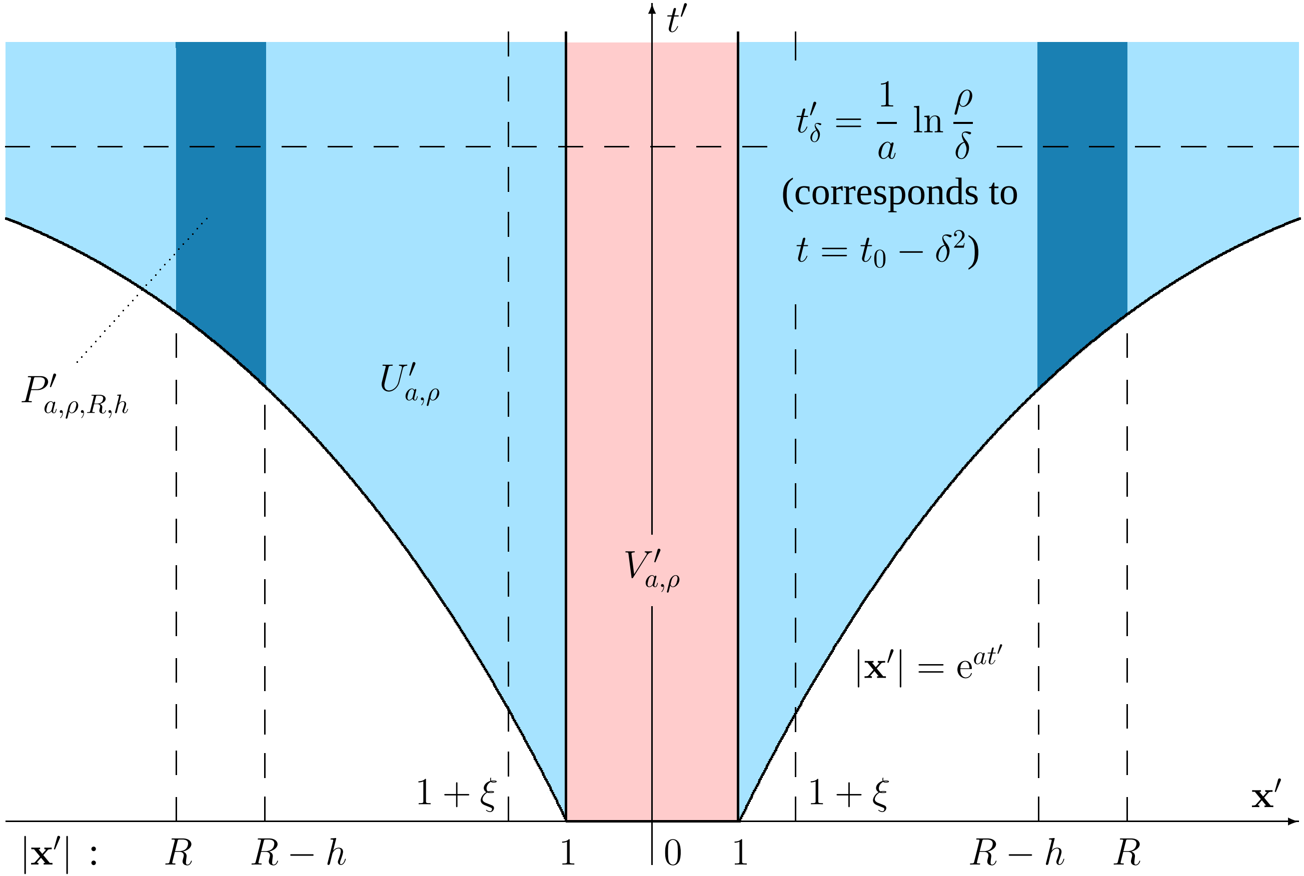
The time interval $(t_0-\rho^2, t_0)$ on the t–axis now corresponds to the interval $\big(0,\ \infty\big)$ on the t'–axis. Equations (5.3) represent a one–to–one transformation of the parabolic region $V_{a,\rho}$ in the $\mathbf{x}, t$–space onto the infinite stripe

$$V'_{a,\rho} \ :=\ \big\{(\mathbf{x}', t') \in \mathbb{R}^4;\ 0 < t' < \infty,\ |\mathbf{x}'| < 1\big\}$$

in the $\mathbf{x}', t'$–space. Similarly, (2.1) is a one–to–one transformation of set $U_{a,\rho}$ in the $\mathbf{x}, t$–space onto

$$U'_{a,\rho} \ :=\ \big\{(\mathbf{x}', t') \in \mathbb{R}^4;\ 0 < t' < \infty,\ 1 < |\mathbf{x}'| < \mathrm{e}^{at'}\big\}$$

in the $\mathbf{x}', t'$–space.



t'
$t'_\delta = \frac{1}{a} \ln \frac{\rho}{\delta}$
(corresponds to
$t = t_0 - \delta^2$)
$P'_{a,\rho,R,h}$
$U'_{a,\rho}$
$V'_{a,\rho}$
$|\mathbf{x}'| = e^{at'}$
$1+\xi$
$1+\xi$
$\mathbf{x}'$
$|\mathbf{x}'|$:
R
$R-h$
1
0
1
$R-h$
R

If we put $\quad \mathbf{v}(\mathbf{x},t) = \frac{1}{\theta(t)}\, \mathbf{v}'(\mathbf{x}',t'), \qquad p(\mathbf{x},t) = \frac{1}{\theta^2(t)}\, p'(\mathbf{x}',t'),$

then functions $\mathbf{v}'$, p' represent a suitable weak solution of the system of equations

$$\partial_{t'}\mathbf{v}' + \mathbf{v}'\cdot\nabla'\mathbf{v}' \;=\; -\nabla' p' + \nu\Delta'\mathbf{v}' - a\mathbf{v}' - a\mathbf{x}'\cdot\nabla'\mathbf{v}', \tag{5.4}$$

$$\mathrm{div}'\,\mathbf{v}' \;=\; 0 \tag{5.5}$$

for $\mathbf{x}' \in \mathbb{R}^3$ and $t' > 0$. Functions $\mathbf{v}'$ and p' satisfy the analog of the generalized energy inequality:

$$\begin{aligned}
&\|\varphi\mathbf{v}'(\,.\,,t')\|^2_{2;\,B_R(\mathbf{0})} + 2\nu\int_{t'_\delta}^{t'} \|\varphi\nabla'\mathbf{v}'(\,.\,,\tau)\|^2_{2;\,B_R(\mathbf{0})}\,\mathrm{d}\tau \;\leq\; \|\varphi\mathbf{v}'(\,.\,,t'_\delta)\|^2_{2;\,B_R(\mathbf{0})} \\
&\quad -\nu\int_{t'_\delta}^{t'}\int_{B_R(\mathbf{0})} \nabla'|\mathbf{v}'|^2\cdot\nabla'\varphi^2\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\tau + \int_{t'_\delta}^{t'}\int_{B_R(\mathbf{0})} \big(|\mathbf{v}'|^2 + 2p'\big)\,\big(\mathbf{v}'\cdot\nabla'\varphi^2\big)\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\tau \\
&\quad + \int_{t'_\delta}^{t'}\int_{B_R(\mathbf{0})} \big[\,a\varphi^2|\mathbf{v}'|^2 + \big(a\mathbf{x}'\cdot\nabla'\varphi^2\big)\,|\mathbf{v}'|^2\,\big]\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\tau
\end{aligned} \tag{5.6}$$

for a.a. $t'_\delta > a^{-1}\,\ln R$, all $t' \geq t'_\delta$ and all $\varphi \in C_0^\infty(B_R(\mathbf{0}))$.

The first estimate of A_δ^{II}. We have

$$
\begin{aligned}
A_\delta^{II} \;&=\; \frac{1}{\delta^2}\int_{t'_\delta}^{\infty}\int_{B_1(\mathbf{0})} |\mathbf{v}'|^3 \, \mathrm{d}\mathbf{x}' \; 2a\,\rho^2 \, \mathrm{e}^{-2at'} \; \mathrm{d}t' \\
&\leq\; 2a\int_{t'_\delta}^{\infty}\int_{B_1(\mathbf{0})} |\mathbf{v}'|^3 \, \mathrm{d}\mathbf{x}' \, \mathrm{d}t' \;\leq\; 2a\int_{t'_\delta}^{\infty}\biggl(\int_{B_1(\mathbf{0})} |\mathbf{v}'|^6 \, \mathrm{d}\mathbf{x}'\biggr)^{\frac{1}{4}} \biggl(\int_{B_1(\mathbf{0})} |\mathbf{v}'|^2 \, \mathrm{d}\mathbf{x}'\biggr)^{\frac{3}{4}} \mathrm{d}t' \\
&\leq\; C\int_{t'_\delta}^{\infty} \bigl(\|\mathbf{v}'\|^2_{2;\,B_1(\mathbf{0})} + \|\nabla'\mathbf{v}'\|^2_{2;\,B_1(\mathbf{0})}\bigr)^{3/4} \, \|\mathbf{v}'\|^{3/2}_{2;\,B_1(\mathbf{0})} \; \mathrm{d}t' \\
&\leq\; C\biggl(\int_{t'_\delta}^{\infty} \|\nabla'\mathbf{v}'\|^2_{2;\,B_1(\mathbf{0})} \, \mathrm{d}t'\biggr)^{\frac{3}{4}} \biggl(\int_{t'_\delta}^{\infty} \|\mathbf{v}'\|^6_{2;\,B_1(\mathbf{0})} \, \mathrm{d}t'\biggr)^{\frac{1}{4}} + C\int_{t'_\delta}^{\infty} \|\mathbf{v}'\|^3_{2;\,B_1(\mathbf{0})} \, \mathrm{d}t',
\end{aligned}
\tag{5.7}
$$

where C depends only on a.

Recall that $d := \frac{1}{5}h$. In order to estimate the integrals on the right hand side, we use the generalized energy inequality (5.6) with $\varphi(\mathbf{x}') = \varphi_{R,d}(\mathbf{x}')$, where $0 \leq \varphi_{R,d} \leq 1$ and

$$
\varphi_{R,d}(\mathbf{x}') = 1 \;\text{ for } |\mathbf{x}'| < R - 3d, \qquad \varphi_{R,d}(\mathbf{x}') = 0 \;\text{ for } |\mathbf{x}'| > R - 2d.
$$

A cut–off function η_ξ and related estimates. Let $\xi > 0$ and η_ξ be an infinitely differentiable cut–off function in $\mathbb{R}^3$ with values in the interval $[0,1]$, such that

$$\eta_\xi = 1 \ \text{ in } \ B_1(\mathbf{0}), \qquad \eta_\xi = 0 \ \text{ in } \ \mathbb{R}^3 \smallsetminus B_{1+\xi}(\mathbf{0}).$$

Let $\mu > 0$. Using the continuous imbedding $W^{1,2}(B_{1+\xi}(\mathbf{0})) \hookrightarrow L^2(B_{1+\xi}(\mathbf{0}))$, we derive the estimates

$$\begin{aligned} \|\eta_\xi \mathbf{v}'\|^2_{2;\, B_{1+\xi}(\mathbf{0})} \ &\leq \ \sum_{i=1}^3 \|\eta_\xi v_i'\|^2_{2;\, B_{1+\xi}(\mathbf{0})} \\ &\leq \ \sum_{i=1}^3 (1+\xi)^2 \, \|\nabla'(\eta_\xi v_i')\|^2_{2;\, B_{1+\xi}(\mathbf{0})} \ = \ (1+\xi)^2 \, \|\nabla'(\eta_\xi \mathbf{v}')\|^2_{2;\, B_{1+\xi}(\mathbf{0})} \\ &\leq \ (1+\xi)^2 \, (1+\mu) \, \|\eta_\xi \nabla' \mathbf{v}'\|^2_{2;\, B_{1+\xi}(\mathbf{0})} + \frac{(1+\xi)^2 \, c_2(\mu)}{\xi^2} \, \|\mathbf{v}'\|^2_{2;\, B_{1+\xi}(\mathbf{0}) \smallsetminus B_1(\mathbf{0})}. \end{aligned} \tag{5.8}$$

We further assume that numbers ξ and μ are chosen so small that $\xi < 1$ and $(1+\mu)(1+\xi)^2 a < 2\nu$.

An auxiliary inequality. For $0 < R_1 < R_2$, we denote

$$\begin{aligned} M_{R_1,R_2}(t) &:= \bigl\{\mathbf{x} \in \mathbb{R}^3;\ R_1\,\theta(t) < |\mathbf{x}-\mathbf{x}_0| < R_2\,\theta(t)\bigr\}, \\ M'_{R_1,R_2} &:= \bigl\{\mathbf{x}' \in \mathbb{R}^3;\ R_1 < |\mathbf{x}'| < R_2\bigr\}. \end{aligned}$$

We have

$$\begin{aligned} \int_{t'_\delta}^{\infty} \int_{M'_{1,R}} |\mathbf{v}'|^2 \,\mathrm{d}\mathbf{x}'\,\mathrm{d}t' &= \int_{t_0-\delta^2}^{t_0} \theta^{-3}(t) \int_{M_{1,R}(t)} |\mathbf{v}|^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ &\leq C\, c_3^{2/r}(\delta)\, c_4^{1-2/r}(\delta), \end{aligned} \tag{5.9}$$

where

$$c_3(\delta) := \int_{t_0-\delta^2}^{t_0} \biggl(\int_{M_{1,R}(t)} |\mathbf{v}|^s \,\mathrm{d}\mathbf{x} \biggr)^{r/s} \mathrm{d}t \qquad \text{and} \qquad c_4(\delta) := \int_{t_0-\delta^2}^{t_0} \theta^{-\frac{6r}{s(r-2)}}(t)\,\mathrm{d}t.$$

$c_3(\delta) \longrightarrow 0$ because $\bigl\{(\mathbf{x},t) \in \mathbb{R}^4;\ t_0-\delta^2 < t < t_0,\ \mathbf{x} \in M_{1,R}(t)\bigr\} \subset U_{a,\rho}$.

$c_4(\delta) \longrightarrow 0$ because $-\dfrac{6r}{s(r-2)} = -2 + 2\,\dfrac{\kappa}{\kappa+3/s} > -2$.

The right hand side of inequality (5.6). The right hand side of (5.6) can be split to the sum

$$\|\varphi_{R,d}\mathbf{v}'(\,.\,,t'_\delta)\|^2_{2;\,B_R(\mathbf{0})} + K^I_\delta + K^{II}_\delta + K^{III}_\delta + K^{IV}_\delta + K^V_\delta\,,$$

where

$$\begin{aligned}
K^I_\delta &:= -\nu\int_{t'_\delta}^{t'}\int_{B_R(\mathbf{0})} \nabla'|\mathbf{v}'|^2\cdot\nabla'\varphi^2_{R,d}\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\tau,\\
K^{II}_\delta &:= \int_{t'_\delta}^{t'}\int_{B_R(\mathbf{0})} |\mathbf{v}'|^2\,(\mathbf{v}'\cdot\nabla'\varphi^2_{R,d})\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\tau,\\
K^{III}_\delta &:= \int_{t'_\delta}^{t'}\int_{B_R(\mathbf{0})} 2p'\,(\mathbf{v}'\cdot\nabla'\varphi^2_{R,d})\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\tau,\\
K^{IV}_\delta &:= \int_{t'_\delta}^{t'}\int_{B_R(\mathbf{0})} a\eta^2_\xi\,|\mathbf{v}'|^2\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\tau,\\
K^V_\delta &:= \int_{t'_\delta}^{t'}\int_{B_R(\mathbf{0})} \bigl[a(\varphi^2_{R,d}-\eta^2_\xi)+(a\mathbf{x}'\cdot\nabla'\varphi^2_{R,d})\bigr]\,|\mathbf{v}'|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}\tau.
\end{aligned}$$

$$K^I_\delta \;:=\; -\nu \int_{t'_\delta}^{t'} \int_{B_R(\mathbf{0})} \nabla'|\mathbf{v}'|^2 \cdot \nabla' \varphi^2_{R,d} \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\tau \;=\; \nu \int_{t'_\delta}^{t'} \int_{B_R(\mathbf{0})} |\mathbf{v}'|^2 \, \Delta' \varphi^2_{R,d} \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\tau.$$

Since $\Delta'\varphi^2_{R,d}(\mathbf{x}')$ is supported for $R-3d \le |\mathbf{x}'| \le R-2d$, we get

$$\begin{aligned} K^I_\delta \;&\le\; \max|\Delta'\varphi^2_{R,d}| \int_{t'_\delta}^{\infty} \int_{M'_{R-3d,R-2d}} |\mathbf{v}'|^2 \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\tau \\ &\le\; \max|\Delta'\varphi^2_{R,d}| \int_{t'_\delta}^{\infty} \int_{M'_{1,R}} |\mathbf{v}'|^2 \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\tau \\ &\longrightarrow\; 0 \qquad \text{for } \delta \to 0+ \end{aligned} \tag{5.10}$$

because of (5.9).

The next term is

$$
\begin{aligned}
{\color{red}K_\delta^{II}} &:= \int_{t'_\delta}^{t'} \int_{B_R(\mathbf{0})} |\mathbf{v}'|^2 \, (\mathbf{v}' \cdot \nabla' \varphi_{R,d}^2) \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\tau \;\leq\; C \int_{t'_\delta}^{t'} \int_{M'_{R-3d,R-2d}} |\mathbf{v}'|^3 \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\tau \\
&= C \int_{t_0-\delta^2}^{t_0} \theta^{-2}(t) \int_{M_{R-3d,R-2d}(t)} |\mathbf{v}|^3 \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \\
&\leq \int_{t_0-\delta^2}^{t_0} \biggl(\int_{M_{R-3d,R-2d}(t)} |\mathbf{v}|^s \, \mathrm{d}\mathbf{x} \biggr)^{3/s} \theta^{1-9/s}(t) \, \mathrm{d}t \\
&\leq C \, c_3^{3/r}(\delta) \, c_5^{1-3/r}(\delta),
\end{aligned} \tag{5.11}
$$

where

$$
c_5(\delta) \;:=\; \int_{t_0-\delta^2}^{t_0} \theta^{\frac{s-9}{s}\,\frac{r}{r-3}}(t) \, \mathrm{d}t.
$$

$c_5(\delta) \longrightarrow 0$ for $\delta \longrightarrow 0+$ because $\quad \dfrac{s-9}{s} \, \dfrac{r}{r-3} = -2 + \dfrac{3}{1-3/r} \Bigl(1 - \dfrac{2}{r} - \dfrac{3}{s} \Bigr) > -2.$

Hence $K_\delta^{II} \longrightarrow 0$ for $\delta \longrightarrow 0+$.

In order to estimate the integral with pressure, we need the inequality

$$\int_{M'_{R-3d,R-2d}} |p'|^{\frac{s}{s-1}}\,\mathrm{d}\mathbf{x}' \;\le\; c_6 \int_{M'_{R-5d,R}} |\mathbf{v}'|^{\frac{2s}{s-1}}\,\mathrm{d}\mathbf{x}' + c_7 \left(\int_{M'_{R-5d,R}} |p'|\,\mathrm{d}\mathbf{x}' \right)^{\frac{s}{s-1}}$$

for a.a. $t' \in (0,\infty)$. The procedure is longer and technical. Finally, we obtain:

$$K_\delta^{III} \;:=\; \int_{t'_\delta}^{t'} \int_{B_R(\mathbf{0})} 2p'\,(\mathbf{v}'\cdot\nabla'\varphi_{R,d}^2)\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\tau \;\longrightarrow\; 0 \qquad \text{for } \delta \to 0+\,.$$

The next integral on the right hand side of (5.6) can be estimated by means of (5.8):

$$\begin{aligned} K_\delta^{IV} \;&:=\; \int_{t'_\delta}^{t'} \int_{B_R(\mathbf{0})} a\eta_\xi^2\,|\mathbf{v}'|^2\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\tau \\ &\le\; a(1+\xi)^2(1+\mu) \int_{t'_\delta}^{t'} \|\eta_\xi \nabla'\mathbf{v}'\|_{2;\,B_{1+\xi}(\mathbf{0})}\,\mathrm{d}t' + c_8(\delta), \end{aligned}$$

where $\quad c_8(\delta) \;:=\; \dfrac{(1+\xi)^2\,c_2(\mu)}{\xi^2} \displaystyle\int_{t'_\delta}^{t'} \|\mathbf{v}'\|_{2;\,M'_{1,1+\xi}}^2\,\mathrm{d}\mathbf{x}'\,\mathrm{d}t' \;\to\; 0 \quad$ because of (5.9).

Finally, we have

$$
\begin{aligned}
K_\delta^V \;&:=\; \int_{t'_\delta}^{t'} \int_{B_R(\mathbf{0})} \big[a(\varphi_{R,d}^2 - \eta_\xi^2) + (a\mathbf{x}' \cdot \nabla' \varphi_{R,d}^2)\big]\, |\mathbf{v}'|^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}\tau \\
&\leq\; \int_{t'_\delta}^{\infty} \int_{B_R(\mathbf{0})} a(\varphi_{R,d}^2 - \eta_\xi^2)\, |\mathbf{v}'|^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}\tau \\
&\leq\; \int_{t'_\delta}^{\infty} \int_{M'_{1,R}} a(\varphi_{R,d}^2 - \eta_\xi^2)\, |\mathbf{v}'|^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}\tau \\
&\longrightarrow\; 0 \qquad \text{for } \delta \to 0+ \qquad \text{because of (5.9).}
\end{aligned}
$$

Thus, we obtain the inequality

$$
K_\delta^I + \ldots + K_\delta^V \;\leq\; a(1+\xi)^2(1+\mu) \int_{t'_\delta}^{t'} \|\eta_\xi \nabla' \mathbf{v}'\|_{2;\, B_{1+\xi}(\mathbf{0})}\, \mathrm{d}t' + c_9(\delta),
$$

where $c_9(\delta) \to 0$ **as** $\delta \to 0+$.

Substituting this to (5.6), we obtain

$$\|\varphi_{R,d}\mathbf{v}'(\,.\,,t')\|^2_{2;\,B_{R-2d}(\mathbf{0})} + 2\nu \int_{t'_\delta}^{t'} \int_{M'_{1,R-2d}} (\varphi^2_{R,d} - \eta^2_\xi)\, |\nabla'\mathbf{v}'(\,.\,,\tau)|^2 \,\mathrm{d}\mathbf{x}'\,\mathrm{d}\tau$$

$$+ \big[2\nu - a(1+\xi)^2(1+\mu)\big] \int_{t'_\delta}^{t'} \int_{B_{1+\xi}(\mathbf{0})} \eta^2_\xi\, |\nabla'\mathbf{v}'(\,.\,,\tau)|^2 \,\mathrm{d}\mathbf{x}'\,\mathrm{d}\tau$$

$$\leq\ \|\varphi_{R,d}\mathbf{v}'(\,.\,,t'_\delta)\|^2_{2;\,B_{R-2d}(\mathbf{0})} + c_9(\delta).$$

This yields

$$\|\varphi_{R,d}\mathbf{v}'(\,.\,,t')\|^2_{2;\,B_{R-2d}(\mathbf{0})}\ \leq\ \|\varphi_{R,d}\mathbf{v}'(\,.\,,t'_\delta)\|^2_{2;\,B_{R-2d}(\mathbf{0})} + c_9(\delta), \tag{5.12}$$

$$2\nu \int_{t'_\delta}^{\infty} \int_{M'_{1,R-2d}} (\varphi^2_{R,d} - \eta^2_\xi)\, |\nabla'\mathbf{v}'(\,.\,,\tau)|^2 \,\mathrm{d}\mathbf{x}'\,\mathrm{d}\tau$$

$$+ \big[2\nu - a(1+\xi)^2(1+\mu)\big] \int_{t'_\delta}^{\infty} \int_{B_{1+\xi}(\mathbf{0})} \eta^2_\xi\, |\nabla'\mathbf{v}'(\,.\,,\tau)|^2 \,\mathrm{d}\mathbf{x}'\,\mathrm{d}\tau$$

$$\leq\ \|\varphi_{R,d}\mathbf{v}'(\,.\,,t'_\delta)\|^2_{2;\,B_{R-2d}(\mathbf{0})} + c_9(\delta). \tag{5.13}$$

Using the integrability of $\|\varphi_{R,d}\mathbf{v}'(\,.\,,s)\|^2_{2;\,B_{R-2d}(\mathbf{0})}$, as a function of s, in the interval $(a^{-1}\,\ln R,\,\infty)$, and estimate (5.12), we can prove that

$$\|\varphi_{R,d}\mathbf{v}'(\,.\,,s)\|_{2;\,B_{R-2d}(\mathbf{0})} \to 0 \qquad \text{for} \;\; s\to\infty.$$

Consequently, since $t'_\delta \to \infty$ for $\delta \to 0+$, the right hand sides of (5.12) and (5.13) tend to zero if $\delta \to 0+$. We denote the right hand sides by $c_{10}(\delta)$.

Final estimates of A^{II}_δ. The integral of $\|\nabla'\mathbf{v}'\|^2_{2;\,B_1(\mathbf{0})}$ on the right hand side of (5.7) can be estimated by means of (5.13):

$$\int_{t'_\delta}^{\infty} \|\nabla'\mathbf{v}'\|^2_{2;\,B_1(\mathbf{0})}\,\mathrm{d}t' \;\leq\; \frac{c_{10}(\delta)}{2\nu - a(1+\xi)^2(1+\mu)}. \tag{5.14}$$

The integral of $\|\mathbf{v}'\|^6_{2;\,B_1(\mathbf{0})}$ on the right hand side of (5.7) can be estimated by means of (5.12), (5.8) and (5.13):

$$
\begin{aligned}
\int_{t'_\delta}^{\infty} \|\mathbf{v}'\|_{2;\,B_1(\mathbf{0})}^6 \, \mathrm{d}t' \;&\leq\; c_{10}^2(\delta) \int_{t'_\delta}^{\infty} \|\mathbf{v}'\|_{2;\,B_1(\mathbf{0})}^2 \, \mathrm{d}t' \\
&\leq\; c_{10}^2(\delta)\,(1+\xi)^2 \int_{t'_\delta}^{\infty} \Big[(1+\mu)\,\|\eta_\xi \nabla' \mathbf{v}'\|_{2;\,B_{1+\xi}(\mathbf{0})}^2 + \frac{c_2(\mu)}{\xi^2}\,\|\mathbf{v}'\|_{2;\,M'_{1,1+\xi}}^2 \Big] \, \mathrm{d}t' \\
&\leq\; c_{10}^2(\delta)\,(1+\xi)^2 \left[\frac{(1+\mu)\,c_{10}(\delta)}{2\nu - a(1+\xi)^2(1+\mu)} + \frac{c_2(\mu)}{\xi^2} \int_{t'_\delta}^{\infty} \|\mathbf{v}'\|_{2;\,M'_{1,1+\xi}}^2 \, \mathrm{d}t' \right].
\end{aligned}
$$

The integral of $\|\mathbf{v}'\|_{2;\,M'_{1,1+\xi}}^2$ tends to zero for $\delta \to 0+$ due to (5.9). Thus, we obtain

$$
\int_{t'_\delta}^{\infty} \|\mathbf{v}'\|_{2;\,B_1(\mathbf{0})}^6 \, \mathrm{d}t' \;\longrightarrow\; 0 \qquad \text{for } \delta \to 0+\,. \tag{5.15}
$$

The integral of $\|\mathbf{v}'\|_{2;\,B_1(\mathbf{0})}^3$ on the right hand side of (5.7) can be estimated similarly as the integral of $\|\mathbf{v}'\|_{2;\,B_1(\mathbf{0})}^6$. Hence we also have

$$
\int_{t'_\delta}^{\infty} \|\mathbf{v}'\|_{2;\,B_1(\mathbf{0})}^3 \, \mathrm{d}t' \;\longrightarrow\; 0 \qquad \text{for } \delta \to 0+\,. \tag{5.16}
$$

It follows from (5.7), (5.14), (5.15) and (5.16) that

$$\lim_{\delta \to 0+} A_\delta^{II} = 0. \tag{5.17}$$

Conclusion. We observe from (5.2) and (5.17) that function v satisfies condition (3.2). Hence $(\mathbf{x}_0, t_0)$ is a regular point of solution v.

The proof is completed. ■