Some recent results on regularity criteria for weak solutions of the Navier-Stokes equations

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Lecture 1 – Contents

1. **Weak solution** to the Navier–Stokes equations. Basic properties. **Energy inequality** and **strong energy inequality**.
   Interior regularity under Serrin’s condition.

2. **Suitable weak solution** to the Navier–Stokes equations. **Generalized energy inequality**. The notion of a **regular** or **singular point** of a suitable weak solution.
   Equivalence of some definitions.
   The 1–dimensional Hausdorff measure of the set of possible singular points.

3. A brief survey of known criteria for regularity at the space–time point \((x_0, t_0)\) (from Serrin, Caffarelli–Kohn–Nirenberg to some recent results).

4. Principles of proofs of some recently obtained criteria.
1. Weak solution to the Navier–Stokes equations

\( \Omega \) \ldots a domain in \( \mathbb{R}^3 \) with a Lipschitz–continuous boundary

\( T > 0, \ Q_T := \Omega \times (0, T) \)

We deal with the Navier–Stokes initial–boundary value problem for viscous incompressible fluid

\[
\begin{align*}
\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} &= -\nabla p + \nu \Delta \mathbf{v} + \mathbf{f} & \text{in } Q_T, \quad (1.1) \\
\text{div } \mathbf{v} &= 0 & \text{in } Q_T, \quad (1.2) \\
\mathbf{v} &= 0 & \text{on } \partial \Omega \times (0, T), \quad (1.3) \\
\mathbf{v} &= \mathbf{v}_0 & \text{in } \Omega \times \{0\}. \quad (1.4)
\end{align*}
\]

(H. Navier 1824, G. Stokes 1845)
First qualitative results on the existence of solutions: J. Leray in the 30–ties of the 20th century.

Leray introduced the notion of the weak solution of the boundary–value problem (1.1)–(1.4). (In fact, Leray studied the case $\Omega = \mathbb{R}^3$. The case of a bounded domain $\Omega$ was treated by E. Hopf in 1951.)

A weak solution to problem (1.1)–(1.4). Let $v_0 \in L^2_\sigma(\Omega)$ and $f \in L^2(Q_T)$. A vector function $v \in L^2(0, T; W^{1,2}_0(\Omega)) \cap L^\infty(0, T; L^2_\sigma(\Omega))$ is said to be a weak solution of the problem (1.1)–(1.4) if for all $\phi \in C_0^{\infty}(\Omega \times [0, T))$:

$$
\int_0^T \int_\Omega \left[ v \cdot \partial_t \phi - \nu \nabla v : \nabla \phi - v \cdot \nabla v \cdot \phi \right] \, dx \, dt
= -\int_0^T \int_\Omega f \cdot \phi \, dx \, dt - \int_\Omega v_0 \cdot \phi(., 0) \, dx. \tag{1.5}
$$

1. Weak solution to the Navier–Stokes equations
**Remark: function space** $L^q_\sigma(\Omega)$. Let $C^\infty_{0,\sigma}(\Omega)$ be the linear space of all infinitely differentiable divergence–free vector functions in $\Omega$ with a compact support in $\Omega$. For $1 \leq q \leq \infty$, we denote by $L^q_\sigma(\Omega)$ the closure of $C^\infty_{0,\sigma}(\Omega)$ in $L^q(\Omega)$.

**Remark: characterization of the space** $L^q_\sigma(\Omega)$. Assume that $\Omega$ has a locally Lipschitzian boundary and $1 < q < \infty$. Let $L^q_{\text{div}}(\Omega)$ be the space of functions $v \in L^q(\Omega)$ such that $\text{div } v \in L^q(\Omega)$. One can prove that

1) The space $C^\infty(\overline{\Omega})$ is dense in $L^q_{\text{div}}(\Omega)$.

2) The mapping $\gamma_n : v \mapsto v \cdot n$ defined on $C^\infty(\overline{\Omega})$ can be extended to a continuous linear mapping from $L^q_{\text{div}}(\Omega)$ to $W^{-1/q,q}(\partial \Omega)$.

The space $L^q_\sigma(\Omega)$ can now be characterized as a space of functions from $L^q_{\text{div}}(\Omega)$, whose divergence equals zero in $\Omega$ (in the sense of distributions) and such that $\gamma_n v = 0$ (the zero element of $W^{-1/q,q}(\partial \Omega)$).
Lemma 1 (Hopf 1951, Prodi 1959, Serrin 1963). The weak solution $v$ to problem
(1.1)–(1.4) can be redefined on a set of zero Lebesgue measure so that $v(.,t) \in \mathbb{L}^2(\Omega)$ for all $t \in [0,T)$ and for all $\varphi \in C_{0,\sigma}^\infty(\Omega \times [0,T))$:

$$
\int_0^t \int_\Omega \left[ v \cdot \partial_\tau \varphi - \nu \nabla v : \nabla \varphi - v \cdot \nabla v \cdot \varphi \right] \, dx \, d\tau \\
= - \int_0^t \int_\Omega f \cdot \varphi \, dx \, d\tau + \int_\Omega v(.,t) \cdot \varphi(.,t) \, dx - \int_\Omega v_0 \cdot \varphi(.,0) \, dx.
$$

(1.6)

Principle of the proof: We use a $C^1$ function $\theta_h$ as on the figure. We use (1.6) with $\varphi(x,\tau) \theta_h(\tau)$ instead of $\varphi(x,\tau)$, and we consider the limit for $h \to 0$. 

1. Weak solution to the Navier–Stokes equations
Theorem 1 (existence of a weak solution – Leray 1934, Hopf 1951, et al). Let $\Omega$ be a domain in $\mathbb{R}^3$, $T > 0$, $v_0 \in L^2_\sigma(\Omega)$ and $f \in L^2(Q_T)$. Then there exists at least one weak solution $v$ to problem (1.1)–(1.4). The solution satisfies

- the energy inequality (EI)

$$\|v(., t)\|_2^2 + 2\nu \int_0^t \|\nabla v(., \tau)\|_2^2 d\tau$$

$$\leq \|v_0\|_2^2 + 2 \int_0^t (v(., \tau), f(., \tau))_2 d\tau$$  \hspace{1cm} (1.7)

for all $t \in [t, T)$,

- $\lim_{t \to 0^+} \|v(., t) - v_0\|_2 = 0$.

Open questions:

- Does each weak solution satisfy (EI), or even the energy equality (EE)?
- Is the weak solution unique?
- Is the weak solution regular provided that $v_0$ and $f$ are regular?
(EI) does not exclude e.g. this behaviour of the kinetic energy $E(t) := \|v(\cdot, t)\|_2^2$:

The inequality, which excludes the growth of $E(t)$, is the so called strong energy inequality (SEI):

$$\|v(\cdot, t)\|_2^2 + 2\nu \int_s^t \|\nabla v(\cdot, \tau)\|_2^2 \, d\tau \leq \|v(\cdot, s)\|_2^2 + 2 \int_s^t (v(\cdot, \tau), f(\cdot, \tau))_2 \, d\tau$$  \hspace{1cm} (1.8)

for a.a. $s \in [0, T)$ and all $t \in [s, T)$.

**Question:** Does the solution, provided by Theorem 1, satisfy (SEI)?
Partial answers regarding (EE): Serrin (1963): If \( v \in L^r(0, T; L^s(\Omega)) \), where \( 2/r + 3/s \leq 1 \), \( 3 \leq s \leq \infty \), \( 2 \leq r \leq \infty \) then \( v \) satisfies (EE).

- Shinbrot (1974), Taniuchi (1997): If \( v \in L^r(0, T; L^s(\Omega)) \), where \( 2/r + 3/s \leq 1 + 1/s \), \( 4 \leq s \leq \infty \), then \( v \) satisfies (EE).

- Farwig and Taniuchi (2010): observed that if \( v \in L^r(0, T; L^s(\Omega)) \), where \( 2/r + 3/s \leq 1 + 1/r \), \( 4 \leq r \leq \infty \), then \( v \) satisfies (EE).

As to (SEI), Leray (1934), Galdi and Maremonti (1986), Miyakawa and Sohr (1988), Farwig, Kozono and Sohr (2005) proved: Weak solution \( v \) can be constructed so that it satisfies not only (EI), but also (SEI).

Partial answer to the question of uniqueness:

Theorem 2 (Prodi 1959, Lions and Prodi 1959, et al). Let \( u \) and \( v \) be two weak solutions of the problem (1.1)–(1.4), with the same data \( v_0 \) and \( f \). Assume that

1) \( u \) satisfies (EI),

2) \( v \) satisfied at least one of the conditions

(a) \( v \in L^r(0, T; L^s(\Omega)) \) for some \( r, s \) satisfying \( 2/r + 3/s = 1, 3 < s \leq \infty \)

(b) \( v \in L^\infty(0, T; L^3(\Omega)) \) and \( v(\cdot, t) \) is right–continuous in the norm of \( L^3(\Omega) \) in dependence on \( t \) for \( 0 \leq t < T \).

Then \( u = v \) a.e. in \( Q_T \).

Kozono and Sohr (1996): if \( \Omega \) is a domain with a ,,smooth” bounded boundary, then the condition of right–continuity in condition (ii) can be omitted.

1. Weak solution to the Navier–Stokes equations
Assume, for simplicity, that $f \equiv 0$.

**Theorem 3 (interior regularity – Serrin 1963).** Let $v$ be a weak solution to (1.1)–(1.4) with $f \equiv 0$. Assume, in addition, that there exists a sub–domain $\Omega'$ of $\Omega$ and $0 \leq t_1 < t_2 \leq T$ so that

(a) $v \in L^r(t_1, t_2; L^s(\Omega'))$ for some $r, s$ satisfying $\frac{2}{r} + \frac{3}{s} = 1$, $3 < s \leq \infty$.

Then, given any bounded domain $\Omega'' \subset \overline{\Omega''} \subset \Omega'$ and $0 < \delta < (t_2 - t_1)/2$, each space derivative of $v$ is bounded in $\overline{\Omega''} \times [t_1 + \delta, t_2 - \delta]$.

If, in addition,

(b) $\partial_t v \in L^2(t_1, t_2; L^q(\Omega'))$ for some $q \geq 1$

then each space derivative of $v$ is absolutely continuous function of $t$.

**Remark.** The analogous result in the case $\Omega = \mathbb{R}^3$ and $s = 3$ follows from the work of Escauriaza, Seregin, Šverák (2003).
**Remark: interior regularity of $\partial_t v$ and $p$.** Condition (a) implies that $\partial_t v$ and $p$ have all spatial derivatives in $L^\alpha(t_1 + \delta, t_2 - \delta; L^\infty(\Omega''))$ for each $\alpha \in [1, 2)$, (see J. N., Penel 2001, Kučera, Skalák 2003.)

**Principle of the proof.**

Let $\zeta > 0$ be so small that $U_{4\zeta}(\Omega'') \subset \Omega'$,

Denote by $\psi$ an infinitely differentiable cut–off function defined in $\mathbb{R}^3$ such that $0 \leq \psi \leq 1$ and

$$
\psi = \begin{cases} 
1 & \text{on } U_{\zeta}(\Omega''), \\
0 & \text{on } \mathbb{R}^3 \setminus U_{3\zeta}(\Omega''). 
\end{cases}
$$

The product $\psi p$ satisfies

$$
\psi(x) p(x, t) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x - y|} \left[ \Delta(\psi p) \right](y, t) \, dy.
$$

If we use the integration by parts and the equation

$$
\Delta p = -\partial_i \partial_j (v_i v_j),
$$
we get (for $x \in \Omega''$)

$$
\psi(x) p(x, t) = p^I(x, t) + p^{II}(x, t)
$$

where

$$
p^I(x, t) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x-y|} \left[ \partial_i \partial_j (\psi v_i v_j) \right](y, t) \, dy,
$$

$$
p^{II}(x, t) = -\frac{1}{2\pi} \int_{\mathbb{R}^3} \frac{x_i - y_i}{|x-y|^3} \left[ (\partial_i \psi) v_i v_j \right](y, t) \, dy
$$

\[ - \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x-y|} \left[ (\partial_i \partial_j \psi) v_i v_j \right](y, t) \, dy + \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{x_i - y_i}{|x-y|^3} \left[ (\partial_i \psi) p \right](y, t) \, dy \]

\[ + \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x-y|} \left[ \Delta \psi p \right](y, t) \, dy. \]

Using the boundedness of $v$ and its spatial derivatives on $\text{supp} \psi \times (t_1, t_2)$, we obtain

$$
|\nabla^k p^I(x, t) \leq C(k).
$$

1. Weak solution to the Navier–Stokes equations
The integrals in $p^{II}$ can be considered only for $y \in U_{3\zeta}(\Omega'') \setminus U_{\zeta}(\Omega'')$ where $v$ and its spatial derivatives are bounded and $|x - y| \geq \zeta$. Thus,

$$|\nabla^k p^{II}(x, t)| \leq C(k) \int_{\text{supp } \nabla \psi} |p(y, t)| \, dy + C(k).$$

Hence

$$\int_{t_1}^{t_2} \left[ \max_{x \in \Omega'} |\nabla^k p^{II}(x, t)| \right] \alpha \, dt \leq C(k) \int_{t_1}^{t_2} \left( \int_{\text{supp } \nabla \psi} |p(y, t)| \, dy \right) \alpha \, dt + C(k)$$

$$\leq C(k) \int_{t_1}^{t_2} \left( \int_{U_{\zeta}(\Omega''')} |p(y, t)|^\beta \, dy \right) ^{\alpha/\beta} \, dt + C(k)$$

where $\beta$ is chosen so that $2/\alpha + 3/\beta = 3$. Due to the results of Taniuchi (1997) and Kozono (1998), $p \in L^\alpha(t_1, t_2; L^\beta(U_{\zeta}(\Omega'')))$ for $1 < \alpha < 2$, $3/2 < \beta < 3$ such that $2/\alpha + 3/\beta = 3$. Hence the last integral is finite.

**Remark.** If $\Omega = \mathbb{R}^3$ then one can use a little different technique to show that $\partial_t v$ and $p$ have all spatial derivatives in $L^\infty(t_1 + \delta, t_2 - \delta; L^\infty(\Omega'')).$
**Corollary.** If \((x_1, t_1), (x_2, t_2) \in \Omega'' \times (t_1 + \delta, t_2 - \delta)\) then

\[
|v(x_1, t_1) - v(x_2, t_2)| \leq |v(x_1, t_1) - v(x_2, t_1)| + |v(x_2, t_1) - v(x_2, t_2)| \\
\leq C |x_1 - x_2| + \int_{t_2}^{t_1} \partial_t v(x_2, t) \, dt \\
\leq C |x_1 - x_2| + \int_{t_2}^{t_1} \|\partial_t v(., t)\|_{\infty; \Omega''} \, dt \\
\leq C |x_1 - x_2| + \left(\int_{t_1}^{t_2} \|\partial_t v(., t)\|_{\infty; \Omega''} \, dt\right)^{1/\alpha} |t_1 - t_2|^{(\alpha-1)/\alpha} \\
\leq C |x_1 - x_2| + C |t_1 - t_2|^{(\alpha-1)/\alpha}.
\]

This implies the Hölder–continuity of \(v\) in \(\Omega'' \times (t_1 + \delta, t_2 - \delta)\).
2. A suitable weak solution of the problem (1.2)–(1.4)

L. Caffarelli, R. Kohn and L. Nirenberg (1983) called a weak solution $v$ of (1.1)–(1.4) a suitable weak solution if an associated pressure $p$ belongs to $L^{5/4}(Q_T)$ and the pair $(v, p)$ satisfies the so called generalized energy inequality (GEI)

$$2\nu \int_0^T \int_\Omega |\nabla v|^2 \varphi \, dx \, dt \leq \int_0^T \int_\Omega \left[ |v|^2 (\partial_t \varphi + \nu \Delta \varphi) + (|v|^2 + 2p) v \cdot \nabla \varphi \right] \, dx \, dt$$

$$+ \int_0^T \int_\Omega 2v \cdot f \varphi \, dx \, dt$$

(2.1)

for every non–negative function $\varphi$ from $C^\infty_0(Q_T)$.

C-K-N proved the existence of a suitable weak solution in the case when $\Omega$ is either $\mathbb{R}^3$ or a “smooth” bounded domain in $\mathbb{R}^3$. (The proof is based on the applications of the so called “retarded mollifications” in the nonlinear term . . . $\Psi_\delta(v) \cdot \nabla v$.)

(See also V. Scheffer 1977 for the proof in the case $f = 0$.)
C-K-N defined a **regular point** of a weak solution \( v \) as a point in \( Q_T \) such that there exists a neighbourhood \( U \) of this point, where \( v \) is essentially bounded.

A point in \( Q_T \) that is not regular is called **singular**.

\[ S(v) \ldots \] the set of all singular points of solution \( v \) in \( Q_T \)

Clearly, since the set of regular points is open in \( Q_T \), the set \( S(v) \) of singular points is closed in \( Q_T \).

Put \( Q_r^*(x, t) := B_r(x) \times (t - \frac{7}{8}r^2, t + \frac{1}{8}r^2) \)

**Lemma 2 (C-K-N 1983).** Let \( v \) be a suitable weak solution of the problem (1.1)–(1.4). There exists a constant \( \epsilon > 0 \) such that if

\[
\limsup_{r \to 0^+} \frac{1}{r} \int \int_{Q_r^*(x_0, t_0)} |\nabla v|^2 \, dx \, dt \leq \epsilon
\]  

(2.2)

then \( (x_0, t_0) \notin S(v) \).
Theorem 4 (C-K-N 1983). Let \( v \) be a suitable weak solution of the problem (1.1)–(1.4). Then the 1–dimensional Hausdorff measure of \( S(v) \) is zero.

**Principle of the proof.**

\[
(x_0, t_0) \in S(v) \implies \limsup_{r \to 0^+} \frac{1}{r} \int \int_{Q_r^*(x_0, t_0)} |\nabla v|^2 \, dx \, dt > \epsilon
\]

Let \( U \) be a neighbourhood of \( S(v) \) in \( Q_T \) and \( \delta > 0 \).

To each \((x_0, t_0) \in S(v)\) choose \( Q_r^*(x, t) \subset U \) (with \( r < \delta \)) such that

\[
\frac{1}{r} \int \int_{Q_r^*(x_0, t_0)} |\nabla v|^2 \, dx \, dt > \epsilon.
\]

Let us denote by \( \mathcal{J} \) the family of all such cylinders. Due to Vitali’s covering lemma, there exists an at most countable sub–family \( \mathcal{J}' = \{ Q_{r_i}^*(x_i, t_i) \} \) of \( \mathcal{J} \) such that

\[
Q_{r_i}^*(x_i, t_i) \cap Q_{r_j}^*(x_j, t_j) = \emptyset \quad \text{for } i \neq j,
\]

\[
\forall Q_r^*(x, t) \in \mathcal{J} \quad \exists Q_{r_i}^*(x_i, t_i) \in \mathcal{J}' : Q_r^*(x, t) \subset Q_{5r_i}^*(x_i, t_i).
\]

Consequently: \( S(v) \subset \bigcup_i Q_{5r_i}^*(x_i, t_i) \). Moreover,
\[
\sum_{i} 5r_i \leq \frac{5}{\epsilon} \sum_{i} \int_{Q_{r_i}(x_i,t_i)} |\nabla v|^2 \, dx \, dt \leq \frac{5}{\epsilon} \int_{U} |\nabla v|^2 \, dx \, dt,
\]
\[
\sum_{i} (5r_i)^3 \leq 125\delta^2 \sum_{i} 5r_1 \leq \frac{125\delta^2}{\epsilon} \int_{U} |\nabla v|^2 \, dx \, dt.
\]
Since \( \delta > 0 \) can be arbitrarily small, we deduce that the 3D Lebesgue measure of \( S_{CKN}(v) \) is zero. Thus, neighbourhood \( U \) can be chosen so that its 3D Lebesgue measure is arbitrarily small. Hence the right hand side can be arbitrarily small. This implies that \( \mathcal{P}^1(S(v)) = 0 \). Consequently, \( \mathcal{H}^1(S(v)) = 0 \).

**What is a real regularity of a suitable weak solution in the neighbourhood of C-K-N’s regular point \((x_0, t_0)\)?**

**Answer:** there exists \( R > 0 \) and \( \delta > 0 \) such that

a) \( v \) and its all spatial derivatives are in \( L^\infty(B_R(x_0) \times (t_0 - \delta, t_0 + \delta)) \),

b) \( \partial_t v \) and \( p \) have all spatial derivatives in \( L^\alpha(t_0 - \delta, t_0 + \delta; L^\infty(B_R(x_0))) \) for each \( \alpha \in [1, 2) \) if \( \Omega \) is bounded and for \( \alpha = \infty \) if \( \Omega = \mathbb{R}^3 \).

c) \( v \) is Hölder–continuous in \( B_R(x_0) \times (t_0 - \delta, t_0 + \delta) \).
Later improvements or modifications:

F. Lin (1996) – used the same assumptions on domain $\Omega$ as in C–K–N, the definition of a suitable weak solution requires the pressure to be in $L^{3/2}(Q_T)$. Considered the case $f = 0$.

**Theorem 5 (Lin 1996).** Let $(v, p)$ be a suitable weak solution of the problem (1.1)–(1.4). There exists a constant $\epsilon > 0$ such that if

$$\frac{1}{\delta^2} \int_{t_0 - \delta^2}^{t_0} \int_{B_\delta(x_0)} (|v|^3 + |p|^{3/2}) \ dx \ dt \leq \epsilon$$

(2.3)

for some $\delta > 0$ then $v \in C^\alpha(B_{\delta/2}(x_0) \times (t_0, t_0 - \frac{1}{4}\delta^2))$ for some $\alpha > 0$.

**Corollary.** (2.3) implies that $(x_0, t_0)$ is a regular point in the sense of C-K-N.

**Principle of the proof.** There exist numbers $\tau > 0$ and $0 < \rho_1 < \rho_2$ so that $v$ is bounded with all its spatial derivatives in $[B_{\rho_2}(x_0) \setminus B_{\rho_1}(x_0)] \times (t_0 - \tau, t_0 + \tau)$. 

2. A suitable weak solution
We use a cut–off function $\eta \in C^\infty(\mathbb{R}^3)$ such that $0 \leq \eta \leq 1$, $\eta = 1$ in $B_{\rho_1}(x_0)$ and $\eta = 0$ outside $B_{\rho_2}(x_0)$.

We put $u = \eta v - V$, where $V$ is the correction such that $\text{div} \ u = 0$. (Function $V$ satisfies $\text{div} \ V = \nabla \eta \cdot v$; it can be constructed so that its support is in $[B_{\rho_2}(x_0) \setminus B_{\rho_1}(x_0)] \times (t_0 - \tau, t_0 + \tau)$.)

Functions $u$, $\eta p$ satisfy the Navier–Stokes equation with the right hand side $h \in L^\alpha(t_0 - \tau, t_0 + \tau; \mathcal{W}^k,\infty(B_{\rho_2}(x_0)))$. 

2. A suitable weak solution
Function \( u \) satisfies the boundary condition \( u = 0 \) on \( \partial B_{\rho_2}(x_0) \times (t_0 - \tau, t_0 + \tau) \).

Using the results on the local existence of strong solutions, one can show that there exist \( \tau, \tau'' \in (0, \tau) \) so that the same Navier–Stokes problem as the one satisfied by functions \( u, \eta p \), has a “smooth” solution \( w, q \) on the time interval \( (t_0 - \tau', t_0 + \tau'') \), satisfying the boundary condition \( w = 0 \) on \( \partial B_{\rho_2}(x_0) \times (t_0 - \tau, t_0 + \tau'') \).

Moreover, \( w = u \) at time \( t = t_0 - \tau' \).

In fact, one obtains \( u \in L^\infty(t_0 - \tau', t_0 + \tau''; W^{1,2}(B_{\rho_2}(x_0))). \)

Since \( v \) is a suitable weak solution, one can verify that solution \( u \) satisfies (SEI).

Consequently, due to theorems on uniqueness, one can identify solutions \( u \) and \( w \) in \( B_{\rho_2}(x_0) \times (t_0 - \tau', t_0 + \tau'') \).

Consequently, \( v \in L^\infty(t_0 - \tau', t_0 + \tau''; W^{1,2}(B_{\rho_1}(x_0))), \) which means that \( v \) satisfies Serrin’s regularity condition in \( B_{\rho_1}(x_0) \times (t_0 - \tau', t_0 + \tau'') \).

**Remark.** Lin also proved that \((2.2) \implies [(2.3) \text{ holds for some } \delta > 0]\).
Y. Taniuchi (1997) considered domain $\Omega$ in $\mathbb{R}^3$ that is either smooth bounded, or smooth exterior, or a half–space, or the whole space $\mathbb{R}^3$.

Provided $v_0 \in L^2_0(\Omega)$ and $f \in L^2(Q_T)$, Taniuchi proved the existence of a suitable weak solution.

**Lemma 3 (Taniuchi 1997).** If $v$ is a weak solution and $K$ is a bounded sub–domain of $\Omega$, $v \in L^r(\epsilon, T; L^s(K))$ with

$$1 < r, s < \infty, \quad \frac{2}{r} + \frac{2}{s} \leq 1, \quad \frac{1}{r} + \frac{3}{s} \leq 1$$

(2.4)

then $v$ satisfies (GEE) in $K \times (\epsilon, T)$. Moreover, if $v \in L^r(0, T; L^r(\Omega))$ with $r, s$ satisfying (2.4) then $v$ satisfies both (SEE) and (GEE).

**Lemma 4 (Taniuchi 1997).** If $v$ is a weak solution, $D$ is a bounded sub–domain of $\Omega$ and $0 < \epsilon < T$ then $p$ and $\partial_t v$ can be taken so that

$$p \in L^r(\epsilon, T; L^s(D)) \quad \text{for} \quad \frac{2}{r} + \frac{3}{s} = 3, \quad 1 < r \leq 2, \quad 1 < s < 3,$$

$$\partial_t v \in L^r(\epsilon, T; L^s(D)) \quad \text{for} \quad \frac{2}{r} + \frac{3}{s} = 4, \quad 1 < r \leq 2, \quad 1 < s \leq \frac{3}{2}.$$
O. A. Ladyzhenskaya and G. Seregin (1999) consider a bounded domain $\Omega \in \mathbb{R}^3$, they also consider $f \neq 0$, $f \in M_{L^2,\gamma}(Q_T)$ (for some $\gamma > 0$; $M_{L^2,\gamma}$ denotes the Morrey space), and they use the same definition of a suitable weak solution as Lin. Ladyzhenskaya and Seregin define a **regular point** of a suitable weak solution $(v, p)$ to be such a point in $Q_T$ that there exists its neighbourhood $U$ in $Q_T$, where $v$ is Hölder–continuous.

**Theorem 6 (Ladyzhenskaya, Seregin 1999).** Let $(v, p)$ be a suitable weak solution of the problem (1.1)–(1.4). There exists a constant $\epsilon > 0$ such that if

$$\limsup_{r \to 0^+} \frac{1}{r} \int_{t_0-r^2}^{t_0} \int_{B_r(x_0)} |\nabla v|^2 \, dx \, dt \leq \epsilon$$

(2.5)

then $(x_0, t_0)$ is a (SL)–regular point of solution $(v, p)$.

The proof is based on the estimates of solution $v$ in parabolic Campanato spaces. (See e.g. W. Schlag, *Comm. PDE* 21, 1996, 1141-1175.) L-S do not prove the existence of a suitable weak solution.
Remark: Basic information on the Morrey and Campanato spaces.
(See e.g. Kufner et al: Function Spaces.)

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$. For $\lambda \geq 0$ and $1 \leq p < \infty$, we set

$$ML_{p,\lambda}(\Omega) := \left\{ u \in L^p(\Omega); \ M\|u\|_{p,\lambda} := \left( \sup_{x \in \Omega, \ r > 0} \frac{1}{r^\lambda} \int_{B_r(x) \cap \Omega} |u(y)|^p \, dy \right)^{1/p} < \infty \right\},$$

$$CL_{p,\lambda}(\Omega) := \left\{ u \in L^p(\Omega); \ [u]_{p,\lambda} < \infty \right\},$$

$$[u]_{p,\lambda} := \left( \sup_{x \in \Omega, \ r > 0} \frac{1}{r^\lambda} \int_{B_r(x) \cap \Omega} |u(y) - u_r(x)|^p \, dy \right)^{1/p},$$

$$C\|u\|_{p,\lambda} := \|u\|_p + [u]_{p,\lambda}$$

- $ML_{p,0}(\Omega) \nRightarrow L^p(\Omega)$, $ML_{p,N}(\Omega) \nRightarrow L^\infty(\Omega)$

- If $1 \leq p \leq q < \infty$, $\lambda \geq 0$, $\mu \geq 0$, $\frac{\lambda - N}{p} \leq \frac{\mu - N}{q}$ then

$ML_{q,\mu}(\Omega) \nRightarrow ML_{p,\lambda}(\Omega)$. 

2. A suitable weak solution
• $\mathcal{ML}_{p,\lambda}(\Omega) = \{0\}$ for $\lambda > N$

• If $1 \leq p \leq q < \infty$, $\lambda \geq 0$, $\mu \geq 0$, $\frac{\lambda - N}{p} \leq \frac{\mu - N}{q}$ then $\mathcal{CL}_{q,\mu}(\Omega) \hookrightarrow \mathcal{CL}_{p,\lambda}(\Omega)$.

• $\mathcal{CL}_{p,\lambda}(\Omega) \rightleftharpoons \mathcal{ML}_{p,\lambda}(\Omega)$ for $0 \leq \lambda \leq N$

• $\mathcal{CL}_{p,\lambda}(\Omega) \rightleftharpoons C^{0,\alpha}(\Omega)$ with $\alpha = \frac{\lambda - N}{p}$ provided $\lambda \in (N, N + p)$.

2. A suitable weak solution
R. Farwig, H. Kozono and H. Sohr (2005) considered an arbitrary domain $\Omega$ in $\mathbb{R}^3$ with a uniformly $C^2$–boundary, $0 < T \leq \infty$, $v_0 \in L^2_\sigma(\Omega)$, $f \in L^{5/4}(0, T; L^2(\Omega))$. They proved the existence of a suitable weak solution $(v, p)$ with $v$, $\partial_t v$, $\nabla v$, $\nabla^2 v$, $\nabla p$ in $L^{5/4}(\epsilon, T'; L^2(\Omega) + L^{5/4}(\Omega))$, where $0 < \epsilon < T' < T$. The solution satisfies (GEI) in the form

$$\|\varphi v(., t)\|_{2; \Omega}^2 + 2\nu \int_s^t \|\varphi \nabla v(., \tau)\|_{2; \Omega}^2 \, d\tau \leq \|\varphi v(., s)\|_{2; \Omega}^2 + \int_s^t (\varphi f, \varphi v) \, d\tau$$

$$-\nu \int_s^t \int_\Omega |v|^2 \cdot \nabla \varphi^2 \, dx \, d\tau + \int_s^t \int_\Omega (|v|^2 + 2p) (v \cdot \nabla \varphi^2) \, dx \, d\tau$$

for a.a. $s \in (0, T)$, all $t \in [s, T)$ and all $\varphi \in C^\infty_0(\mathbb{R}^3)$ and (SEI) in the form

$$\|v(., t)\|_2^2 + 2\nu \int_s^t \|\nabla v(., \tau)\|_2^2 \, d\tau \leq \|v(., s)\|_2^2 + 2 \int_s^t (v, f) \, d\tau$$

for a.a. $s \in [0, T)$ (including $s = 0$) and all $t \in [s, T)$.

Note that $\epsilon$ can be considered to be $= 0$ if $v_0 \in D(\tilde{A}^{5/4})$.

$T'$ can be considered to be $= T$ if $T < \infty$.  

2. A suitable weak solution
J. Wolf (2007) considered a general domain $\Omega \subset \mathbb{R}^3$, $f = 0$.

Using the pressure representation $p = p_0 + \partial_t \tilde{p}_h$ where $p_0 \in L^{4/3}(0, \infty; L^2(\Omega))$ and $\tilde{p}_h \in C(Q)$ being harmonic, Wolf proved the existence of the so called generalized suitable weak solution in $Q := \Omega \times (0, \infty)$, which is defined to be a weak solution in $Q$ such that the function $V := v + \nabla \tilde{p}_h$ satisfies the identity

$$
\int_0^\infty \int_\Omega \left[ -V \partial_t \phi + v \cdot \nabla v \cdot \phi + \nu \nabla V : \nabla \phi \right] \, dx \, dt = \int_0^\infty \int_\Omega p_0 \div \phi \, dx \, dt \quad (2.6)
$$

for all $\phi \in C_0^\infty(Q)$.

**Remark.** Integral equation (2.6) formally follows from the Navier–Stokes equation if we use the representation $p = p_0 + \partial_t \tilde{p}_h$, the identities

$$
\langle \nabla p, \phi \rangle = -\langle p, \div \phi \rangle = - \int_0^T \int_\Omega p_0 \div \phi \, dx \, dt + \int_0^T \int_\Omega \tilde{p}_h \div \partial_t \phi \, dx \, dt
$$

(where $\langle \nabla p, \phi \rangle$ denotes the distribution $\nabla p$, applied to function $\phi$), and the fact that $\tilde{p}_h$ is harmonic.
Furthermore, Wolf proved that his solution satisfies

\[
\int_{\Omega} |V(t)|^2 \varphi(t) \, dx + 2\nu \int_{0}^{t} \int_{\Omega} |\nabla V|^2 \varphi \, dx \, ds \\
\leq \int_{0}^{t} \int_{\Omega} \left[ |V|^2 (\partial_t \varphi + \nu \Delta \varphi) + (|V|^2 + 2p_0) V \cdot \nabla \varphi \right] \, dx \, ds \\
+ 2 \int_{0}^{t} \int_{\Omega} (\nabla \tilde{p}_h \times \text{curl } V) \cdot V \varphi \, dx \, ds
\]

(2.7)

for every non–negative function \( \varphi \) from \( C^\infty_0(Q_T) \).

Inequality (2.7) formally follows from (2.6) if we choose \( \varphi = V \varphi \theta_{t,\delta} \), where \( \theta_{\delta} \) is an appropriate smooth cut–off function of one variable (equal to 1 on the interval \([0, t]\) and equal to zero on the interval \([t + \delta, \infty)\)), and pass to zero with \( \delta \).

**Theorem 7 (Wolf 2007).** Let \( v \) be a generalized suitable weak solution of the problem (1.1)–(1.4). There exists a constant \( \epsilon > 0 \) such that if

\[
\limsup_{r \to 0+} \frac{1}{r} \int_{t_0-r^2}^{t_0} \int_{B_r(x_0)} \left| \text{curl } v \times \frac{v}{|v|} \right|^2 \, dx \, dt \leq \epsilon
\]

(2.8)

then \((x_0, t_0)\) is a (Lin)–regular point of solution \( v \).
3. A brief survey of further criteria for regularity at a point 
\((x_0, t_0)\) for weak or suitable weak solutions

- **S. Takahashi** (1990) proved that if the norm of solution \(v\) in \(L^{r}(t_0 - \rho^2, t_0; L^{s}(B_{\rho}(x_0)))\) (where \(2/r + 3/s \leq 1, \ 3 < s \leq \infty\)) is less than or equal to \(\epsilon\) then 
\((x_0, t_0)\) is a regular point of solution \(v\).

- **H. Kozono** (1998) proved that if \(v\) is a weak solution satisfying
\[
\sup_{t_0-\sigma<t<t_0+\sigma} \| v(\cdot, t) \|_{L^{3}_{w}(B_{\rho}(x_0))} \leq \epsilon
\]
for some \(\sigma > 0\) and \(\rho > 0\) then \(\partial_t v\) and \(\nabla^k v\) \((k = 0, 1, 2)\) are bounded in some
neighbourhood of point \((x_0, t_0)\).

- **J. Nečas, J.N.** (2002) have shown that if \(v\) is a suitable weak solution then the condition
\[
\lim_{\delta \to 0+} \frac{1}{\delta^2} \int_{t_0-\delta^2}^{t_0} \int_{B_{\delta}(x_0)} |v|^3 \, dx \, dt = 0
\]
implies that \((x_0, t_0)\) is a regular point.
• **G. Seregin, V. Šverák** (2005) have shown that if \( v \) is a suitable weak solution satisfying
\[
\frac{1}{\delta^2} \int_{t_0-\delta^2}^{t_0} \int_{B_\delta(x_0)} |v|^3 \, dx \, dt \leq \epsilon, \tag{3.3}
\]
for all \( \delta > 0 \) “sufficiently small” then \((x_0, t_0)\) is a regular point.

• An improvement of the last criterion has been obtained by **J. Wolf** (2010), requiring the validity of
\[
\frac{1}{\delta^2} \int_{t_0-\delta^2}^{t_0} \int_{B_\delta(x_0)} |v|^3 \, dx \, dt \leq \epsilon \tag{3.4}
\]
for at least one \( \delta > 0 \).

• Further generalizations and modifications of the aforementioned regularity criteria can be found in the paper by **G. Seregin, V. Šverák** (2005).

• Another generalization of the CKN–condition for suitable weak solutions has been proven by **A. Mahalov, B. Nicoalenko, G. Seregin** (2008), where the authors replace \( \nabla v \) by a quantity \( d(v) \), which is in some sense related to \( \nabla v \).
• **S. Gustafson, K. Kang and T.-P. Tsai** (2006): \( \mathbf{v} \) is a suitable weak solution, \( \mathbf{f} \in M L_{2,\gamma}(Q_T) \) for some \( \gamma > 0 \). If

\[
\limsup_{\rho \to 0^+} \rho^{1-\left(\frac{2}{r} + \frac{3}{s}\right)} \| \mathbf{v} - \overline{\mathbf{v}}_\rho \|_{L^r(t_0-\rho^2,t_0; L^s(B_\rho(x_0)))} \leq \epsilon,
\]

for some \( r, s \in [1, \infty] \), satisfying \( 1 \leq \frac{2}{r} + \frac{3}{s} \leq 2 \),

where \( \overline{\mathbf{v}}_\rho := \frac{1}{|B_\rho(x_0)|} \int_{B_\rho(x_0)} \mathbf{v} \, dx \),

then \((x_0, t_0)\) is a regular point.

• **S. Gustafson, K. Kang and T.-P. Tsai** (2006) also formulated a criterion in terms of vorticity:

\[
\limsup_{\rho \to 0^+} \rho^{2-\left(\frac{2}{r} + \frac{3}{s}\right)} \| \text{curl} \mathbf{v} \|_{L^r(t_0-\rho^2,t_0; L^s(B_\rho(x_0)))} \leq \epsilon
\]

for some \( r, s \in [1, \infty] \), satisfying \( 2 \leq \frac{2}{r} + \frac{3}{s} \leq 3 \).
**Remark.** Put

\[ x - x_0 = \rho x', \quad t_0 - t = t_0 - \rho^2 t', \quad v(x, t) = \frac{1}{\rho} v'(x', t'), \quad p(x, t) = \frac{1}{\rho^2} p'(x', t'). \]

Then \( v', p' \) satisfy the Navier–Stokes equation and the equation of continuity in \( B_1(0) \times (t_0 - 1, t_0) \).

Furthermore,

\[
\frac{1}{\rho} \int_{t_0 - \rho^2}^{t_0} \int_{B_\rho(x_0)} |\nabla v|^2 \, dx \, dt = \int_{t_0 - 1}^{t_0} \int_{B_1(0)} |\nabla' v'|^2 \, dx' \, dt',
\]

\[
\frac{1}{\rho^2} \int_{t_0 - \rho^2}^{t_0} \int_{B_\rho(x_0)} |v|^3 \, dx \, dt = \int_{t_0 - 1}^{t_0} \int_{B_1(0)} |v'|^3 \, dx' \, dt'.
\]

Similarly, one obtains

\[
\rho^{1 - \left(\frac{2}{r} + \frac{3}{s}\right)} \left\| v - \overline{v}_\rho \right\|_{L^r(t_0 - \rho^2, t_0; L^s(B_\rho(x_0)))} = \left\| v' - \overline{v}'_1 \right\|_{L^r(t_0 - 1, t_0; L^s(B_1(0)))},
\]

\[
\rho^{2 - \left(\frac{2}{r} + \frac{3}{s}\right)} \left\| \text{curl} v \right\|_{L^r(t_0 - \rho^2, t_0; L^s(B_\rho(x_0)))} = \left\| \text{curl}' v' \right\|_{L^r(t_0 - 1, t_0; L^s(B_1(0)))}.
\]

These formulas show that the mentioned criteria are “scale invariant”.

3. A brief survey of further criteria for regularity at a point \((x_0, t_0)\)
Theorem 8 (J.N. 2011). If \( v \) is a suitable weak solution, satisfying the condition

\[
\liminf_{t \to t_0^-} \| v(\cdot, t) \|_{3; B_\delta(x_0)} \leq \epsilon,
\]

for some \( \delta > 0 \) then \( (x_0, t_0) \notin S(v) \).

3. A brief survey of further criteria for regularity at a point \((x_0, t_0)\)
J.N. (2012): $a > 0$, $0 < \rho < \sqrt{t_0}$, $R > 1$, $0 < h < R - 1$; we denote

$$U_{a,\rho} := \left\{ (x, t); t_0 - \rho^2 < t < t_0 \text{ and } \sqrt{2a(t_0 - t)} < |x - x_0| < \sqrt{2a\rho} \right\},$$

$$P_{a,\rho,R,h} := \left\{ (x, t); t_0 - \rho^2/R^2 < t < t_0 \text{ and } (R - h)\sqrt{2a(t_0 - t)} < |x - x_0| < R\sqrt{2a(t_0 - t)} \right\},$$

3. A brief survey of further criteria for regularity at a point $(x_0, t_0)$
Theorem 9 (J.N. 2012). Let $v$ be a suitable weak solution of (1.1)–(1.3), $p$ be an associated pressure, $(x_0, t_0) \in \mathbb{R}^3 \times (0, T)$, $0 < a < 2\nu$ and $0 < \rho < \sqrt{t_0}$. Suppose that

(i) function $v$ satisfies the integrability condition in set $U_{a, \rho}$:

$$\int_{t_0-\rho^2}^{t_0} \left( \int_{\sqrt{2a(t_0-t)} < |x-x_0| < \sqrt{2a\rho}} |v(x, t)|^s \, dx \right)^{r/s} \, dt < \infty \quad (3.6)$$

for some $r, s$, satisfying $3 \leq r < \infty$, $3 < s < \infty$, $\frac{2}{r} + \frac{3}{s} < 1$,

(ii) there exist real numbers $R > 1$, $0 < h < R - 1$, such that function $p$ satisfies the integrability condition in set $P_{a, \rho, R, h}$:

$$\int_{t_0-\rho^2/R^2}^{t_0} \left( \int_{(R-h)\sqrt{2a(t_0-t)} < |x-x_0| < R\sqrt{2a(t_0-t)}} |p(x, t)|^\beta \, dx \right)^{\alpha/\beta} \, dt < \infty \quad (3.7)$$

for some $\alpha, \beta$, satisfying $\frac{r}{r-1} \leq \alpha < \infty$, $\frac{3}{2} < \beta < \infty$, $\frac{2}{\alpha} + \frac{3}{\beta} < 2$.

Then $(x_0, t_0)$ is a regular point of solution $v$. 

3. A brief survey of further criteria for regularity at a point $(x_0, t_0)$
We call \( A \subset S_{t_0}(v) \) a separated subset of \( S_{t_0}(v) \) if \( \overline{A} \cap \overline{S_{t_0}(v)} \setminus A = \emptyset \).

A separated subset of \( S_{t_0}(v) \) is a closed set in \( \mathbb{R}^3 \).

\( B_1 \) denotes a ball in \( \mathbb{R}^3 \) with radius 1.

**Lemma 5.** There exist \( \sigma > 0, c_1 > 0 \) and \( \delta_1 > 0 \) such that if \( A \) is a nonempty separated subset of \( S_{t_0}(v) \) such that \( U_{1/2}(A) \subset B_1 \), and \( \epsilon_1 > 0, \rho_1 \in (0, 1/2) \) are given numbers, then there exists \( \theta > 0 \) such that the inequality

\[
\|v(\cdot, t^*)\|_{3; U_{\rho_1}(A)} < \delta_1
\]

(4.1)

for some \( t^* \in [t_0 - \sigma, t_0) \) implies that

\[
\|v(\cdot, t)\|_{3; U_{\rho_1/2}(A)} \leq c_1 \|v(\cdot, t^*)\|_{3; U_{\rho_1}(A)} + \epsilon_1
\]

(4.2)

for all \( t \in (t^*, t^* + \theta) \cap (t_0 - \sigma, t_0 + \sigma) \).
Note that $\theta$ is independent of $t^*$. 

4. Principle of the proof of Theorem 8
Now, we prove Theorem 8 by contradiction: Suppose that to each \( \epsilon > 0 \) there exists a singular point \((x_0, t_0) \in \mathbb{R}^3 \times (0, T)\) of solution \(v\) such that (3.5) holds. Inequality (3.5) implies that there exists \( \rho_2 > 0 \) such that

\[
\liminf_{t \to t_0^-} \|v(. , t)\|_{3; B_{\rho}(x_0)} < \epsilon
\]

holds for all \( \rho \in (0, \rho_2) \). It means that there exists a sequence \( t_n \nearrow t_0 \) such that

\[
\|v(. , t_n)\|_{3; B_{\rho}(x_0)} < \epsilon. \tag{4.3}
\]

There exists a separated subset \( \mathcal{A} \) of \( \mathcal{S}_{t_0}(v) \) such that \( x_0 \in \mathcal{A} \subset B_{\rho/2}(x_0) \). We can assume without loss of generality that \( t_n \in (t_0 - \sigma, t_0) \). Inequality (4.3) implies that

\[
\|v(. , t_n)\|_{3; U_{\rho/2}(\mathcal{A})} \leq \epsilon. \tag{4.4}
\]

If \( \epsilon \) is chosen so small that \( \epsilon \leq \delta_1 \) (where \( \delta_1 \) is the number from Lemma 5) and if \( \epsilon_1 \) is a positive number then Lemma 5 provides the existence of \( \theta > 0 \) such that
\[ \|v(\cdot, t)\|_{3; U_{\rho/4}(A)} \leq c_1 \|v(\cdot, t_n)\|_{3; U_{\rho/2}(A)} + \epsilon_1 \leq c_1 \epsilon + \epsilon_1 \]

for all \( t \in (t_n, t_n + \theta) \cap (t_0 - \sigma, t_0 + \sigma) \). However, if \( \epsilon_1 \) and \( \epsilon \) are chosen so small that \( c_1 \epsilon + \epsilon_1 < \epsilon_3 \), where \( \epsilon_3 \) is the number on the right hand side of (3.1), then

\[ \|v(\cdot, t)\|_{3; B_{\rho/4}(x_0)} \leq \|v(\cdot, t)\|_{3; U_{\rho/4}(A)} \leq \epsilon_3 \]

for all \( t \in (t_0 - \sigma_0, t_0 + \sigma_0) \) for some \( \sigma_0 > 0 \). Since the \( L^3 \)–norm dominates the weak \( L^3 \)–norm, inequality (3.1) is fulfilled. Consequently, \((x_0, t_0)\) cannot be a singular point of solution \( v \).

This is the contradiction.

The proof is completed.
5. Principle of the proof of Theorem 9

The used regularity criterion. We will prove that solution \( v \) satisfies the regularity criterion

\[
\lim_{\delta \to 0^+} \frac{1}{\delta^2} \int_{t_0 - \delta^2}^{t_0} \int_{B_\delta(x_0)} |v|^3 \, dx \, dt = 0. 
\]  

(3.1)

We denote

\[
\theta(t) := \sqrt{2a(t_0 - t)},
\]

and derive an estimate of

\[
\lim_{\delta \to 0^+} \frac{1}{\delta^2} \int_{t_0 - \delta^2}^{t_0} \int_{B_\delta(x_0)} |v|^3 \, dx \, dt = \lim_{\delta \to 0^+} \left[ A_\delta^I + A_\delta^{II} \right],
\]  

(5.1)

where

\[
A_\delta^I := \frac{1}{\delta^2} \int_{t_0 - \delta^2}^{t_0} \int_{\theta(t) < |x - x_0| < \delta} |v|^3 \, dx \, dt,
\]

\[
A_\delta^{II} := \frac{1}{\delta^2} \int_{t_0 - \delta^2}^{t_0} \int_{|x - x_0| < \theta(t)} |v|^3 \, dx \, dt.
\]
An estimate of $A^I_\delta$. The integral in $A^I_\delta$ is an integral over a subset of $U_{a,\rho}$. Hence $A^I_\delta$ can be estimated as follows:

\[
A^I_\delta = \frac{1}{\delta^2} \int_{t_0-\delta^2}^{t_0} \int_{\theta(t)<|x-x_0|<\delta} |v|^3 \, dx \, dt
\]

\[
\leq \frac{1}{\delta^2} \int_{t_0-\delta^2}^{t_0} \left( \int_{\theta(t)<|x-x_0|<\delta} |v|^s \, dx \right)^{\frac{3}{s}} \left( \frac{4\pi \delta^3}{3} \right)^{1-\frac{3}{s}} \, dt
\]

\[
\leq \left( \frac{4\pi}{3} \right)^{1-\frac{3}{s}} \delta^{3 \left( 1-\frac{2}{r} - \frac{3}{s} \right)} \left[ \int_{t_0-\delta^2}^{t_0} \left( \int_{\theta(t)<|x-x_0|<\delta} |v|^s \, dx \right)^{\frac{r}{s}} \, dt \right]^\frac{3}{r}.
\]

The limit of the right hand side, for $\delta \to 0^+$, equals zero due to condition (2.2) and the assumptions on $r$ and $s$. Hence we have

\[
\lim_{\delta \to 0^+} A^I_\delta = 0. \tag{5.2}
\]

5. Principle of the proof of Theorem 9
**Transformation to the new coordinates** $x', t'$. We use coordinates $x'$ and $t'$, which are related to $x$ and $t$ through the formulas

$$
x' = \frac{x - x_0}{\theta(t)}, \quad t' = \int_{t_0 - \rho^2}^{t} \frac{ds}{\theta^2(s)} = \frac{1}{2a} \ln \frac{\rho^2}{t_0 - t}.
$$

(5.3)

Then

$$t = t_0 - \rho^2 e^{-2at'} \quad \text{and} \quad \theta(t) = \sqrt{2a} \rho e^{-at'}.
$$

The time interval $(t_0 - \rho^2, t_0)$ on the $t$–axis now corresponds to the interval $(0, \infty)$ on the $t'$–axis. Equations (5.3) represent a one–to–one transformation of the parabolic region $V_{a,\rho}$ in the $x, t$–space onto the infinite stripe

$$V_{a,\rho}': = \{(x', t') \in \mathbb{R}^4; 0 < t' < \infty, |x'| < 1\}$$

in the $x', t'$–space. Similarly, (2.1) is a one–to–one transformation of set $U_{a,\rho}$ in the $x, t$–space onto

$$U_{a,\rho}' := \{(x', t') \in \mathbb{R}^4; 0 < t' < \infty, 1 < |x'| < e^{at'}\}$$

in the $x', t'$–space.
5. Principle of the proof of Theorem 9

\[ t'_\delta = \frac{1}{a} \ln \frac{\rho}{\delta} \]
(corresponds to \( t = t_0 - \delta^2 \))
If we put
\[ v(x, t) = \frac{1}{\theta(t)} v'(x', t'), \quad p(x, t) = \frac{1}{\theta^2(t)} p'(x', t'), \]
then functions \( v', p' \) represent a suitable weak solution of the system of equations
\[ \partial_t v' + v' \cdot \nabla' v' = -\nabla' p' + \nu \Delta' v' - a v' - a x' \cdot \nabla' v', \quad (5.4) \]
\[ \text{div}' v' = 0 \quad (5.5) \]
for \( x' \in \mathbb{R}^3 \) and \( t' > 0 \). Functions \( v' \) and \( p' \) satisfy the analog of the generalized energy inequality:
\[
\| \varphi v'(\cdot, t') \|_{2; B_R(0)}^2 + 2 \nu \int_{t'_\delta}^{t'} \| \varphi \nabla' v'(\cdot, \tau) \|_{2; B_R(0)}^2 \, d\tau \leq \| \varphi v'(\cdot, t'_\delta) \|_{2; B_R(0)}^2 \\
- \nu \int_{t'_\delta}^{t'} \int_{B_R(0)} |\nabla' v'|^2 \cdot \nabla' \varphi^2 \, dx' \, d\tau + \int_{t'_\delta}^{t'} \int_{B_R(0)} (|v'|^2 + 2p') (v' \cdot \nabla' \varphi^2) \, dx' \, d\tau \\
+ \int_{t'_\delta}^{t'} \int_{B_R(0)} \left[ a \varphi^2 |v'|^2 + (a x' \cdot \nabla' \varphi^2) |v'|^2 \right] \, dx' \, d\tau \quad (5.6)
\]
for a.a. \( t'_\delta > a^{-1} \ln R \), all \( t' \geq t'_\delta \) and all \( \varphi \in C_0^\infty(B_R(0)) \).
The first estimate of $A^I_{\delta}$. We have

$$A^I_{\delta} = \frac{1}{\delta^2} \int_{t'_\delta}^{\infty} \int_{B_1(0)} |v'|^3 \, dx' \, 2a \rho^2 e^{-2at'} \, dt'$$

$$\leq 2a \int_{t'_\delta}^{\infty} \int_{B_1(0)} |v'|^3 \, dx' \, dt' \leq 2a \int_{t'_\delta}^{\infty} \left( \int_{B_1(0)} |v'|^6 \, dx' \right)^{\frac{1}{4}} \left( \int_{B_1(0)} |v'|^2 \, dx' \right)^{\frac{3}{4}} dt'$$

$$\leq C \int_{t'_\delta}^{\infty} \left( \|v'\|_{2; B_1(0)}^2 + \|\nabla' v'\|_{2; B_1(0)}^2 \right)^{\frac{3}{4}} \|v'\|_{2; B_1(0)}^{\frac{3}{2}} \, dt'$$

$$\leq C \left( \int_{t'_\delta}^{\infty} \|\nabla' v'\|_{2; B_1(0)}^2 \, dt' \right)^{\frac{3}{4}} \left( \int_{t'_\delta}^{\infty} \|v'\|_{2; B_1(0)}^6 \, dt' \right)^{\frac{1}{4}} + C \int_{t'_\delta}^{\infty} \|v'\|_{2; B_1(0)}^3 \, dt',$$

where $C$ depends only on $a$. (5.7)

Recall that $d := \frac{1}{5} h$. In order to estimate the integrals on the right hand side, we use the generalized energy inequality (5.6) with $\varphi(x') = \varphi_{R,d}(x')$, where $0 \leq \varphi_{R,d} \leq 1$ and

$$\varphi_{R,d}(x') = 1 \text{ for } |x'| < R - 3d, \quad \varphi_{R,d}(x') = 0 \text{ for } |x'| > R - 2d.$$
A cut–off function $\eta_\xi$ and related estimates. Let $\xi > 0$ and $\eta_\xi$ be an infinitely differentiable cut–off function in $\mathbb{R}^3$ with values in the interval $[0, 1]$, such that

$$\eta_\xi = 1 \text{ in } B_1(0), \quad \eta_\xi = 0 \text{ in } \mathbb{R}^3 \setminus B_{1+\xi}(0).$$

Let $\mu > 0$. Using the continuous imbedding $W^{1,2}(B_{1+\xi}(0)) \hookrightarrow L^2(B_{1+\xi}(0))$, we derive the estimates

$$\|\eta_\xi v'\|_{2; B_{1+\xi}(0)}^2 \leq \sum_{i=1}^{3} \|\eta_\xi v'_i\|_{2; B_{1+\xi}(0)}^2$$

$$\leq \sum_{i=1}^{3} (1 + \xi)^2 \|\nabla'(\eta_\xi v'_i)\|_{2; B_{1+\xi}(0)}^2 = (1 + \xi)^2 \|\nabla'(\eta_\xi v')\|_{2; B_{1+\xi}(0)}^2$$

$$\leq (1 + \xi)^2 (1 + \mu) \|\eta_\xi \nabla' v'\|_{2; B_{1+\xi}(0)}^2 + \frac{(1 + \xi)^2 c_2(\mu)}{\xi^2} \|v'\|_{2; B_{1+\xi}(0) \setminus B_1(0)}^2. \quad (5.8)$$

We further assume that numbers $\xi$ and $\mu$ are chosen so small that $\xi < 1$ and $(1 + \mu)(1 + \xi)^2a < 2\nu$. 

5. Principle of the proof of Theorem 9
An auxiliary inequality. For $0 < R_1 < R_2$, we denote

$$M_{R_1,R_2}(t) := \{ x \in \mathbb{R}^3; R_1 \theta(t) < |x - x_0| < R_2 \theta(t) \},$$

$$M'_{R_1,R_2} := \{ x' \in \mathbb{R}^3; R_1 < |x'| < R_2 \}.$$

We have

$$\int_{t_0}^{\infty} \int_{M'_{1,R}} |v'|^2 \, dx' \, dt' = \int_{t_0 - \delta^2}^{t_0} \theta^{-3}(t) \int_{M_{1,R}(t)} |v|^2 \, dx \, dt$$

$$\leq C \, c_3^{2/r}(\delta) \, c_4^{1-2/r}(\delta), \quad (5.9)$$

where

$$c_3(\delta) := \int_{t_0 - \delta^2}^{t_0} \left( \int_{M_{1,R}(t)} |v|^s \, dx \right)^{r/s} \, dt$$

and

$$c_4(\delta) := \int_{t_0 - \delta^2}^{t_0} \theta^{-\frac{6r}{s(r-2)}}(t) \, dt.$$

$c_3(\delta) \to 0$ because $\{ (x, t) \in \mathbb{R}^4; t_0 - \delta^2 < t < t_0, \, x \in M_{1,R}(t) \} \subset U_{a,\rho}$.

$c_4(\delta) \to 0$ because $-\frac{6r}{s(r-2)} = -2 + 2 \frac{\kappa}{\kappa + 3/s} > -2$.  

5. Principle of the proof of Theorem 9
The right hand side of inequality (5.6). The right hand side of (5.6) can be split to the sum

\[
\| \varphi_{R,d} \mathbf{v}' (\cdot, t'_\delta) \|_{2; B_R(0)}^2 + K_{\delta}^I + K_{\delta}^{II} + K_{\delta}^{III} + K_{\delta}^{IV} + K_{\delta}^{V},
\]

where

\[
K_{\delta}^I := -\nu \int_{t'_\delta}^{t'} \int_{B_R(0)} \nabla' |\mathbf{v}'|^2 \cdot \nabla' \varphi_{R,d}^2 \, d\mathbf{x}' \, d\tau,
\]

\[
K_{\delta}^{II} := \int_{t'_\delta}^{t'} \int_{B_R(0)} |\mathbf{v}'|^2 (\mathbf{v}' \cdot \nabla' \varphi_{R,d}^2) \, d\mathbf{x}' \, d\tau,
\]

\[
K_{\delta}^{III} := \int_{t'_\delta}^{t'} \int_{B_R(0)} 2p' (\mathbf{v}' \cdot \nabla' \varphi_{R,d}^2) \, d\mathbf{x}' \, d\tau,
\]

\[
K_{\delta}^{IV} := \int_{t'_\delta}^{t'} \int_{B_R(0)} a\eta_\xi^2 |\mathbf{v}'|^2 \, d\mathbf{x}' \, d\tau,
\]

\[
K_{\delta}^{V} := \int_{t'_\delta}^{t'} \int_{B_R(0)} [a(\varphi_{R,d}^2 - \eta_\xi^2) + (a\mathbf{x}' \cdot \nabla' \varphi_{R,d}^2)] |\mathbf{v}'|^2 \, d\mathbf{x} \, d\tau.
\]
\[ K^I_{\delta} := -\nu \int_{t_\delta'}^{t'} \int_{B_R(0)} \nabla' |v'|^2 \cdot \nabla' \varphi_{R,d}^2 \, dx' \, d\tau = \nu \int_{t_\delta'}^{t'} \int_{B_R(0)} |v'|^2 \Delta' \varphi_{R,d}^2 \, dx' \, d\tau. \]

Since \( \Delta' \varphi_{R,d}^2(x') \) is supported for \( R - 3d \leq |x'| \leq R - 2d \), we get

\[ K^I_{\delta} \leq \max |\Delta' \varphi_{R,d}^2| \int_{t_\delta'}^{t'} \int_{M_{R-3d,R-2d}} |v'|^2 \, dx' \, d\tau \]

\[ \leq \max |\Delta' \varphi_{R,d}^2| \int_{t_\delta'}^{t'} \int_{M_{1,R}} |v'|^2 \, dx' \, d\tau \]

\[ \longrightarrow 0 \quad \text{for} \quad \delta \to 0^+ \]  

(5.10)

because of (5.9).
The next term is
\[
K_{\delta}^{II} := \int_{t_0}^{t'} \int_{B_R(0)} |v'|^2 (v' \cdot \nabla' \varphi_{R,d}^2) \, dx' \, d\tau \leq C \int_{t_0}^{t'} \int_{M_{R-3d,R-2d}} |v'|^3 \, dx' \, d\tau
\]
\[
= C \int_{t_0 - \delta^2}^{t_0} \theta^{-2}(t) \int_{M_{R-3d,R-2d}(t)} |v|^3 \, dx \, dt
\]
\[
\leq \int_{t_0 - \delta^2}^{t_0} \left( \int_{M_{R-3d,R-2d}(t)} |v|^s \, dx \right)^{3/s} \theta^{1-9/s}(t) \, dt
\]
\[
\leq C c_3^{3/r}(\delta) c_5^{1-3/r}(\delta),
\]
where
\[
c_5(\delta) := \int_{t_0 - \delta^2}^{t_0} \theta^{s-9/s} \frac{r}{r-3}(t) \, dt.
\]
\[
c_5(\delta) \to 0 \text{ for } \delta \to 0^+ \text{ because } \frac{s-9}{s} \frac{r}{r-3} = -2 + \frac{3}{1-3/r} \left(1 - \frac{2}{r} - \frac{3}{s}\right) > -2.
\]
Hence \( K_{\delta}^{II} \to 0 \) for \( \delta \to 0^+ \).

5. Principle of the proof of Theorem 9
In order to estimate the integral with pressure, we need the inequality
\[
\int_{M'_{R-3d,R-2d}} |p'|^{\frac{s}{s-1}} \, dx' \leq c_6 \int_{M'_{R-5d,R}} |v'|^{\frac{2s}{s-1}} \, dx' + c_7 \left( \int_{M'_{R-5d,R}} |p'| \, dx' \right)^{\frac{s}{s-1}}
\]
for a.a. \( t' \in (0, \infty) \). The procedure is longer and technical. Finally, we obtain:

\[
K^\text{III}_\delta := \int_{t'_\delta}^{t'} \int_{B_R(0)} 2p' (v' \cdot \nabla' \varphi_{R,d}^2) \, dx' \, d\tau \longrightarrow 0 \quad \text{for } \delta \to 0^+.
\]

The next integral on the right hand side of (5.6) can be estimated by means of (5.8):

\[
K^\text{IV}_\delta := \int_{t'_\delta}^{t'} \int_{B_R(0)} a \eta_\xi^2 |v'|^2 \, dx' \, d\tau \leq a(1 + \xi)^2 (1 + \mu) \int_{t'_\delta}^{t'} \| \eta_\xi \nabla' v' \|_{2; B_{1+\xi}(0)} \, dt' + c_8(\delta),
\]

where

\[
c_8(\delta) := \frac{(1 + \xi)^2}{\xi^2} c_2(\mu) \int_{t'_\delta}^{t'} \| v' \|^2_{2; M'_{1,1+\xi}} \, dx' \, dt' \to 0 \quad \text{because of (5.9)}.
\]
Finally, we have

\[
K_\delta^V := \int_{t'_\delta}^{t'} \int_{B_R(0)} \left[ a(\varphi^2_{R,d} - \eta^2_\xi) + (a\mathbf{x}' \cdot \nabla' \varphi^2_{R,d}) \right] |\mathbf{v}'|^2 \, d\mathbf{x} \, d\tau
\]

\[
\leq \int_{t'_\delta}^{\infty} \int_{B_R(0)} a(\varphi^2_{R,d} - \eta^2_\xi) |\mathbf{v}'|^2 \, d\mathbf{x} \, d\tau
\]

\[
\leq \int_{t'_\delta}^{\infty} \int_{M'_{1,R}} a(\varphi^2_{R,d} - \eta^2_\xi) |\mathbf{v}'|^2 \, d\mathbf{x} \, d\tau
\]

\[\longrightarrow 0 \quad \text{for} \; \delta \to 0+ \quad \text{because of (5.9)}.
\]

Thus, we obtain the inequality

\[
K_\delta^I + \ldots + K_\delta^V \leq a(1 + \xi)^2(1 + \mu) \int_{t'_\delta}^{t'} \|\eta_\xi \nabla' \mathbf{v}'\|_{L^2; B_{1+\xi}(0)} \, dt' + c_9(\delta),
\]

where \(c_9(\delta) \to 0\) as \(\delta \to 0+\).
Substituting this to (5.6), we obtain

\[
\| \varphi_{R,d} \mathbf{v}'(. , t') \|_{B_{R-2d}(0)}^2 + 2\nu \int_{t'_\delta}^{t'} \int_{M_{1,R-2d}} (\varphi_{R,d}^2 - \eta_\xi^2) |\nabla' \mathbf{v}'(. , \tau)|^2 \, d\mathbf{x}' \, d\tau \\
+ [2\nu - a(1 + \xi)^2(1 + \mu)] \int_{t'_\delta}^{t'} \int_{B_{1+\xi}(0)} \eta_\xi^2 |\nabla' \mathbf{v}'(. , \tau)|^2 \, d\mathbf{x}' \, d\tau \\
\leq \| \varphi_{R,d} \mathbf{v}'(. , t'_\delta) \|_{B_{R-2d}(0)}^2 + c_9(\delta).
\]

This yields

\[
\| \varphi_{R,d} \mathbf{v}'(. , t') \|_{B_{R-2d}(0)}^2 \leq \| \varphi_{R,d} \mathbf{v}'(. , t'_\delta) \|_{B_{R-2d}(0)}^2 + c_9(\delta), \tag{5.12}
\]

\[
2\nu \int_{t'_\delta}^{\infty} \int_{M'_{1,R-2d}} (\varphi_{R,d}^2 - \eta_\xi^2) |\nabla' \mathbf{v}'(. , \tau)|^2 \, d\mathbf{x}' \, d\tau \\
+ [2\nu - a(1 + \xi)^2(1 + \mu)] \int_{t'_\delta}^{\infty} \int_{B_{1+\xi}(0)} \eta_\xi^2 |\nabla' \mathbf{v}'(. , \tau)|^2 \, d\mathbf{x}' \, d\tau \\
\leq \| \varphi_{R,d} \mathbf{v}'(. , t'_\delta) \|_{B_{R-2d}(0)}^2 + c_9(\delta). \tag{5.13}
\]
Using the integrability of $\|\varphi_{R,d}v'(.,s)\|_{2;B_{R-2d}(0)}^2$, as a function of $s$, in the interval $(a^{-1}\ln R, \infty)$, and estimate (5.12), we can prove that

$$\|\varphi_{R,d}v'(.,s)\|_{2;B_{R-2d}(0)}^2 \to 0$$

for $s \to \infty$.

Consequently, since $t'_\delta \to \infty$ for $\delta \to 0^+$, the right hand sides of (5.12) and (5.13) tend to zero if $\delta \to 0^+$. We denote the right hand sides by $c_{10}(\delta)$.

**Final estimates of $A^I_{\delta}$**. The integral of $\|\nabla'v'|^2_{2;B_{1}(0)}$ on the right hand side of (5.7) can be estimated by means of (5.13):

$$\int_{t'_\delta}^{\infty} \|\nabla'v'|^2_{2;B_{1}(0)} \, dt' \leq \frac{c_{10}(\delta)}{2\nu - a(1 + \xi)^2(1 + \mu)}. \quad (5.14)$$

The integral of $\|v'|^6_{2;B_{1}(0)}$ on the right hand side of (5.7) can be estimated by means of (5.12), (5.8) and (5.13):
\[ \int_{t'_\delta}^{\infty} \| v' \|^6_{2; B_1(0)} \, dt' \leq c_{10}^2(\delta) \int_{t'_\delta}^{\infty} \| v' \|^2_{2; B_1(0)} \, dt' \]
\[ \leq c_{10}^2(\delta) (1 + \xi)^2 \int_{t'_\delta}^{\infty} \left[ (1 + \mu) \| \eta_\xi \nabla' v' \|^2_{2; B_{1+\xi}(0)} + \frac{c_2(\mu)}{\xi^2} \| v' \|^2_{2; M'_{1,1+\xi}} \right] \, dt' \]
\[ \leq c_{10}^2(\delta) (1 + \xi)^2 \left[ \frac{(1 + \mu) c_{10}(\delta)}{2\nu - a(1 + \xi)^2(1 + \mu)} + \frac{c_2(\mu)}{\xi^2} \int_{t'_\delta}^{\infty} \| v' \|^2_{2; M'_{1,1+\xi}} \, dt' \right]. \]

The integral of \( \| v' \|^2_{2; M'_{1,1+\xi}} \) tends to zero for \( \delta \to 0^+ \) due to (5.9). Thus, we obtain
\[ \int_{t'_\delta}^{\infty} \| v' \|^6_{2; B_1(0)} \, dt' \longrightarrow 0 \quad \text{for } \delta \to 0^+. \quad (5.15) \]

The integral of \( \| v' \|^3_{2; B_1(0)} \) on the right hand side of (5.7) can be estimated similarly as the integral of \( \| v' \|^6_{2; B_1(0)} \). Hence we also have
\[ \int_{t'_\delta}^{\infty} \| v' \|^3_{2; B_1(0)} \, dt' \longrightarrow 0 \quad \text{for } \delta \to 0^+. \quad (5.16) \]

5. Principle of the proof of Theorem 9
It follows from (5.7), (5.14), (5.15) and (5.16) that

$$\lim_{\delta \to 0^+} A^{II}_\delta = 0.$$  \hfill (5.17)

**Conclusion.** We observe from (5.2) and (5.17) that function $v$ satisfies condition (3.2). Hence $(x_0, t_0)$ is a regular point of solution $v$.

**The proof is completed.**