Some recent results on regularity criteria for weak solutions of the Navier-Stokes equations

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1. Regularity criteria imposing conditions on various quantities (not only velocity) – a brief survey

We assume that $\mathbf{v}$ is a weak solution of the Navier–Stokes initial–boundary value problem in Q_T and $\boldsymbol{\omega} := \mathbf{curl}\ \mathbf{v}$.

Regularity via certain integrability of vorticity:

J. T. Beale, T. Kato, A. Majda (1984): $\Omega = \mathbb{R}^3$, proved that the inequality

$$\int_0^T \|\boldsymbol{\omega}(t)\|_\infty \, \mathrm{d}t \; < \; \infty$$

implies regularity.

Later improvement:

H. Kozono, T. Ogawa, Y. Taniuchi (2003): $\Omega = \mathbb{R}^3$, the L^∞–norm can be replaced by the $\dot{B}^0_{\infty,\infty}$–norm (in the homogeneous Besov space $\dot{B}^0_{\infty,\infty}(\mathbb{R}^3)$) in the BKM–condition.

Regularity via the direction of vorticity:

P. Constantin, C. Fefferman (1993): *Assume that there exist constants* $C,\ M > 0$ *such that*

$$|\sin\varphi| \ \le\ C\,|\mathbf{y}-\mathbf{x}|, \tag{1.1}$$

for all $\mathbf{x},\mathbf{y} \in \Omega_t^M$, $0 < t < T$, *where* φ *is the angle between* $\boldsymbol{\omega}(\mathbf{x},t)$ *and* $\boldsymbol{\omega}(\mathbf{y},t)$ *and*

$$\Omega_t^M \ :=\ \{\mathbf{x}\in\Omega;\ |\boldsymbol{\omega}(\mathbf{x},t)| \ge M\}.$$

Then solution $\mathbf{v}$ *is regular in* Q_T.

Later improvements:

- **H. Beirao da Veiga, L. Berselli** (2002): Inequality (1.1) can be replaced by
$$|\sin\varphi| \ \le\ C\,|\mathbf{y}-\mathbf{x}|^{1/2},$$
- **Z. Grujic, A. Ruzmaikina** (2004): assumed that $q \ge 2$, $\boldsymbol{\omega}_0 \in \mathbf{L}^1(\Omega)\cap\mathbf{L}^q(\Omega)$,
$$\text{a) } \|\boldsymbol{\omega}\|_q^{q/(q-1)} \in L^1(0,T), \qquad \text{b) } |\sin\varphi| \ \le\ C\,|\mathbf{y}-\mathbf{x}|^{1/q}$$
for all $\mathbf{x}\in\Omega_t^M$, $0<t<T$. These assumptions imply that solution $\mathbf{v}$ has no singular points in Q_T.

Regularity via the direction of velocity:

A. Vasseur (2009): $\Omega = \mathbb{R}^3$, $\mathbf{f} = \mathbf{0}$, $\mathbf{v}$ is supposed to be a Leray–Hopf weak solution such that $\mathrm{div}\,(\mathbf{v}/|\mathbf{v}|) \in L^r(0,T;\, L^s(\mathbb{R}^3))$ with

$$\frac{2}{r} + \frac{3}{s} \leq \frac{1}{2}, \quad s \geq 6, \quad r \geq 4, \quad 0 < T \leq \infty.$$

Such a solution is smooth in Q_T.

Regularity via the eigenvalues or eigenvectors of the rate of deformation tensor:

J.N. and P. Penel (2001): *If D is an open sub–domain of Q_T, $(\mathbf{v}, p)$ is a suitable weak solution, $\zeta_1 \leq \zeta_2 \leq \zeta_3$ are the eigenvalues of the tensor $\mathbb{D} := (\nabla \mathbf{v})_s$ and $\zeta_2 = \zeta_2^I + \zeta_2^{II}$ where*

(i) *one of the functions ζ_1, $(\zeta_2)_+$, ζ_3 belongs to $L^{s,r}(D)$ for some $r \in [1, \infty]$, $s \in (\frac{3}{2}, \infty]$, satisfying $2/r + 3/s \leq 2$,*

then solution $\mathbf{v}$ is regular in D.

An assumption can also be made only on the eigenvectors of tensor $\mathbb{D}$.

Improvements of Serrin's regularity condition:

- **H. Kozono, T. Ogawa, Y. Taniuchi** (2003): regularity in $L^2(0,T;\ \dot{B}^0_{\infty,\infty})(\mathbb{R}^3)$.
- **H. Kim** (2007): Ω either bounded smooth or the whole space $\mathbb{R}^3$, if a strong solution $\mathbf{v}$ blows up at time T^* then $\int_0^{T^*} \|\mathbf{v}(t)\|^r_{L^s_w}\,\mathrm{d}t = \infty$, where $2/r+3/s = 1$, $3 < s \leq \infty$.

Regularity via jumps of the $B^{-1}_{\infty,\infty}$–norm:

A. Cheskidov, R. Shvydkoy (2010): $\Omega = \mathbb{R}^3$, if jumps of a weak solution in the $B^{-1}_{\infty,\infty}$–norm do not exceed certain constant (a multiple of viscosity) then the solution is smooth.

A logarithmically improved Serrin's criterion:

S. Montgomery-Smith (2007): $\Omega = \mathbb{R}^3$,

$$\int_0^T \frac{\|\mathbf{v}(t)\|^r_s}{1+\ln^+\|\mathbf{v}(t)\|_s}\,\mathrm{d}t \;<\; \infty$$

with $2 < r < \infty$, $3 < s < \infty$, $2/r + 3/s = 1$, implies regularity of weak solution $\mathbf{v}$.

Regularity beyond Serrin's condition:

- **R. Farwig, H. Kozono, H. Sohr** (2007): *Ω is a domain in $\mathbb{R}^3$ with a smooth boundary, $\mathbf{f} = \mathbf{0}$, $\mathbf{v}$ is a weak solution satisfying (SEI), $\mathbf{v}_0 \in \mathbf{L}^2_\sigma(\Omega)$, $4 < s < \infty$, $3 < q < 6$, $2/s + 3/q = 1$. Then*

  (i) *To given $r \in [1, s)$ such that $2/r + 3/q = 1 + \alpha$, $0 \leq \alpha \leq 2(1 - 1/s)$ there exists a constant $C = C(\mathbf{v}_0, \Omega, r, s) > 0$ such that if*

  $$\|\mathbf{v}(t)\|_{L^r(0,T;\,\mathbf{L}^q(\Omega))} \ \leq \ C$$

  then solution $\mathbf{v}$ is smooth in Q_T (in the sense that $\mathbf{v} \in L^s(0, T;\ \mathbf{L}^q(\Omega))$).

  (ii) *If to each $T_1 \in (0, T)$ there exists $0 < \delta(T_1) < T_1$ such that*

  $$\mathbf{v} \in L^s(T_1 - \delta, T_1;\ \mathbf{L}^q(\Omega))$$

  then solution $\mathbf{v}$ is smooth in Q_T (in the sense that $\mathbf{v} \in L^s_{loc}((0, T);\ \mathbf{L}^q(\Omega))$).

Here, since $2/r + 3/q > 1$ (in condition (i)) or $\mathbf{v}$ satisfies only the left–ward Serrin condition in $(0, T)$, the results go beyond Serrin's condition.

- **R. Farwig, H. Kozono, H. Sohr** (2007): *Ω a bounded domain in $\mathbb{R}^3$ with a smooth boundary, $\mathbf{f} = \mathbf{0}$, $\mathbf{v}$ is a weak solution satisfying (SEI), $\mathbf{v}_0 \in \mathbf{L}^2_\sigma(\Omega)$, $4 < s < \infty,\ 3 < q < 6,\ 2/s + 3/q = 1$.*

  Then there exists $C = C(\Omega, q)$ such that if $0 < t < T_1 < T$, $0 \leq \alpha \leq 2(1-1/s)$ and at least one of the conditions

  (i) $\displaystyle\int_t^{T_1} \|\mathbf{v}(\tau)\|_q^r \, \mathrm{d}\tau \leq C(T_1 - t)$ *with* $\displaystyle\frac{2}{r} + \frac{3}{q} = 1 + \alpha,\ 1 \leq r \leq s,$

  (ii) $\displaystyle\int_t^{T_1} (T-\tau)^{r/s} \, \|\mathbf{v}(t)\|_q^r \, \mathrm{d}\tau \leq C(T_1 - t)$ *with* $\displaystyle\frac{2}{r} + \frac{3}{q} = 1 + \alpha,\ 1 \leq r \leq s$

  holds then solution $\mathbf{v}$ is smooth in Q_T (in the sense that $\mathbf{v} \in L^s(T_1, T;\ \mathbf{L}^q(\Omega))$).

- **P. Han** *(2009): Ω has a smooth bounded boundary or it is a half–space. $\mathbf{v}_0 \in \mathbf{L}^2_\sigma(\Omega) \cap \mathbf{L}^3(\Omega)$. $\mathbf{v}$ a weak solution satisfying (SEI),*

$$\int_t^{T_1} \|\mathbf{v}(\tau)\|_3^r \, \mathrm{d}\tau \ < \ C\,(T_1 - t)$$

  with $1 \leq r < \infty$ then $\mathbf{v}$ is regular in some left neighbourhood of T_1, including point T_1.

Regularity via the kinetic energy:

R. Farwig, H. Kozono, H. Sohr (2008): Ω *is a bounded domain with a smooth boundary,* $\mathbf{f} = \mathbf{0}$*,* $\mathbf{v}$ *is a weak solution satisfying (SEI),* $0 \leq a < b \leq T \leq \infty$*. If* $\|\mathbf{v}(t)\|_2^2$ *is Hölder continuous (as a function of time* t*) in* (a,b) *with exponent* $\alpha \in (\frac{1}{2}, 1)$ *then* $\mathbf{v}$ *is smooth in* Q_T*.*

Serrin–type conditions imposed on the gradient of velocity or on the vorticity:

- **H. Beirão da Veiga** (1995): *If* $\Omega = \mathbb{R}^3$, $\mathbf{f} = \mathbf{0}$, $\boldsymbol{\omega}_0 = \mathbf{curl}\, v_0 \in \mathbf{L}^2_\sigma(\mathbb{R}^3)$, $\boldsymbol{\omega} \in L^r(0,T;\, \mathbf{L}^s(\mathbb{R}^3))$ *with* $1 < r < \infty$, $\frac{3}{2} < s < \infty$, $2/r + 3/s \leq 2$ *or if the norm of* $\boldsymbol{\omega}$ *in* $L^\infty(0,T;\, \mathbf{L}^{3/2}(\mathbb{R}^3))$ *is sufficiently small then solution* $\mathbf{v}$ *is smooth in* Q_T*.*
- **D. Chae, H. J. Choe** (1999): proved the same result, imposing conditions only on two components of vorticity or on the gradients of only two components of velocity.

These results have been later generalized to other domains Ω.

Regularity in terms of pressure:

We assume that $\mathbf{v}$ is a weak solution and p is an associated pressure. Pressure p can be considered in the class $L^{3/2}(Q_T)$.

- **D. Chae, J. Lee** (2001): *If* $\Omega = \mathbb{R}^3$, $2/r + 3/s < 2$, $s > \frac{3}{2}$, $p \in L^r(0, T;\, L^s(\mathbb{R}^3))$ *solution* $\mathbf{v}$ *is smooth in* Q_T.
- **G. P.. Galdi, L. Berselli** (2002): Ω *is the whole space* $\mathbb{R}^3$*, a half–space, a smooth bounded domain, or a smooth exterior domain. If* $p \in L^r(0, T;\, L^s(\Omega))$ *with* $2/r + 3/s \le 2$, $\frac{3}{2} < s < \infty$ *then solution* $\mathbf{v}$ *is smooth in* Q_T.

Remark. The pressure is given uniquely up to an additive constant, depending possibly on t. If p does not satisfy the aforementioned assumptions then one cannot correct it, adding any function $p^*(t)$ to p.

- **K. Kang–J. Lee** (2006): extended the previous result to the case $\frac{3}{2} < s \le \infty$.
- **Ch. Qionglei, Z. Zhifei** (2007): $\Omega = \mathbb{R}^3$, $\mathbf{f} = \mathbf{0}$, considered $p \in L^1(0, T;\, \dot{B}^0_{\infty,\infty}(\mathbb{R}^3))$ and proved the regularity of solution $\mathbf{v}$.

- **T. Suzuki** (2012): *Ω bounded smooth domain in $\mathbb{R}^3$, $p \in L^{s,\infty}(0,T;\, L^{q,\infty}(\Omega))$, with the norm in this space "sufficiently small", $2/s + 3/q = 2$, $\frac{4}{2} \le q \le 3$, $2 \le s \le \frac{5}{2}$ then $\mathbf{v}$ is smooth in Q_T.*

  T. Suzuki also used conditions, imposed on ∇p.

- **G. Seregin, V. Šverák** (2002): $\Omega = \mathbb{R}^3$, $\mathbf{f} = \mathbf{0}$

  A function $g : \mathbb{R}^3 \times (0,\infty) \to [0,\infty)$ is said to satisfy condition (C) if to each $t_0 > 0$ there exists $R_0(t_0) > 0$ such that

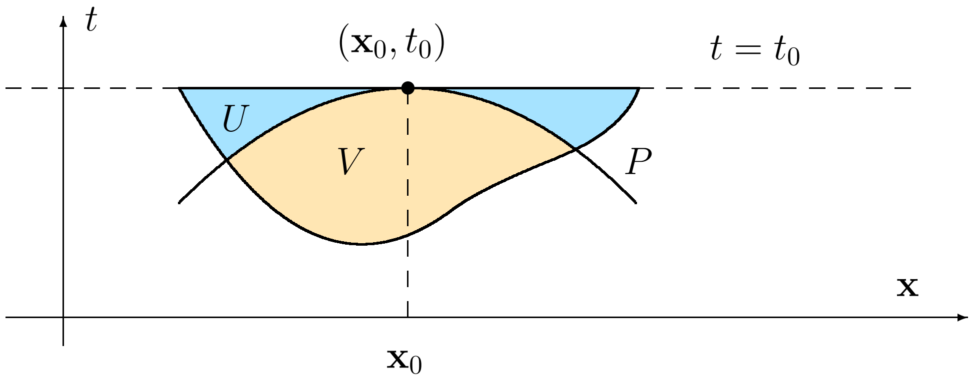
$$a)\ \sup_{\mathbf{x}_0 \in \mathbb{R}^3} \ \sup_{t_0 - R_0^2 \le t \le t_0} \int_{B_{R_0}(\mathbf{x})} \frac{g(\mathbf{x},t)}{|\mathbf{x}-\mathbf{x}_0|}\, \mathrm{d}\mathbf{x} \ < \ \infty,$$

$b)$ for each fixed $\mathbf{x}_0 \in \mathbb{R}^3$ and each fixed $R \in (0, R_0]$, the function

$$t \mapsto \int_{B_R(\mathbf{x}_0)} \frac{g(\mathbf{x},t)}{|\mathbf{x}-\mathbf{x}_0|}\, \mathrm{d}\mathbf{x} \text{ is continuous at } t_0 \text{ from the left.}$$

If either $p \ge g$ in Q_T or $p + \frac{1}{2}|\mathbf{v}|^2 \le g$ in Q_T for some function g satisfying condition (C) then solution $\mathbf{v}$ is smooth in Q_T.

J. Nečas and J.N. (2002): *Assume that* $(\mathbf{v}, p)$ *is a suitable weak solution.*



P *is an arbitrarily wide space–time paraboloid.*

If $p_- \in L^{\beta,\alpha}(V)$ *with* $\dfrac{2}{\alpha} + \dfrac{3}{\beta} \leq 2,\ \alpha \geq \frac{3}{2}, \beta > \frac{3}{2}$

and $\mathbf{v} \in L^{s,r}(U)$ *with* $\dfrac{2}{r} + \dfrac{3}{s} \leq 1,\ r \geq 3, s > 3,$

then $(\mathbf{x}_0, t_0)$ *is a regular point.*

2. Regularity criteria based on conditions imposed only on some components

Regularity via two components of velocity:

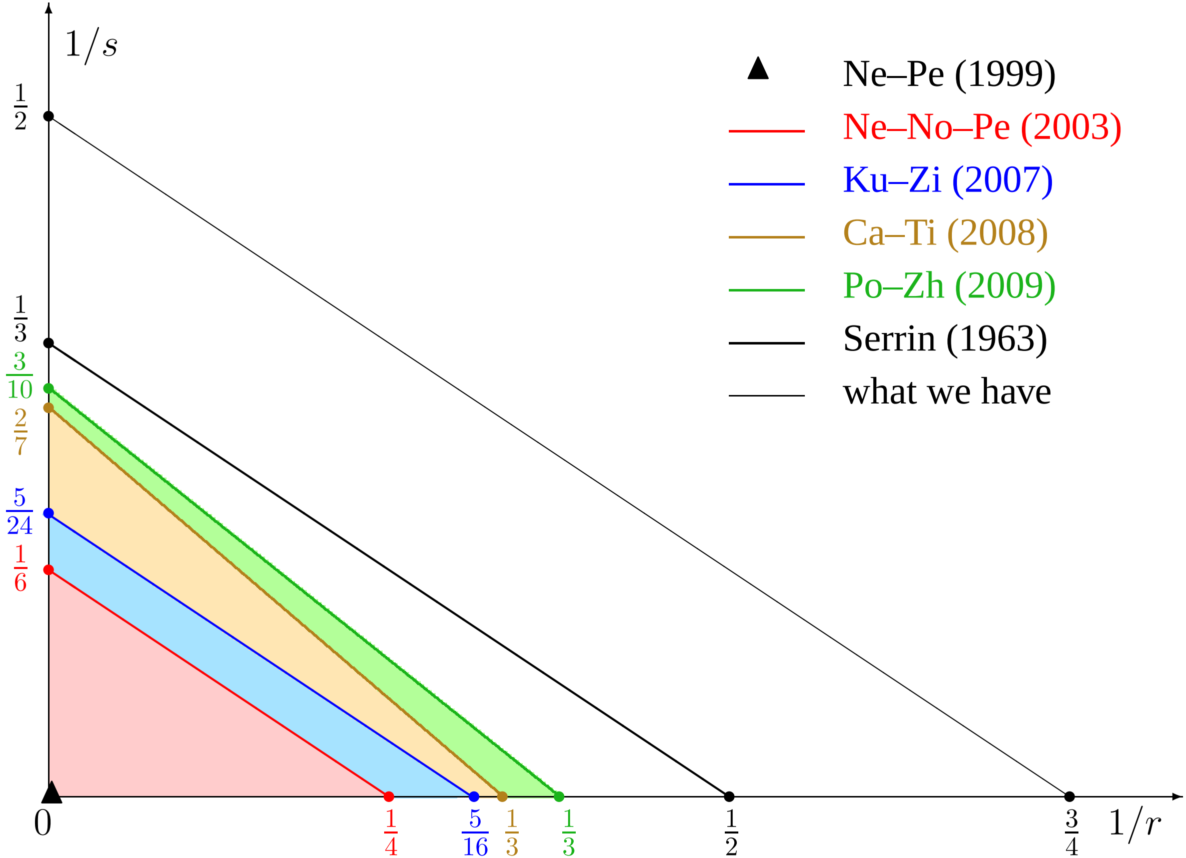
- **H. O. Bae and H. J. Choe** (2000, 2005): $\mathbf{v}$ is a suitable weak solution, imposed Serrin's conditions only on two components of velocity in a sub–domain D of Q_T, proved that there are no singular points in D.
- **J.N. and P. Penel** (2002): proved an analogous result.

Regularity via one component of velocity:

- **J.N. and P. Penel** (1999): $\mathbf{v}$ is suitable weak solution, one component of $\mathbf{v}$ is assumed to be essentially bounded in a space–time domain $D \subset \Omega \times (0, T)$. Then $\mathbf{v}$ has no singular points in D.
- **J.N., A. Novotný and P. Penel** (2003): $\mathbf{v}$ is suitable weak solution, one component of $\mathbf{v}$ is assumed to be in $L^r(a, b;\ L^s(\Omega'))$, where $0 \leq a < b \leq T$, Ω' is a

sub–domain of Ω, $r \in [4, \infty]$, $s \in (6, \infty]$ satisfying $2/r + 3/s \le \frac{1}{2}$. Then $\mathbf{v}$ has no singular points in $\Omega' \times (a, b)$.

- **J.N. and P. Penel (2002):** certain anisotropic criteria, interpolating between Serrin's condition with $2/r' + 3/s' \le 1$ imposed on two velocity components and Serrin's condition with $2/r'' + 3/s'' \le \frac{1}{2}$ imposed on one velocity component.
- **I. Kukavica and M. Ziane** (2007): $\Omega = \mathbb{R}^3$, v_3 is only assumed to be in $L^r(0, T;\, L^s(\mathbb{R}^3))$, where $2/r + 3/s = \frac{5}{8}$ for $r \in [\frac{16}{5}, \infty)$ and $s \in (\frac{24}{5}, \infty]$.
- **C. Cao and E. Titi** (2008): the authors consider the spatially periodic problem in $\mathbb{R}^3$ and use the condition $2/r + 3/s < \frac{2}{3} + 2/(3s)$, $s > \frac{7}{2}$.
- **M. Pokorný and Y. Zhou** (2009): the exponents r, s are supposed to satisfy the conditions $2/r + 3/s \le \frac{3}{4} + 1/(2s)$, $s > \frac{10}{3}$.


1/s
1/2
1/3
3/10
2/7
5/24
1/6
0
1/4
5/16
1/3
1/3
1/2
3/4
1/r
Ne–Pe (1999)
Ne–No–Pe (2003)
Ku–Zi (2007)
Ca–Ti (2008)
Po–Zh (2009)
Serrin (1963)
what we have

Regularity via one velocity component in the radially symmetric case:

J.N. and M. Pokorný (2000) (Later improved by **M. Pokorný**): $\mathbf{v}$ *is an axially symmetric suitable weak solution,* $\mathbf{f} = \mathbf{0}$, $(v_r)_- \in L^r(a,b;\ L^s(\Omega'))$, *where* $0 \leq a < b \leq T$, Ω' *is an axially symmetric sub–domain of* Ω, $r \in [2,\infty]$, $s \in (3,\infty]$ *satisfying* $2/r + 3/s \leq 1$. *Then* $\mathbf{v}$ *has no singular points in* $\Omega' \times (a,b)$.

Alternatively: $v_\vartheta \in L^r(a,b;\ L^s(\Omega'))$, *where*

a) either $s \in [6,\infty]$, $r \in [\frac{20}{7},\infty]$, $2/r + 3/s \leq \frac{7}{10}$,

b) or $s \in [\frac{24}{5},6)$, $r \in (10,\infty]$, $2/r + 3/s \leq 1 - \frac{9}{5s}$,

Then $\mathbf{v}$ *has no singular points in* $\Omega' \times (a,b)$.

Regularity via some components of the gradient of velocity:

- **H. Beirao da Veiga** (1995)
- **D. Chae, H. J. Choe** (1999)
- **P. Penel, M. Pokorný** (2004)
- **I. Kukavica, M. Ziane** (2006, 2007)
- **Y. Zhou, M. Pokorný** (2009, 2010)

Open problem:

Can the regularity of a weak solution be controlled by just one component of vorticity?

Principles of proofs of some one–velocity–component criteria

I. Application of equation for vorticity

Assume that e.g. the component v_3 of velocity $\mathbf{v}$ is “smooth” in a space–time cylinder $\Omega' \times (t_1, t_2)$, where $\Omega' \subset \Omega$ and $0 \leq t_1 < t_2 \leq T$.

We know (from the so called “Theorem of structure”) that

$$(t_1, t_2) \;=\; \bigcup_{\gamma \in \Gamma} (a_\gamma, b_\gamma) \,\cup\, G,$$

where set Γ is at most countable, set G is of measure zero and solution $\mathbf{v}$ is “smooth” on each time interval (a_γ, b_γ). Thus singularities can appear only at times $t \in G$.

Assume that t_0 is one of the time instants b_γ. Let $t_0 - \tau$ be any point in (a_γ, b_γ). Then we know that "smooth" on the time interval $[t_0 - \tau, t_0)$.

Let $\mathbf{x}_0 \in \Omega'$. In order to work in a bounded domain, we choose a ball $B = B_R(\mathbf{x}_0) \subset \Omega'$, and use a cut–off function η, that is a function from $C_0^\infty(B)$, equal to one in $B_{R-\epsilon}(\mathbf{x}_0)$, and such that $0 \leq \eta \leq 1$ in $B_R(\mathbf{x}_0) \setminus B_{R-\epsilon}(\mathbf{x}_0)$.

Multiplying the Navier–Stokes equation by function η, we obtain the equations of the same type, i.e.

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla(\eta p) + \nu \Delta \mathbf{u} + \mathbf{h} \qquad \text{in } B \times (t_0 - \tau, t_0), \tag{2.1}$$

$$\operatorname{div} \mathbf{u} = 0 \qquad \text{in } B \times (t_0 - \tau, t_0), \tag{2.2}$$

for the function $\mathbf{u} := \eta \mathbf{v} - \mathbf{V}$, where $\mathbf{V}$ is an appropriate correction that guarantees the validity of equation (2.2). (Concretely, $\operatorname{div} \mathbf{V} = \nabla \eta \cdot \mathbf{v}$.) Function $\mathbf{V}$ can be constructed so that $\operatorname{supp} \mathbf{V} \subset [B_R(\mathbf{x}_0) \setminus B_{R-\epsilon}(\mathbf{x}_0)] \times (t_0 - \tau, t_0)$.

Function $\mathbf{u}$, and all its derivatives, are equal to zero on $\partial B \times (t_0 - \tau, t_0)$.

Function $\mathbf{h}$ depends on $\mathbf{V}$, η, and on the values of functions $\mathbf{v}$, p in $[B_R(\mathbf{x}_0) \setminus B_{R-\epsilon}(\mathbf{x}_0)] \times (t_0 - \tau, t_0)$.

Denote by ω_3 the third component of vorticity $\boldsymbol{\omega}$ ($= \mathbf{curl}\ \mathbf{u}$). We have

$$\partial_t \omega_3 + \mathbf{u} \cdot \nabla \omega_3 \;=\; \partial_1 h_2 - \partial_2 h_1 + \boldsymbol{\omega} \cdot \nabla u_3 + \nu \Delta \boldsymbol{\omega}. \tag{2.3}$$

We denote by $|||\,.\,|||_{a,b}$ the norm in $L^a(t_0 - \tau, t_0;\ L^b(B))$ and $|||\,.\,|||_{(\infty,2)\cap(2,6)} := |||\,.\,|||_{\infty,2} + |||\,.\,|||_{2,6}$.

Lemma 1. *Suppose that* $u_3 \in L^r(t_0 - \tau, t_0;\ L^s(\mathbb{R}^3))$ *for* $2 \leq r \leq +\infty,\ 3 < s \leq +\infty,\ 2/r + 3/s \leq 1$. *Then there exist positive constants* c_1 *and* c_2 *such that*

$$|||\omega_3|||_{(\infty,2)\cap(2,6)} \;\leq\; c_1\, |||u_3|||_{r,s}\, |||\nabla u|||_{(\infty,2)\cap(2,6)}^{2/r+3/s} + c_2. \tag{2.4}$$

Equation (2.1) can be written in the form

$$\partial_t \mathbf{u} + F_1(\mathbf{u}) + \dots + F_6(\mathbf{u}) \;=\; \mathbf{h} - \nabla(\eta p + \tfrac{1}{2}\, u_3^2) + \nu \Delta \mathbf{u}, \tag{2.5}$$

where

$$\begin{aligned} F_1(\mathbf{u}) &= (\omega_2\, u_3\,,\ -\omega_1\, u_3\,,\ 0), & F_4(\mathbf{u}) &= (u_2\, \partial_1 u_2\,,\ u_2\, \partial_2 u_2\,,\ 0),\\ F_2(\mathbf{u}) &= (-\omega_3\, u_2\,,\ \omega_3\, u_1\,,\ 0), & F_5(\mathbf{u}) &= (0\,,\ 0\,,\ u_1\, \partial_1 u_3),\\ F_3(\mathbf{u}) &= (u_1\, \partial_1 u_1\,,\ u_1\, \partial_2 u_1\,,\ 0), & F_6(\mathbf{u}) &= (0\,,\ 0\,,\ u_2\, \partial_2 u_3). \end{aligned}$$

- $|||F_1(u)|||^2_{2,2} \leq |||u_3|||^2_{r,s}\ |||\nabla \mathbf{u}|||^2_{a,b}$, where

$$\frac{2}{r} + \frac{2}{a} = 1, \qquad \frac{2}{s} + \frac{2}{b} = 1 \qquad \text{and} \qquad \frac{2}{a} + \frac{3}{b} = \frac{5}{2} - \left(\frac{2}{r} + \frac{3}{s}\right) \geq \frac{3}{2}.$$

- Similarly, functions $F_5(\mathbf{u})$ and $F_6(\mathbf{u})$ can also be estimates by means of "good properties" of u_3.

- Function $F_2(\mathbf{u})$ can be estimated, using inequality (2.4):

$$|||F_2(\mathbf{u})|||^2_{2,2} \ = \ \int_{t_0-\tau}^{t_0} \int_B \omega_3^2\, (u_1^2 + u_2^2)\, \mathrm{d}\mathbf{x}\, \mathrm{d}t \ \leq \ |||\omega_3|||^2_{a,b}\ |||\mathbf{u}|||^2_{\alpha,\beta}$$

where $a > 2,\ b > 2,\ 2/a + 3/b = \frac{3}{2},\ \alpha = \dfrac{2a}{a-2},\ \beta = \dfrac{2b}{b-2}$.

Using the imbedding $W_0^{1,\frac{3\beta}{\beta+3}}(B) \hookrightarrow L^\beta(B)$, the last estimate gives:

$$|||F_2(\mathbf{u})|||^2_{2,2} \ \leq \ C\, |||\omega_3|||^2_{(\infty,2)\cap(2,6)}\ |||\nabla \mathbf{u}|||^2_{\alpha,\frac{3\beta}{\beta+3}}\ .$$

Since $|||\nabla\mathbf{u}|||^2_{\alpha,\frac{3\beta}{\beta+3}} \le C\,|||\nabla\mathbf{u}|||_{(\infty,2)\cap(2,6)}\,|||\nabla\mathbf{u}|||_{2,2}$, we obtain

$$
\begin{aligned}
|||F_2(\mathbf{u})|||^2_{2,2} &\le C\,|||\omega_3|||^2_{(\infty,2)\cap(2,6)}\,|||\nabla\mathbf{u}|||_{(\infty,2)\cap(2,6)} \\
&\le C\left(c_1^2\,|||u_3|||^2_{r,s}\,|||\nabla\mathbf{u}|||^{2(2/r+3/s)}_{(\infty,2)\cap(2,6)} + c_3^2\right)|||\nabla\mathbf{u}|||_{(\infty,2)\cap(2,6)} \\
&\le C\,|||u_3|||^2_{r,s}\,|||\nabla\mathbf{u}|||^{2(2/r+3/s)+1}_{(\infty,2)\cap(2,6)} + C\,|||\nabla\mathbf{u}|||_{(\infty,2)\cap(2,6)}.
\end{aligned}
$$

- $$
\begin{aligned}
|||F_3(\mathbf{u})|||^2_{2,2} &= \int_{t_0-\tau}^{t_0}\int_B \left[(u_1^2\,(\partial_1 u_1)^2 + u_1^2\,(\partial_2 u_1)^2\right]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&= -\frac{1}{3}\int_{t_0-\tau}^{t_0}\int_B \left[u_1^3\,\partial_1^2 u_1 + u_1^3\,\partial_2^2 u_1\right]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t.
\end{aligned}
$$

  Using the equation $\partial_1^2 u_1 + \partial_2^2 u_1 = -\partial_{31}u_3 - \partial_2\omega_3$ (which follows from the equation of continuity), we transform the last integral to

$$
\frac{1}{3}\int_{t_0-\tau}^{t_0}\int_B \left[u_1^3\,\partial_{31}u_3 + u_1^3\,\partial_2\omega_3\right]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t.
$$

- Finally, we use the estimate

$$|||\nabla \mathbf{u}|||_{\infty,2}^2 + |||\nabla^2 \mathbf{u}|||_{2,2}^2 \;\leq\; C\,|||F_1(\mathbf{u}) + \cdots + F_6(\mathbf{u})|||_{2,2}^2 \\ +\; C\,|||\mathbf{h}|||_{2,2} + C\,\|\nabla \mathbf{u}(.\,, t_0 - \tau)\|_2^2.$$

  The norm $|||F_1(\mathbf{u}) + \cdots + F_6(\mathbf{u})|||_{2,2}^2$ can be estimated by something that can be absorbed by the left hand side.

Conclusion: In this way, we can e.g. prove the result obtained by Ne–No–Pe (2003), i.e. that if $v_3 \in L^r(t_1, t_2;\, \Omega')$, where $2/r + 3/s \leq \frac{1}{2}$, then solution $\mathbf{v}$ is smooth in $\Omega' \times (t_1, t_2)$.

II. Application of the multiplicative Gagliardo–Nirenberg inequality

$$u \in W^{1,2}(\mathbb{R}^3): \qquad \|u\|_6 \;\leq\; C\, \|\partial_1 u\|_2^{1/3}\, \|\partial_2 u\|_2^{1/3}\, \|\partial_3 u\|_2^{1/3}$$

Assume that $\Omega = \mathbb{R}^3$, $\nu = 1$ and $\mathbf{f} = \mathbf{0}$. Multiplying the Navier–Stokes equation by $\Delta\mathbf{v}$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\frac{1}{2}\,\|\nabla\mathbf{v}(t)\|_2^2 + \|\Delta\mathbf{v}\|_2^2 \;=\; \int_{\mathbb{R}^3} (\mathbf{v}\cdot\nabla\mathbf{v})\cdot\Delta\mathbf{v}\;\mathrm{d}\mathbf{x}.$$

The right hand side can be rewritten in the form $K_1 + K_2 + K_3$, where

$$K_1 = -\sum_{j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3}(\partial_k v_3)(\partial_3 v_j)(\partial_k v_j)\;\mathrm{d}\mathbf{x} + \frac{1}{2}\sum_{j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3}(\partial_3 v_1)(\partial_k v_j)(\partial_k v_j)\;\mathrm{d}\mathbf{x},$$

$$K_2 = -\sum_{i=1}^{2}\sum_{j=1}^{3}\sum_{k=1}^{3}\int_{\mathbb{R}^3}(\partial_k v_i)(\partial_i v_j)(\partial_k v_j)\;\mathrm{d}\mathbf{x},$$

$$K_3 = \frac{1}{2}\sum_{j=1}^{3}\int_{\mathbb{R}^3}\big[(\partial_1 v_1) + (\partial_2 v_2)\big]\,(\partial_3 v_j)(\partial_3 v_j)\;\mathrm{d}\mathbf{x}.$$

All the terms contain derivatives with respect to x_1 or x_2.

Thus, denoting $\nabla_h = (\partial_1, \partial_2)$, we get

$$\begin{aligned}
\frac{1}{2}\,\|\nabla\mathbf{v}(t)\|_2^2 + \int_0^t \|\nabla^2\mathbf{v}(s)\|_2^2\,\mathrm{d}s \;\le\; & C\int_0^t\int_{\mathbb{R}^3} |\nabla_h\mathbf{v}|\,|\nabla\mathbf{v}|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}s + \frac{1}{2}\,\|\nabla\mathbf{v}_0\|_2^2 \\
\le\; & C\int_0^t \|\nabla_h\mathbf{v}(s)\|_2\,\|\nabla\mathbf{v}(s)\|_2^{1/2}\,\|\nabla\mathbf{v}(s)\|_6^{3/2}\,\mathrm{d}s + \frac{1}{2}\,\|\nabla\mathbf{v}_0\|_2^2 \\
\le\; & C + C\,|||\nabla_h\mathbf{v}|||_{\infty,2}\,|||\nabla\nabla_h\mathbf{v}|||_{2,2}\,|||\nabla\mathbf{v}|||_{2,2}^{1/2}\left(\int_0^t \|\nabla^2\mathbf{v}(s)\|_2^2\,\mathrm{d}s\right)^{1/4} \\
\le\; & C + C\,J^2(t)\left(\int_0^t \|\nabla^2\mathbf{v}(s)\|_2^2\,\mathrm{d}s\right)^{1/4}, \qquad (2.6)
\end{aligned}$$

where $J^2(t) \;:=\; \sup_{0<s<t}\,\|\nabla_h\mathbf{v}(s)\|_2^2 + \int_0^t \|\nabla\nabla_h\mathbf{v}(s)\|_2^2\,\mathrm{d}s$.

In order to estimate $J^2(t)$, we multiply the Navier–Stokes equation by $\Delta_h\mathbf{v}$ and integrate in $\mathbb{R}^3$. We obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\frac{1}{2}\,\|\nabla_h\mathbf{v}(t)\|_2^2 + \|\nabla\nabla_h\mathbf{v}\|_2^2 \;=\; \int_{\mathbb{R}^3} (\mathbf{v}\cdot\nabla\mathbf{v})\cdot\Delta_h\mathbf{v}\,\mathrm{d}\mathbf{x}, \qquad (2.7)$$

where $\int_{\mathbb{R}^3} (\mathbf{v} \cdot \nabla \mathbf{v}) \cdot \Delta_h \mathbf{v} \, \mathrm{d}\mathbf{x} = \sum_{i=1}^{2} \sum_{j=1}^{2} \int_{\mathbb{R}^3} v_i (\partial_i v_j)(\Delta_h v_j) \, \mathrm{d}\mathbf{x}$

$$+ \sum_{i=1}^{3} \int_{\mathbb{R}^3} v_i (\partial_i v_3) \Delta_h v_3 \, \mathrm{d}\mathbf{x} + \sum_{j=1}^{2} \int_{\mathbb{R}^3} v_3 (\partial_3 v_j) \Delta_h v_j) \, \mathrm{d}\mathbf{x}$$

$$= J_1 + J_2 + J_3.$$

$$\begin{aligned} J_1 &= \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} \int_{\mathbb{R}^3} (\partial_3 v_2)(\partial_j v_i)(\partial_j v_i) \, \mathrm{d}\mathbf{x} \\ &\quad + \int_{\mathbb{R}^3} \big[-(\partial_3 v_3)(\partial_1 v_1)(\partial_2 v_2) + (\partial_3 v_3)(\partial_2 v_1)(\partial_1 v_2) \big] \, \mathrm{d}\mathbf{x} \\ &= -\sum_{i=1}^{2} \sum_{j=1}^{2} \int_{\mathbb{R}^3} v_3 (\partial_3 \partial_j v_i)(\partial_j v_i) \, \mathrm{d}\mathbf{x} \\ &\quad + \int_{\mathbb{R}^3} v_3 \big[(\partial_1 \partial_3 v_1)(\partial_2 v_2) + (\partial_2 \partial_3 v_2)(\partial_1 v_1) - (\partial_2 \partial_3 v_1)(\partial_1 v_2) \\ &\qquad\qquad - (\partial_1 \partial_3 v_2)(\partial_2 v_1) \big] \, \mathrm{d}\mathbf{x} \end{aligned}$$

$$J_2 = \sum_{i=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3} v_3(\partial_i\partial_k v_3)(\partial_k v_i)\,\mathrm{d}\mathbf{x}$$

One can derive that

$$|J_1+J_2| \le C\,\|v_3\|_s\,\|\nabla\nabla_h\mathbf{v}\|_2^{1+3/s}\,\|\nabla_h\mathbf{v}\|_2^{1-3/s} \le \frac{1}{4}+C\,\|v_3\|_s^{2s/(s-3)}\,\|\nabla_h\mathbf{v}\|_2^2,$$

$$|J_3| \le C\int_{\mathbb{R}^3}|v_3|\,|\nabla\mathbf{v}|\,|\nabla\nabla_h\mathbf{v}|\,\mathrm{d}\mathbf{x} \le C\,\|v_3\|_s\,\|\nabla\nabla_h\mathbf{v}\|_2\,\|\nabla\mathbf{v}\|_2^{1-3/s}\,\|\nabla\mathbf{v}\|_6^{3/s}$$

Applying the G–N inequality to $\nabla\mathbf{v}$, we get

$$\begin{aligned}|J_3| &\le C\,\|v_3\|_s\,\|\nabla\nabla_h\mathbf{v}\|_2^{1+2/s}\,\|\nabla\mathbf{v}\|_2^{1-3/s}\,\|\nabla^2\mathbf{v}\|_2^{1/s}\\ &\le C\,\|v_3\|_s^{\frac{2s}{(s-2)}}\,\|\nabla\mathbf{v}\|_2^{\frac{2(s-3)}{(s-2)}}\,\|\nabla^2\mathbf{v}\|_2^{\frac{2}{(s-2)}}+\frac{1}{4}\,\|\nabla\nabla_h\mathbf{v}\|_2^2.\end{aligned}$$

Substituting the estimates of $|J_1+J_2|$ and $|J_3|$ to (2.7), we get

$$\frac{1}{2}\,\|\nabla_h\mathbf{v}(t)\|_2^2+\int_0^t\|\nabla\nabla_h\mathbf{v}(s)\|_2^2\,\mathrm{d}s \le \|\nabla_h\mathbf{v}_0\|_2^2+\frac{1}{2}\int_0^t\|\nabla\nabla_h\mathbf{v}(s)\|_2^2\,\mathrm{d}s$$

$$+ C \int_0^t \|v_3(s)\|_s^{\frac{2s}{(s-3)}} \, \|\nabla_h \mathbf{v}(s)\|_2^2 \, \mathrm{d}s$$

$$+ C \int_0^t \|v_3(s)\|_s^{\frac{2s}{(s-2)}} \, \|\nabla \mathbf{v}(s)\|_2^{\frac{2(s-3)}{(s-2)}} \, \|\nabla^2 \mathbf{v}(s)\|_2^{\frac{2}{(s-2)}} \, \mathrm{d}s$$

Thus,

$$J^2(t) \; \leq \; C + C \int_0^t \|v_3(s)\|_s^{\frac{2s}{(s-3)}} \, \|\nabla_h \mathbf{v}(s)\|_2^2 \, \mathrm{d}s + C \left(\int_0^t \|v_3(s)\|_s^{\frac{2s}{(s-3)}} \, \|\nabla_h \mathbf{v}(s)\|_2^2 \, \mathrm{d}s \right)^{\frac{s-3}{s-2}} \, |||\nabla^2 \mathbf{v}|||_{2,2}^{\frac{2}{s-2}}.$$

Substituting the estimate of $J^2(t)$ to (2.6), we get (after some technical manipulations like e.g. applications of Hölder's and Young's inequalities):

$$\|\nabla \mathbf{v}\|_2^2 + \int_0^t \|\Delta \mathbf{v}(s)\|_2^2 \, \mathrm{d}s \; \leq \; \text{something that enables us to apply Gronwall's inequality.}$$

Conclusion: In this way, we can e.g. prove the result obtained by Zh–Po (2009), i.e. that if $v_3 \in L^r(0,T;\mathbb{R}^3)$, where $2/r + 3/s \leq \frac{3}{4} + 1/(2s)$, then solution $\mathbf{v}$ is smooth in Q_T.

3. Regularity as a result of "smoothness" of a certain spectral projection of vorticity

3.1. Spectral projections of velocity and vorticity

We assume that $\Omega = \mathbb{R}^3$. We denote $Q_T := \mathbb{R}^3 \times (0, T)$.

- The operator $(-\Delta)$, with the domain $W^{2,2}(\mathbb{R}^3)$ (respectively $\mathbf{W}^{2,2}(\mathbb{R}^3)$), is positive and self–adjoint in $L^2(\mathbb{R}^3)$ (respectively in $\mathbf{L}^2(\mathbb{R}^3)$).
- The spectrum of $(-\Delta)$ is continuous and coincides with the interval $[0, \infty)$ on the real axis.
- The Stokes operator $S := \mathbf{curl}^{\,2}$, as an operator in space $\mathbf{L}^2_\sigma(\mathbb{R}^3)$, coincides with the reduction of $(-\Delta)$ to $\mathbf{L}^2_\sigma(\mathbb{R}^3)$.
- $D(S) = \mathbf{W}^{2,2}(\mathbb{R}^3) \cap \mathbf{L}^2_\sigma(\mathbb{R}^3)$

- Operator S is positive. Its spectrum is continuous and coincides with the interval $[0, \infty)$ on the real axis.
- The power $S^{1/4}$ of operator S satisfies the Sobolev–type inequality

$$\|\mathbf{u}\|_{3;\,\mathbb{R}^3} \;\leq\; c_3\,\|S^{1/4}\mathbf{u}\|_{2;\,\mathbb{R}^3} \qquad \text{for } \mathbf{u} \in D(S^{1/4}).$$

- Operator $\mathbf{curl}$, with the domain $D(\mathbf{curl}) := \mathbf{W}^{1,2}_{\sigma}(\mathbb{R}^3)$, is self–adjoint in $\mathbf{L}^2_{\sigma}(\mathbb{R}^3)$.
- The spectrum of $\mathbf{curl}$ is continuous and coincides with the whole real axis.

Principle of the proof. $\mathrm{Sp}(\mathbf{curl})$ is a subset of the real axis.

The residual part is empty, because $\mathbf{curl}$ is self–adjoint. It means that each point $\lambda \in \mathrm{Sp}(\mathbf{curl})$ is either an eigenvalue, or it belongs to $\mathrm{Sp}_c(\mathbf{curl})$ (the continuous spectrum of $\mathbf{curl}$).

If λ is an eigenvalue then λ^2 is an eigenvalue of the Stokes operator S, which is impossible. Thus, $\mathrm{Sp}(\mathbf{curl}) = \mathrm{Sp}_c(\mathbf{curl})$.

Let us show that the spectrum covers the whole real axis.

All points of $\mathrm{Sp}_c(\mathbf{curl})$ are non–isolated, otherwise they would have been the eigenvalues.

Let $\lambda \in \mathrm{Sp}_c(\mathbf{curl})$, $\lambda \neq 0$. There exists a sequence $\{\mathbf{u}_n\}$ on the unit sphere in $\mathbf{L}^2_\sigma(\mathbb{R}^3)$, such that

$$\|\mathbf{curl}\ \mathbf{u}_n - \lambda \mathbf{u}_n\|_{2;\,\mathbb{R}^3} \longrightarrow 0.$$

Let $\xi \in \mathbb{R}$, $\xi >= 0$. Put $\mathbf{u}_n^\xi(\mathbf{x}) := \xi^{3/2}\,\mathbf{u}_n(\xi\mathbf{x})$. Then $\{\mathbf{u}_n^\xi\}$ is a sequence on the unit sphere in $\mathbf{L}^2_\sigma(\mathbb{R}^3)$, satisfying

$$\|\mathbf{curl}\ \mathbf{u}_n^\xi - \xi\lambda \mathbf{u}_n^\xi\|_{2;\,\mathbb{R}^3} \longrightarrow 0.$$

It means that $\xi\lambda$ belongs to $\mathrm{Sp}_c(\mathbf{curl})$ as well. Thus, each real number, with the same sign as λ, is in $\mathrm{Sp}_c(\mathbf{curl})$.

Since the spectrum of $\mathbf{curl}$ is on both sides of 0 on the real axis (because $\mathbf{curl}$ is not a positive or a negative operator), it must cover $(-\infty, 0) \cup (0, \infty)$.

However, $\mathrm{Sp}_c(\mathbf{curl})$ is closed, hence $\mathrm{Sp}_c(\mathbf{curl}) = \mathbb{R}$. ∎

- Let $\{E_\lambda\}$ be the spectral resolution of identity, associated with operator $\mathbf{curl}$. Projection E_λ is strongly continuous in dependence on λ.

- $P^- := E_0 = \int_{-\infty}^{0} \mathrm{d}E_\lambda, \quad P^+ := I - E_0 = \int_0^\infty \mathrm{d}E_\lambda$

- Operators P^- and P^+ are orthogonal projections in $\mathbf{L}^2_\sigma(\mathbb{R}^3)$ such that $I = P^- + P^+$ and $O = P^- P^+$.

- $\mathbf{L}^2_\sigma(\mathbb{R}^3)^- := P^- \mathbf{L}^2_\sigma(\mathbb{R}^3)$ and $\mathbf{L}^2_\sigma(\mathbb{R}^3)^+ := P^+ \mathbf{L}^2_\sigma(\mathbb{R}^3)$

- Operator $\mathbf{curl}$ reduces on each of the spaces $\mathbf{L}^2_\sigma(\mathbb{R}^3)^-$ and $\mathbf{L}^2_\sigma(\mathbb{R}^3)^+$. It is negative on $\mathbf{L}^2_\sigma(\mathbb{R}^3)^-$ and positive on $\mathbf{L}^2_\sigma(\mathbb{R}^3)^+$.

- $A := |\mathbf{curl}| = -\mathbf{curl}\,\big|_{\mathbf{L}^2_\sigma(\mathbb{R}^3)^-} + \mathbf{curl}\,\big|_{\mathbf{L}^2_\sigma(\mathbb{R}^3)^+} = \int_{-\infty}^{\infty} |\lambda| \, \mathrm{d}E_\lambda$

- $A^2 = S$

- Operator A is positive and self–adjoint in $\mathbf{L}^2_\sigma(\mathbb{R}^3)$.

- The resolution of identity associated with operator A is the system of projections $F_\lambda := O$ for $\lambda < 0$, $F_\lambda = E_\lambda - E_{-\lambda}$ for $\lambda > 0$.

- The family of projections $G_\lambda := O$ for $\lambda < 0$, $G_\lambda := F_{\sqrt{\lambda}}$ for $\lambda > 0$, represents the resolution of identity associated with the operator $A^2 \equiv S$.

- $A = \displaystyle\int_0^\infty \lambda \, \mathrm{d}F_\lambda = \int_0^\infty \sqrt{\zeta} \, \mathrm{d}F_{\sqrt{\zeta}} = \int_0^\infty \sqrt{\zeta} \, \mathrm{d}G_\zeta = S^{1/2}$

- $\|\mathbf{u}\|_{3;\,\mathbb{R}^3} \leq c_3 \, \|A^{1/2}\mathbf{u}\|_{2;\,\mathbb{R}^3} \quad$ for $\mathbf{u} \in D(A^{1/2})$

- $\mathbf{v}^- := P^-\mathbf{v}, \quad \mathbf{v}^+ := P^+\mathbf{v}, \quad \boldsymbol{\omega}^- := P^-\boldsymbol{\omega}, \quad \boldsymbol{\omega}^+ := P^+\boldsymbol{\omega}$

- Since operator **curl** commutes with projections P^- and P^+, we have $\boldsymbol{\omega}^- = \mathbf{curl}\;\mathbf{v}^- = -A\mathbf{v}^-$ and $\boldsymbol{\omega}^+ = \mathbf{curl}\;\mathbf{v}^+ = A\mathbf{v}^+$.

3.2. Regularity in dependence on the spectral projection of vorticity

Theorem 1. *Let* $\mathbf{v}$ *be a weak solution to the initial–value Navier–Stokes problem. Assume that*

(i) $(-\Delta)^{1/4}\boldsymbol{\omega}^{+} \in \mathbf{L}^2(Q_T)$

and at least one of the two conditions

(a) $\mathbf{v}_0 \in \mathbf{L}^2_{\sigma}(\mathbb{R}^3)$ *and* $\mathbf{v}$ *satisfies (SEI),*

(b) $\mathbf{v}_0 \in D(A^{1/2})$ *and* $\mathbf{v}$ *satisfies (EI)*

holds. Then the norm $\|A^{1/2}\mathbf{v}\|_{2;\,\mathbb{R}^3}$ *is bounded in each time interval* (ϑ, T)*, where* $0 < \vartheta < T$*. Consequently, solution* $\mathbf{v}$ *has no singular points in* Q_T*.*

Moreover, if condition (b) holds then $\|A^{1/2}\mathbf{v}\|_{2;\,\mathbb{R}^3}$ *is bounded on the whole interval* $(0, T)$*.*

Principle of the proof.

Suppose that condition (a) holds. Solution $\mathbf{v}$ belongs to $L^2(0,T;\ D(A))$, hence there exists $t_0 \in (0,T)$ (arbitrarily close to 0) such that

1) $\mathbf{v}(t_0) \in D(A)$,

2) solution $\mathbf{v}$ satisfies the energy inequality, starting from the time instant t_0.

Due to the theorems on the local in time existence of a strong solution to the Navier–Stokes equations, there exists $\theta > 0$, $t_0 + \theta \leq T$, and a strong solution $\mathbf{v}'$ on the time interval $(t_0, t_0 + \theta)$, satisfying the initial condition $\mathbf{v}'(t_0) = \mathbf{v}(t_0)$.

Due to the theorem on uniqueness, $\mathbf{v}' = \mathbf{v}$ on $(t_0, t_0 + \theta)$. Hence $\mathbf{v}$ is a strong solution on $(t_0, t_0 + \theta)$.

Suppose further that $t \in (t_0, t_0 + \theta)$.

The Navier–Stokes equation (with $\nu = 1$) can also be written in the equivalent form

$$\partial_t \mathbf{v} + \boldsymbol{\omega} \times \mathbf{v} + \mathbf{curl}^{\,2} \mathbf{v} \ = \ -\nabla\big(p + \tfrac{1}{2}|\mathbf{v}|^2\big). \tag{3.1}$$

Multiplying this equation by $A\mathbf{v}$, and integrating in $\mathbb{R}^3$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\frac{1}{2}\,\|A^{1/2}\mathbf{v}\|_{2;\,\mathbb{R}^3}^2 - 2\big(\boldsymbol{\omega}^+ \times \mathbf{v},\, \boldsymbol{\omega}^-\big)_{2;\,\mathbb{R}^3} + \|A^{3/2}\mathbf{v}\|_{2;\,\mathbb{R}^3}^2 \;=\; 0. \tag{3.2}$$

The scalar product $(\boldsymbol{\omega}^+ \times \mathbf{v},\, \boldsymbol{\omega}^-)_{2;\,\mathbb{R}^3}$ can be estimated:

$$\begin{aligned} \big|\big(\boldsymbol{\omega}^+ \times \mathbf{v},\, \boldsymbol{\omega}^-\big)_{2;\,\mathbb{R}^3}\big| \;&\leq\; c_3^3\,\|A^{1/2}\boldsymbol{\omega}^+\|_{2;\,\mathbb{R}^3}\,\|A^{1/2}\mathbf{v}\|_{2;\,\mathbb{R}^3}\,\|A^{1/2}\boldsymbol{\omega}^-\|_{2;\,\mathbb{R}^3} \\ &\leq\; \frac{1}{4}\,\|A^{1/2}\boldsymbol{\omega}^-\|_{2;\,\mathbb{R}^3}^2 + c_3^6\,\|A^{1/2}\mathbf{v}\|_{2;\,\mathbb{R}^3}^2\,\|A^{1/2}\boldsymbol{\omega}^+\|_{2;\,\mathbb{R}^3}^2 \\ &\leq\; \frac{1}{4}\,\|A^{3/2}\mathbf{v}\|_{2;\,\mathbb{R}^3}^2 + c_3^6\,\|A^{1/2}\mathbf{v}\|_{2;\,\mathbb{R}^3}^2\,\|A^{1/2}\boldsymbol{\omega}^+\|_{2;\,\mathbb{R}^3}^2. \end{aligned} \tag{3.3}$$

Equation (3.2) and inequalities (3.3) yield

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\|A^{1/2}\mathbf{v}\|_{2;\,\mathbb{R}^3}^2 + \|A^{3/2}\mathbf{v}\|_{2;\,\mathbb{R}^3}^2 \;\leq\; 4c_3^6\,\|A^{1/2}\mathbf{v}\|_{2;\,\mathbb{R}^3}^2\,\|A^{1/2}\boldsymbol{\omega}^+\|_{2;\,\mathbb{R}^3}^2. \tag{3.4}$$

Since $\|A^{1/2}\boldsymbol{\omega}^+\|_{2;\,\mathbb{R}^3}^2 \;=\; \|(-\Delta)^{1/4}\boldsymbol{\omega}^+\|_{2;\,\mathbb{R}^3}^2 \;\in\; L^1(0,T)$, we can apply Gronwall's inequality and deduce that $\|A^{1/2}\mathbf{v}\|_{2;\,\mathbb{R}^3}$ is bounded in $(t_0, t_0+\theta)$. ■

3.3. Regularity in dependence on one component the spectral projection of vorticity

Theorem 2. *Let* $\mathbf{v}$ *be a weak solution to the Navier–Stokes initial problem. Assume that*

(ii) $(-\Delta)^{3/4}\omega_3^+ \in L^2(Q_T)$

and at least one of the two conditions

(a) $\mathbf{v}_0 \in \mathbf{L}^2_\sigma(\mathbb{R}^3)$ *and* $\mathbf{v}$ *satisfies (SEI),*

(b) $\mathbf{v}_0 \in D(A^{1/2})$ *and* $\mathbf{v}$ *satisfies (EI)*

holds. Then the norm $\|A^{1/2}\mathbf{v}\|_{2;\,\mathbb{R}^3}$ *is bounded in each time interval* (ϑ, T)*, where* $0 < \vartheta < T$*. Consequently, solution* $\mathbf{v}$ *has no singular points in* Q_T*.*

Moreover, if condition (b) holds then $\|A^{1/2}\mathbf{v}\|_{2;\,\mathbb{R}^3}$ *is bounded on the whole interval* $(0, T)$*.*

Principle of the proof.

I. A formal approach.

$$\|A^{1/2}\boldsymbol{\omega}^+\|^2_{2;\,\mathbb{R}^3} \;=\; (A\boldsymbol{\omega}^+,\boldsymbol{\omega}^+)_{2;\,\mathbb{R}^3} \;=\; (\mathbf{curl}\;\boldsymbol{\omega}^+,\boldsymbol{\omega}^+)_{2;\,\mathbb{R}^3}$$

Assume that we can find scalar functions y and z so that

$$\Delta_{2D}y = (-\Delta)^{1/4}(\partial_3\omega_3^+), \qquad \nabla^{\perp}_{2D}z = (-\Delta)^{1/4}\boldsymbol{\omega}_{2D} - \nabla_{2D}y \qquad \text{in } \mathbb{R}^2 \qquad (3.5)$$

for each fixed $x_3 \in \mathbb{R}$. Denote $\mathbf{w} = \mathbf{curl}\;\boldsymbol{\omega}^+$. Then

$$(\mathbf{curl}\;\boldsymbol{\omega}^+,\boldsymbol{\omega}^+)_{2;\,\mathbb{R}^3} \;=\; (\mathbf{w},\boldsymbol{\omega}^+)_{2;\,\mathbb{R}^3} \;=\; \big((-\Delta)^{-1/4}\mathbf{w},\,(-\Delta)^{1/4}\boldsymbol{\omega}^+\big)_{2;\,\mathbb{R}^3}$$

$$= \int_{\mathbb{R}^3}(-\Delta)^{-1/4}\mathbf{w}\cdot\begin{pmatrix}(-\Delta)^{-1/4}\omega_1^+\\(-\Delta)^{-1/4}\omega_2^+\\(-\Delta)^{-1/4}\omega_3^+\end{pmatrix}\,\mathrm{d}\mathbf{x}$$

$$= \int_{\mathbb{R}^3}(-\Delta)^{-1/4}\mathbf{w}\cdot\begin{pmatrix}\partial_2 z + \partial_1 y\\-\partial_1 z + \partial_2 y\\(-\Delta)^{-1/4}\omega_3^+\end{pmatrix}\,\mathrm{d}\mathbf{x}$$

$$= \int_{\mathbb{R}^3} \left[(-\Delta)^{-1/4}\mathbf{w} \cdot \begin{pmatrix} \partial_1 y \\ \partial_2 y \\ (-\Delta)^{-1/4}\omega_3^+ \end{pmatrix} + (-\Delta)^{-1/4}\mathbf{w} \cdot \mathbf{curl} \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} \right] \mathrm{d}\mathbf{x}$$

$$= \int_{\mathbb{R}^3} \left[(-\Delta)^{-1/4}\mathbf{w} \cdot \begin{pmatrix} \partial_1 y \\ \partial_2 y \\ (-\Delta)^{-1/4}\omega_3^+ \end{pmatrix} + (-\Delta)^{-1/4}\,\mathbf{curl}\,\mathbf{w} \cdot \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} \right] \mathrm{d}\mathbf{x}$$

$$= \int_{\mathbb{R}^3} \left[(-\Delta)^{-1/4}\mathbf{w} \cdot \begin{pmatrix} \partial_1 y \\ \partial_2 y \\ (-\Delta)^{-1/4}\omega_3^+ \end{pmatrix} + (-\Delta)^{-1/4}\,\mathbf{curl}^{\,2}\boldsymbol{\omega}^+ \cdot \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} \right] \mathrm{d}\mathbf{x}$$

$$= \int_{\mathbb{R}^3} \left[(-\Delta)^{-1/4}\mathbf{curl}\,\boldsymbol{\omega}^+ \cdot \begin{pmatrix} \partial_1 y \\ \partial_2 y \\ (-\Delta)^{-1/4}\omega_3^+ \end{pmatrix} + (-\Delta)^{3/4}\omega_3^+\, z \right] \mathrm{d}\mathbf{x}.$$

Problem. We need to estimate the L^2–norms of $\nabla_{2D}\, y$ and z by means of the L^2–norms of the right hand sides in (3.5). This is, however, impossible if (3.5) is considered in the whole plane $\mathbb{R}^2$.

II. A correct approach.

Sets K_ξ^{mn}, C^{mn} and the partition of function $\boldsymbol{\omega}^+$. For $m, n \in \mathbb{Z}$ and $\xi \in (-\frac{1}{2}, \infty)$, we denote

$$\begin{aligned} K_\xi^{mn} &:= (m-\xi, m+1+\xi) \times (n-\xi, n+1+\xi) && \subset \mathbb{R}^2, \\ C^{mn} &:= K_2^{mn} \times \mathbb{R} = (m-2, m+3) \times (n-2, n+3) \times \mathbb{R} && \subset \mathbb{R}^3. \end{aligned}$$

K_ξ^{mn} are squares in $\mathbb{R}^2$, C^{mn} are cylinders in $\mathbb{R}^3$.

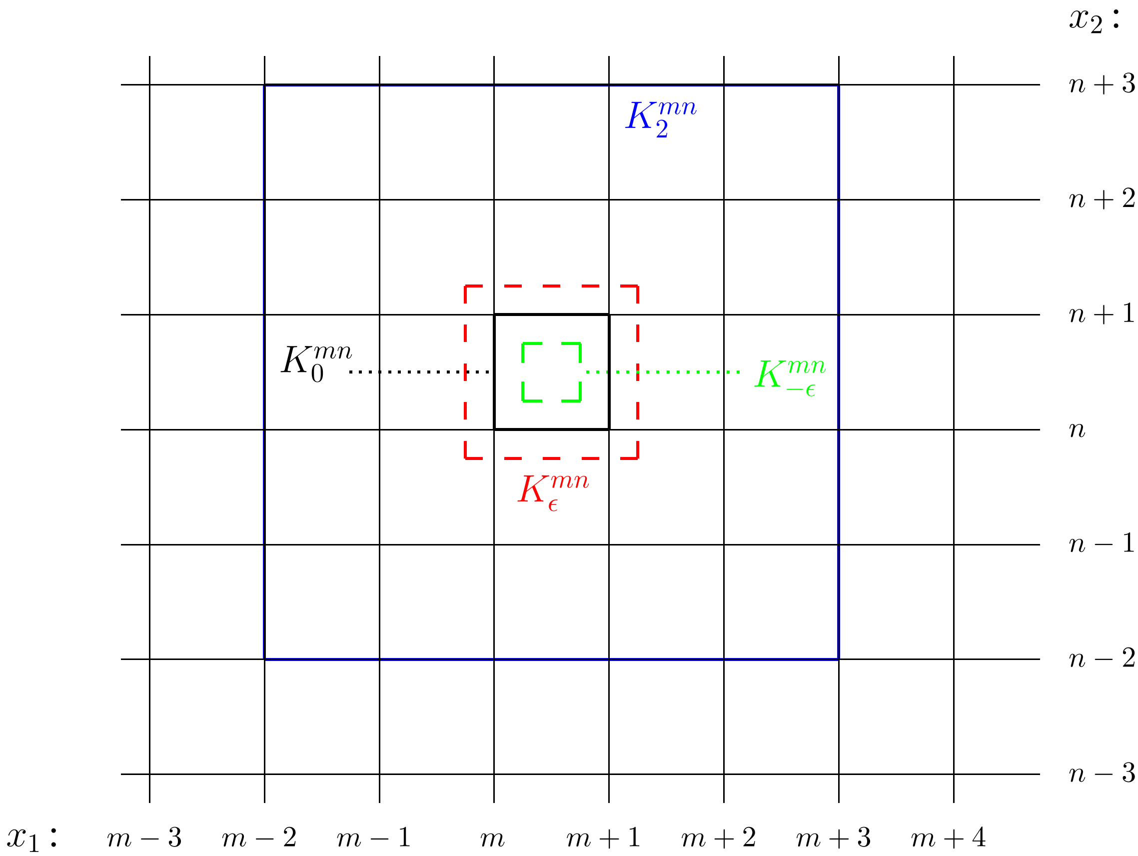
Let $\epsilon \in (0, \frac{1}{8})$ be fixed. There exists a partition of unity $\{\eta^{mn}\}_{m,n\in\mathbb{Z}}$ that consists of infinitely differentiable functions η^{mn} of two variables, such that

a) $\eta^{mn} = 1$ in $K_{-\epsilon}^{mn}$, $\eta^{mn} = 0$ in $\mathbb{R}^2 \setminus K_\epsilon^{mn}$, $0 \le \eta^{mn} \le 1$ in $\mathbb{R}^2$,

b) $\eta^{m+i,n+j}(x_1, x_2) = \eta^{mn}(x_1+i, x_2+j)$ for all $i, j \in \mathbb{Z}$,

c) $\sum_{m,n\in\mathbb{Z}} \eta^{mn} = 1$ in $\mathbb{R}^2$.

We denote by ∇_{2D} the 2D nabla operator (∂_1, ∂_2), and by $\boldsymbol{\omega}_{2D}^+$ the 2D vector field (ω_1^+, ω_2^+).


x_2 :
$n+3$
$n+2$
$n+1$
n
$n-1$
$n-2$
$n-3$
K_2^{mn}
K_0^{mn}
$K_{-\epsilon}^{mn}$
K_ϵ^{mn}
x_1 :
$m-3$
$m-2$
$m-1$
m
$m+1$
$m+2$
$m+3$
$m+4$

Applying successively the procedure of solving the equation $\nabla_{2D} \cdot \mathbf{u} = f$ (the so called Bogovskij operator), we deduce that there exists a system $\{\mathbf{V}^{mn}\}_{m,n\in\mathbb{Z}}$ of 2D vector functions $\mathbf{V}^{mn} = (V_1^{mn}, V_2^{mn})$ defined in $\mathbb{R}^3$ with the properties

d) $\nabla_{2D} \cdot \mathbf{V}^{mn} = -\nabla_{2D}\eta^{mn} \cdot \boldsymbol{\omega}_{2D}^{+}$ in $\mathbb{R}^3$,

e) $\mathrm{supp}\, \mathbf{V}^{mn} \subset [K_{2\epsilon}^{mn} \smallsetminus K_{-2\epsilon}^{mn}] \times \mathbb{R}$,

f) $\sum_{m,n\in\mathbb{Z}} \mathbf{V}^{mn} = \mathbf{0}$ in $\mathbb{R}^3$,

g) $\|\mathbf{V}^{mn}\|_{2;\,C^{mn}} + \|\nabla_{2D}\mathbf{V}^{mn}\|_{2;\,C^{mn}} \leq c\, \|\boldsymbol{\omega}_{2D}^{+}\|_{2;\,C^{mn}}$,

h) $\|\partial_3\mathbf{V}^{mn}\|_{2;\,C^{mn}} \leq c\, \|\partial_3\boldsymbol{\omega}_{2D}^{+}\|_{2;\,C^{mn}}$.

Constant c is always independent of m and n. We can derive from the last two estimates, by interpolation, that

$$\|\mathbf{V}^{mn}\|_{1/2,2;\,C^{mn}} \;\leq\; c\, \|\boldsymbol{\omega}^{+}\|_{1/2,2;\,C^{mn}}. \tag{3.6}$$

For technical reasons, we put $V_3^{mn} := 0$ and we further consider $\mathbf{V}^{mn}$ to be the 3D vector field. Further, we put

$$\boldsymbol{\omega}^{mn} \;:=\; \eta^{mn}\boldsymbol{\omega}^{+} - \mathbf{V}^{mn}.$$

The components of $\boldsymbol{\omega}^{mn}$ are denoted by ω_1^{mn}, ω_2^{mn} and ω_3^{mn}.

By analogy with $\boldsymbol{\omega}_{2D}^{+}$, we also denote $\boldsymbol{\omega}_{2D}^{mn} := (\omega_1^{mn}, \omega_2^{mn})$.

Function $\boldsymbol{\omega}^{mn}$ is divergence–free in $\mathbb{R}^3$, it equals $\boldsymbol{\omega}^{+}$ in $K_{-2\epsilon}^{mn} \times \mathbb{R}$, and its support is a subset of $K_{2\epsilon}^{mn} \times \mathbb{R}$.

Moreover, we have: $\boldsymbol{\omega}^{+} = \sum_{m,n\in\mathbb{Z}} \boldsymbol{\omega}^{mn}$.

The term $\|A^{1/2}\boldsymbol{\omega}^{+}\|_{2;\,\mathbb{R}^3}^2$ can now be written in this form:

$$\|A^{1/2}\boldsymbol{\omega}^{+}\|_{2;\,\mathbb{R}^3}^2 = (A\boldsymbol{\omega}^{+}, \boldsymbol{\omega}^{+})_{2;\,\mathbb{R}^3} = (\mathbf{curl}\ \boldsymbol{\omega}^{+}, \boldsymbol{\omega}^{+})_{2;\,\mathbb{R}^3} = \sum_{m,n\in\mathbb{Z}} \sum_{k,l\in\mathbb{Z}} (\mathbf{curl}\ \boldsymbol{\omega}^{mn}, \boldsymbol{\omega}^{kl})_{2;\,\mathbb{R}^3}$$

$$= \sum_{m,n\in\mathbb{Z}} \sum_{\substack{k\in\{m-1;\,m;\,m+1\} \\ l\in\{n-1;\,n;\,n+1\}}} (\mathbf{curl}\ \boldsymbol{\omega}^{mn}, \boldsymbol{\omega}^{kl})_{2;\,C^{mn}}. \tag{3.7}$$

The last equality holds because the supports of $\boldsymbol{\omega}^{mn}$ and $\boldsymbol{\omega}^{kl}$ have non–empty intersections only if $k \in \{m-1;\, m;\, m+1\}$ and $l \in \{n-1;\, n;\, n+1\}$. In this case, both the supports are subsets of C^{mn}.

Operator $(-\Delta)_{mn}$. We denote by $(-\Delta)_{mn}$ the operator $-\Delta$ with the domain $D((-\Delta)_{mn}) := W^{2,2}(C^{mn}) \cap W_0^{1,2}(C^{mn})$.

$(-\Delta)_{mn}$ is a positive and self–adjoint operator in $L^2(C^{mn})$, with a bounded inverse.

Auxiliary functions y_{mn}^{kl}. Function y_{mn}^{kl} is the solution of the 2D Neumann problem

$$\Delta_{2D} y_{mn}^{kl} = -(-\Delta)_{mn}^{1/4}(\partial_3 \omega_3^{kl}) \text{ in } K_2^{mn}, \qquad \frac{\partial y_{mn}^{kl}}{\partial \mathbf{n}} = 0 \text{ on } \partial K_2^{mn} \tag{3.8}$$

for m, $n \in \mathbb{Z}$, $k \in \{m-1;\, m;\, m+1\}$ and $l \in \{n-1;\, n;\, n+1\}$. Function y_{mn}^{kl} satisfies the estimate

$$\|\nabla_{2D}\, y_{mn}^{kl}\|_{2;\, K_2^{mn}}^2 + \|\nabla_{2D}{}^2 y_{mn}^{kl}\|_{2;\, K_2^{mn}}^2 \;\leq\; c\, \|(-\Delta)_{mn}^{1/4}(\partial_3 \omega_3^{kl})\|_{2;\, K_2^{mn}}^2, \tag{3.9}$$

where c is independent of m, n, k and l. Since $\partial_3 \omega_3^{kl}$ is a function of three variables x_1, x_2, x_3, function y_{mn}^{kl} naturally depends not only on x_1, x_2, but also on x_3. Integrating the last estimate with respect to x_3, we obtain

$$\|\nabla_{2D}{}^2 y_{mn}^{kl}\|_{2;\, C^{mn}}^2 + \|\nabla_{2D}\, y_{mn}^{kl}\|_{2;\, C^{mn}}^2 \;\leq\; c\, \|(-\Delta)_{mn}^{1/4} \partial_3 \omega_3^{kl}\|_{2;\, C^{mn}}^2. \tag{3.10}$$

Auxiliary functions z_{mn}^{kl}. We define function z_{mn}^{kl} to be the solution of the equation

$$\nabla_{2D}{}^{\perp} z_{mn}^{kl} \;=\; (-\Delta)_{mn}^{1/4}\boldsymbol{\omega}_{2D}^{kl} - \nabla_{2D}\, y_{mn}^{kl} \tag{3.11}$$

in K_2^{mn}. (Here, we denote by $\nabla_{2D}^{\perp}$ the operator $(-\partial_2, \partial_1)$.) The solution exists because

$$\nabla_{2D} \cdot \big[(-\Delta)_{mn}^{1/4}\boldsymbol{\omega}_{2D}^{kl} - \nabla_{2D}\, y_{mn}^{kl}\big] \;=\; 0.$$

Solution z_{mn}^{kl} depends not only on x_1, x_2, but also on x_3 because the right hand side of equation (3.7) depends on x_3 as well.

z_{mn}^{kl} ... the so called *stream function* of the 2D vector field $(-\Delta)_{mn}^{1/4}\boldsymbol{\omega}_{2D}^{kl} - \nabla_{2D}\, y_{mn}^{kl}$

For each fixed $x_3 \in \mathbb{R}$, z_{mn}^{kl} satisfies the estimate

$$\|\nabla_{2D}\, z_{mn}^{kl}\|_{2;\, K_2^{mn}} \;\leq\; c\left(\|(-\Delta)_{mn}^{1/4}\boldsymbol{\omega}_{2D}^{kl}\|_{2;\, K_2^{mn}} + \|\nabla_{2D}\, y_{mn}^{kl}\|_{2;\, K_2^{mn}}\right). \tag{3.12}$$

Moreover, z_{mn}^{kl} is constant on ∂C^{mn} ($= \partial K_2^{mn} \times \mathbb{R}$). This follows from the identities

$$\nabla_{2D}{}^{\perp} z_{mn}^{kl} \cdot \mathbf{n} \;=\; (-\Delta)_{mn}^{1/4}\boldsymbol{\omega}_{2D}^{kl} \cdot \mathbf{n} - \nabla_{2D}\, y_{mn}^{kl} \cdot \mathbf{n} \;=\; 0 \qquad \text{on } \partial C^{mn}.$$

Indeed, the second term $\nabla_{2D}\, y^{kl}_{mn} \cdot \mathbf{n}$ equals zero on ∂C^{mn} by definition of y^{kl}_{mn}.

The first term $(-\Delta)^{1/4}_{mn}\boldsymbol{\omega}^{kl}$ is zero on ∂C^{mn} because $\boldsymbol{\omega}^{mn} \in D((-\Delta)_{mn})$, hence $(-\Delta)^{1/4}_{mn}\boldsymbol{\omega}^{mn} \in D((-\Delta)^{3/4}_{mn})$, and functions from $D((-\Delta)^{3/4}_{mn})$ have the trace on ∂C^{mn} equal to zero.

Function z^{kl}_{mn} is unique up to an additive function of t and x_3. We can now choose this function so that $z^{kl}_{mn} = 0$ on ∂C^{mn}. This choice, together with (3.12) and (3.10), implies that

$$\begin{aligned} \|z^{kl}_{mn}\|_{2;\, C^{mn}} &\leq c\left(\|(-\Delta)^{1/4}_{mn}\boldsymbol{\omega}^{kl}_{2D}\|_{2;\, C^{mn}} + \|\nabla_{2D}\, y^{kl}_{mn}\|_{2;\, C^{mn}}\right) \\ &\leq c\left(\|(-\Delta)^{1/4}_{mn}\boldsymbol{\omega}^{kl}\|_{2;\, C^{mn}} + \|(-\Delta)^{1/4}_{mn}(\partial_3\omega^{kl}_3)\|_{2;\, C^{mn}}\right). \end{aligned} \tag{3.13}$$

The estimate of $(\mathbf{curl}\ \boldsymbol{\omega}^{mn}, \boldsymbol{\omega}^{kl})_{2;\, C^{mn}}$. Due to the definition of functions y^{kl}_{mn} and z^{kl}_{mn}, function $(-\Delta)^{1/4}_{mn}\boldsymbol{\omega}^{kl}$ has the form

$$(-\Delta)^{1/4}_{mn}\boldsymbol{\omega}^{kl} = \begin{pmatrix} \partial_1 y^{kl}_{mn} \\ \partial_2 y^{kl}_{mn} \\ (-\Delta)^{1/4}_{mn}\omega^{kl}_3 \end{pmatrix} + \mathbf{curl} \begin{pmatrix} 0 \\ 0 \\ z^{kl}_{mn} \end{pmatrix} \qquad \text{in } C^{mn}.$$

We denote $\mathbf{w}^{mn} \equiv (w_1^{mn}, w_2^{mn}, w_3^{mn}) := \mathbf{curl}\ \boldsymbol{\omega}^{mn}$ and $\mathbf{w}_{2D}^{mn} := (w_1^{mn}, w_2^{mn})$.

We have

$$(\mathbf{curl}\ \boldsymbol{\omega}^{mn}, \boldsymbol{\omega}^{kl})_{2;\, C^{mn}} \;=\; (\mathbf{w}^{mn}, \boldsymbol{\omega}^{kl})_{2;\, C^{mn}} \;=\; \int_{C^{mn}} (-\Delta)_{mn}^{-1/4} \mathbf{w}^{mn} \cdot (-\Delta)_{mn}^{1/4} \boldsymbol{\omega}^{kl}\, \mathrm{d}\mathbf{x}$$

$$= \int_{C^{mn}} \left[(-\Delta)_{mn}^{-1/4} \mathbf{w}^{mn} \cdot \begin{pmatrix} \partial_1 y_{mn}^{kl} \\ \partial_2 y_{mn}^{kl} \\ (-\Delta)_{mn}^{1/4} \omega_3^{kl} \end{pmatrix} + (-\Delta)_{mn}^{-1/4} \mathbf{curl}^{\,2} \boldsymbol{\omega}^{mn} \cdot \begin{pmatrix} 0 \\ 0 \\ z_{mn}^{kl} \end{pmatrix} \right] \mathrm{d}\mathbf{x}$$

$$= \int_{C^{mn}} \left[(-\Delta)_{mn}^{-1/4} \mathbf{w}^{mn} \cdot \begin{pmatrix} \partial_1 y_{mn}^{kl} \\ \partial_2 y_{mn}^{kl} \\ (-\Delta)_{mn}^{1/4} \omega_3^{kl} \end{pmatrix} + (-\Delta)_{mn}^{3/4} \omega_3^{mn}\, z_{mn}^{kl} \right] \mathrm{d}\mathbf{x}.$$

The norms of the components $\partial_1 y_{mn}^{kl}$ and $\partial_2 y_{mn}^{kl}$ are estimated by $c\, \|(-\Delta)_{mn}^{1/4} \partial_3 \omega_3^{kl}\|_{2;\, C^{mn}}^2$, see (3.6).

Thus, each term in this integral is “controlled” by ω_3^+.

Applying the estimates of y_{mn}^{kl}, z_{mn}^{kl} and $\mathbf{V}^{mn}$, we can finally derive the estimate

$$(\mathbf{curl}\ \boldsymbol{\omega}^{mn}, \boldsymbol{\omega}^{kl})_{2;\,C^{mn}} \ \leq\ \delta\, c\, \|\boldsymbol{\omega}^+\|^2_{1/2,2;\,C^{mn}} + c(\delta)\, \|\omega_3^+\|^2_{3/2,2;\,C^{mn}}. \tag{3.14}$$

The estimate of the right hand side of (3.7). The sum $\sum_{m,n\in\mathbb{Z}}$ in (3.7) can be split to twenty five parts, which successively contain the sums over $m = 0 \bmod 5, \dots,$ $m = 4 \bmod 5$ and $n = 0 \bmod 5, \dots, n = 4 \bmod 5$.

Let us consider e.g. the case $m, n \in \mathbb{Z}$, $m = 0 \bmod 5$, $n = 0 \bmod 5$ (i.e. m and n are integer multiples of 5).

Denote the sum over these m, n by $\sum^{(1)}_{m,n\in\mathbb{Z}}$, and the sums over twenty four other possibilities by $\sum^{(2)}_{m,n\in\mathbb{Z}}, \dots, \sum^{(25)}_{m,n\in\mathbb{Z}}$.

The cylinders C^{mn} corresponding to the first case are disjoint and their union equals $\mathbb{R}^3$ up to the set of measure zero. Applying (3.14), we have

$$\sum_{m,n\in\mathbb{Z}}^{(1)} \sum_{\substack{k\in\{m-1;\,m;\,m+1\}\\ l\in\{n-1;\,n;\,n+1\}}} (\mathbf{curl}\;\boldsymbol{\omega}^{mn}, \boldsymbol{\omega}^{kl})_{2;\,C^{mn}}$$

$$\leq\ \delta\, c \sum_{m,n\in\mathbb{Z}}^{(1)} \|\boldsymbol{\omega}^+\|^2_{1/2,2;\,C^{mn}} + c(\delta) \sum_{m,n\in\mathbb{Z}}^{(1)} \|\omega_3^+\|^2_{3/2,2;\,C^{mn}}. \tag{3.15}$$

The L^2–norms and $W^{1,2}$–norms of $\boldsymbol{\omega}^+$ satisfy the identities

$$\sum_{m,n\in\mathbb{Z}}^{(1)} \|\boldsymbol{\omega}^+\|^2_{2;\,C^{mn}} \;=\; \|\boldsymbol{\omega}^+\|^2_{2;\,\mathbb{R}^3} \quad \text{and} \quad \sum_{m,n\in\mathbb{Z}}^{(1)} \|\boldsymbol{\omega}^+\|^2_{1,2;\,C^{mn}} \;=\; \|\boldsymbol{\omega}^+\|^2_{1,2;\,\mathbb{R}^3}.$$

Applying the theorem on interpolation (see e.g. Theorem I.5.1 in the book by Lions, Magenes), we derive that

$$\sum_{m,n\in\mathbb{Z}}^{(1)} \|\boldsymbol{\omega}^+\|^2_{1/2,2;\,C^{mn}} \;\leq\; c\,\|\boldsymbol{\omega}^+\|^2_{1/2,2;\,\mathbb{R}^3}.$$

The norms $\|\omega_3^+\|_{3/2,2;\,C^{mn}}$ and $\|\omega_3^+\|_{3/2,2;\,\mathbb{R}^3}$ satisfy the same inequalities.

Applying these inequalities, and estimating the sums $\sum_{m,n\in\mathbb{Z}}^{(2)}, \ \dots\ , \sum_{m,n\in\mathbb{Z}}^{(25)}$ in the same way as the sum in (3.15), we get

$$\|A^{1/2}\boldsymbol{\omega}^+\|_{2;\,\mathbb{R}^3}^2 \;\leq\; \sum_{m,n\in\mathbb{Z}} \; \sum_{\substack{k\in\{m-1;\,m;\,m+1\}\\ l\in\{n-1;\,n;\,n+1\}}} (\mathbf{curl}\;\boldsymbol{\omega}^{mn},\boldsymbol{\omega}^{kl})_{2;\,C^{mn}}$$

$$\leq\; \delta\, c\, \|\boldsymbol{\omega}^+\|_{1/2,2;\,\mathbb{R}^3}^2 + c(\delta)\, \|\omega_3^+\|_{3/2,2;\,\mathbb{R}^3}^2 .$$

The first term on the right hand side is less than or equal to $\delta\, c\, \big(\|\boldsymbol{\omega}^+\|_{2;\,\mathbb{R}^3}^2 + \|A^{1/2}\boldsymbol{\omega}^+\|_{2;\,\mathbb{R}^3}^2\big)$. Choosing $\delta > 0$ so small that $\delta\, c \leq \frac{1}{2}$, and estimating $\|\omega_3^+\|_{3/2,2;\,\mathbb{R}^3}^2$ from above by $\|\omega_3^+\|_{2;\,\mathbb{R}^3}^2 + \|(-\Delta)^{3/4}\omega_3^+\|_{2;\,\mathbb{R}^3}^2$, we finally obtain

$$\|A^{1/2}\boldsymbol{\omega}^+\|_{2;\,\mathbb{R}^3}^2 \;\leq\; c_4\, \|\boldsymbol{\omega}^+\|_{2;\,\mathbb{R}^3}^2 + c_5\, \|(-\Delta)^{3/4}\omega_3^+\|_{2;\,\mathbb{R}^3}^2 . \tag{3.16}$$

Completion of the proof. Substituting estimate (3.12) to (2.4), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\, \frac{1}{2}\, \|A^{1/2}\mathbf{v}\|_{2;\,\mathbb{R}^3}^2 + \|A^{3/2}\mathbf{v}\|_{2;\,\mathbb{R}^3}^2$$

$$\leq\; 4c_3^6\, \|A^{1/2}\mathbf{v}\|_{2;\,\mathbb{R}^3}^2\, \big(c_4\, \|\boldsymbol{\omega}^+\|_{2;\,\mathbb{R}^3}^2 + c_5\, \|(-\Delta)^{3/4}\omega_3^+\|_{2;\,\mathbb{R}^3}^2\big). \tag{3.17}$$

The proof can now be finished in the same way as the proof of Theorem 1. ■

3.4. Some generalizations of the results from sections 3.2 and 3.3

The "positive" and "negative" parts of the velocity or the vorticity need not be generally separated by point 0 in the spectrum of $\mathbf{curl}$.

Let the point of separation be a, where $a = a(t)$ is a function of t in the interval $(0, T)$ with values in $[-\infty, \infty)$.

We denote by $a^+(t)$ the positive part and by $a^-(t)$ the negative part of $a(t)$.

Recall that $\{E_\lambda\}$ is the spectral resolution of identity, corresponding to the self–adjoint operator $\mathbf{curl}$.

We denote

$$
\begin{aligned}
P^+_{a(t)} &:= I - E_{a(t)} = \int_{a(t)}^{\infty} \mathrm{d}E_\lambda, \\
\mathbf{v}^+_a(t) &:= P^+_{a(t)} \mathbf{v}(t), \\
\boldsymbol{\omega}^+_a(t) &:= P^+_{a(t)} \boldsymbol{\omega}(t) = \mathbf{curl}\ \mathbf{v}^+_a(t).
\end{aligned}
$$

Theorem 3 (generalization of Theorem 1). *Let* $\mathbf{v}$ *be a weak solution to the Navier–Stokes initial value problem. Assume that*

(iii) $a^+ \in L^3(0,T)$ *and* $(-\Delta)^{1/4}\boldsymbol{\omega}_a^+ \in \mathbf{L}^2(Q_T)$,

and at least one of the two conditions

(a) $\mathbf{v}_0 \in \mathbf{L}^2_\sigma(\mathbb{R}^3)$ *and* $\mathbf{v}$ *satisfies (SEI),*

(b) $\mathbf{v}_0 \in D(A^{1/2})$ *and* $\mathbf{v}$ *satisfies (EI)*

holds. Then the norm $\|A^{1/2}\mathbf{v}\|_{2;\,\mathbb{R}^3}$ *is bounded in each time interval* (ϑ, T), *where* $0 < \vartheta < T$. *Consequently, solution* $\mathbf{v}$ *has no singular points in* Q_T.

Moreover, if condition (b) holds then $\|A^{1/2}\mathbf{v}\|_{2;\,\mathbb{R}^3}$ *is bounded on the whole interval* $(0,T)$.

Remark (on the special case $a(t) = -\infty$).

If function a in Theorem 3 is: $a(t) = -\infty$ for all $t \in (0,T)$ then $P^+_{a(t)} = I$ and $\boldsymbol{\omega}_a^+(t) = \boldsymbol{\omega}(t)$ in $(0,T)$.

In this case, condition (iii) is the condition on the whole vorticity $\boldsymbol{\omega}$, and it requires that $\boldsymbol{\omega} \in L^2(0,T;\, D(S^{1/4}))$.

The space $D(S^{1/4})$ is continuously imbedded in $\mathbf{L}^3(\mathbb{R}^3)$.

In this case, our result is in a good agreement with the older result of Beirão da Veiga (1995), which states that if $\boldsymbol{\omega} \in L^2(0,T;\, \mathbf{L}^3(\mathbb{R}^3))$ then solution v has no singular points in Q_T.

Theorem 4 (generalization of Theorem 2). *Let* $\mathbf{v}$ *be a weak solution to the Navier–Stokes initial value problem. Assume that*

(iv) $a^+ \in L^3(0,T)$, $a^- \in L^5(0,T)$ *and* $(-\Delta)^{3/4}\omega^+_{a3} \in L^2(Q_T)$

and at least one of the two conditions (a), (b) holds. Then all the conclusions of Theorem 3 are true.

Note that condition (iv) is not applicable to the case $a \equiv -\infty$.

3.5. Further related remarks

Remark 1 (on the meaning of functions v^+ and $\boldsymbol{\omega}^+$).

The velocity $\mathbf{v}$ and the corresponding vorticity $\boldsymbol{\omega}$ satisfy

$$\mathbf{v} = \int_{-\infty}^{\infty} \mathrm{d}E_\lambda(\mathbf{v}), \qquad \boldsymbol{\omega} = \mathbf{curl}\ \mathbf{v} = \int_{-\infty}^{\infty} \lambda\, \mathrm{d}E_\lambda(\mathbf{v}) = \int_{-\infty}^{\infty} \mathrm{d}E_\lambda(\boldsymbol{\omega}). \tag{3.18}$$

In accordance with the heuristic understanding of the definite integral, we can interpret the first integral in (3.18) as a sum of "infinitely many" contributions $\mathrm{d}E_\lambda(\mathbf{v})$, each of whose is an "infinitely small" Beltrami flow.

Recall that *Beltrami flows* are flows, whose vorticity is parallel to the velocity. Here, concretely, $\mathbf{curl}\ \mathrm{d}E_\lambda(\mathbf{v}) = \lambda\, \mathrm{d}E_\lambda(\mathbf{v})$.

Function v^+ can now be understood to be the sum of only those "infinitely many" "infinitely small" contributions, whose vorticity is a positive multiple of velocity. (We call them the *positive Beltrami flows*.)

Remark 2 (explicit form of spectral projection E_λ).

$\mathcal{F}$ . . . the Fourier transform

$$\begin{aligned} \mathcal{F}\big[\mathbf{curl}\ \mathbf{v}\big](\boldsymbol{\zeta}) &= \mathrm{i}\,\boldsymbol{\zeta} \times \mathcal{F}[\mathbf{v}](\boldsymbol{\zeta}), \\ \mathbf{curl}\ \mathbf{v}(\mathbf{x}) &= \mathcal{F}^{-1}\big[\mathrm{i}\,\boldsymbol{\zeta} \times \mathcal{F}[\mathbf{v}](\boldsymbol{\zeta})\big](\mathbf{x}) \\ \mathbf{curl}\ \mathbf{v}(\mathbf{x}) &= \big[\mathcal{F}^{-1} \circ (\mathrm{i}\,\mathbb{M}) \circ \mathcal{F}\big]\mathbf{v}(\mathbf{x}), \end{aligned}$$

where

$$\mathbb{M} = \begin{pmatrix} 0, & -\zeta_3, & \zeta_2 \\ \zeta_3, & 0, & -\zeta_1 \\ -\zeta_2, & \zeta_1, & 0 \end{pmatrix}.$$

Operator $(\mathrm{i}\,\mathbb{M})$ is self–adjoint in space $\mathbb{R}^3 + \mathrm{i}\,\mathbb{R}^3$. If we denote by $\mathcal{E}_\lambda$ its resolution of identity, then

$$\boxed{E_\lambda \mathbf{v} = \big[\mathcal{F}^{-1} \circ \mathcal{E}_\lambda \circ \mathcal{F}\big]\mathbf{v}.}$$

Operator $(\mathrm{i}\,\mathbb{M})$ has the eigenvalues $-|\boldsymbol{\zeta}|$, 0, and $|\boldsymbol{\zeta}|$. The corresponding eigenvectors are

$$\lambda_1 = -|\boldsymbol{\zeta}| \;:\quad \mathbf{V}_1(\boldsymbol{\zeta}) \;=\; |\boldsymbol{\zeta}|^2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - (\zeta_1+\zeta_2+\zeta_3)\,\boldsymbol{\zeta} + \mathrm{i}\,|\boldsymbol{\zeta}| \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \times \boldsymbol{\zeta},$$

$$= \begin{pmatrix} |\boldsymbol{\zeta}|^2 - (\zeta_1+\zeta_2+\zeta_3)\,\zeta_1 - \mathrm{i}\,|\boldsymbol{\zeta}|\,(\zeta_2-\zeta_3) \\ |\boldsymbol{\zeta}|^2 - (\zeta_1+\zeta_2+\zeta_3)\,\zeta_2 - \mathrm{i}\,|\boldsymbol{\zeta}|\,(\zeta_3-\zeta_1) \\ |\boldsymbol{\zeta}|^2 - (\zeta_1+\zeta_2+\zeta_3)\,\zeta_3 - \mathrm{i}\,|\boldsymbol{\zeta}|\,(\zeta_1-\zeta_2) \end{pmatrix}$$

$$\lambda_2 = 0 \;:\qquad \mathbf{V}_2(\boldsymbol{\zeta}) \;=\; \boldsymbol{\zeta},$$

$$\lambda_3 = -|\boldsymbol{\zeta}| \;:\quad \mathbf{V}_3(\boldsymbol{\zeta}) \;=\; \overline{\mathbf{V}_1(\boldsymbol{\zeta})}.$$

Since $\mathbf{V}_2(\boldsymbol{\zeta}) \cdot \widehat{\mathbf{v}} = 0$, projection $\mathcal{E}_\lambda$ can be expressed:

$$\mathcal{E}_\lambda \widehat{\mathbf{v}} \;=\; \mathbf{0} \qquad \text{for } \lambda < -|\boldsymbol{\zeta}|,$$

$$\mathcal{E}_\lambda \widehat{\mathbf{v}} \;=\; \frac{\mathbf{V}_1(\boldsymbol{\zeta}) \cdot \widehat{\mathbf{v}}}{|\mathbf{V}_1(\boldsymbol{\zeta})|^2}\; \mathbf{V}_1(\zeta) \qquad \text{for } -|\boldsymbol{\zeta}| \le \lambda < |\boldsymbol{\zeta}|,$$

$$\mathcal{E}_\lambda \widehat{\mathbf{v}} \;=\; \widehat{\mathbf{v}} \qquad \text{for } |\boldsymbol{\zeta}| \le \lambda.$$

Remark 3 (flow in the neighbourhood of a singularity).

Theorems 1 and 2 are also true if $\boldsymbol{\omega}^+$ (respectively ω_3^+) is replaced by $\boldsymbol{\omega}^-$ (respectively ω_3^-).

Thus, both the conditions (i) and (ii) show that if weak solution $\mathbf{v}$ has a singular point then the singularity must contemporarily develop in the "positive part" $\mathbf{v}^+$ of function $\mathbf{v}$ as well as in the "negative part" $\mathbf{v}^-$.

(Recall that $\mathbf{v}^+$ represents the contribution to $\mathbf{v}$ coming from the positive Beltrami flows and $\mathbf{v}^-$ is the contribution from the negative Beltrami flows.)

The singularity must even develop at the same spatial point. (This can be proven by an appropriate localization procedure.)

Remark 4 (the role of “large frequencies”)

Suppose, for simplicity, that function a considered in Theorem 3 is positive.

Then projection P_a^+ can be interpreted as a reduction to the positive Beltrami flows with “high frequencies”, concretely the “frequencies” comparable to a and higher.

Theorem 3 shows that if a singularity develops in solution v, then it must especially develop in the part of v (respectively its vorticity $\boldsymbol{\omega}$) that consists of positive Beltrami flows with the “large frequencies” (i.e. $\sim a$ and higher).

Since the functions a_+, $\boldsymbol{\omega}_a^+$ and ω_{a3}^+ can be replaced by a_-, $\boldsymbol{\omega}_a^-$ and ω_{a3}^- in Theorem 3, the singularity must also develop in the part of $\mathbf{v}$ (respectively vorticity $\boldsymbol{\omega}$) that consists of negative Beltrami flows with “large frequencies”. The singularities must appear in both the parts at the same space–time point.

Remark 5 (relation to the helicity)

Note that the so called helicity

$$H(\mathbf{v}) \;:=\; (\mathbf{v},\, \mathbf{curl}\ \mathbf{v})_{2;\,\mathbb{R}^3} \;=\; \int_{\mathbb{R}^3} \mathbf{v}\cdot\mathbf{curl}\ \mathbf{v}\ \mathrm{d}\mathbf{x}$$

can be expressed in the form

$$H(\mathbf{v}) \;=\; H(\mathbf{v}^+) + H(\mathbf{v}^-).$$

The "partial" helicities $H(\mathbf{v}^+)$ and $H(\mathbf{v}^-)$ satisfy the inequalities

$$H(\mathbf{v}^+) \geq 0, \qquad H(\mathbf{v}^-) \leq 0.$$

Consequently, since $A = |\mathbf{curl}\,|$, we have

$$(\mathbf{v},\, A\mathbf{v})_{2;\,\mathbb{R}^3} \;=\; \|A^{1/2}\mathbf{v}\|^2_{2;\,\mathbb{R}^3} \;=\; H(\mathbf{v}^+) - H(\mathbf{v}^-).$$

Thus, **conclusions of Theorems 1–4 imply that both the terms $H(\mathbf{v}^+)$ and $H(\mathbf{v}^-)$ are in $L^\infty(\vartheta, T)$ for each $\vartheta \in (0, T)$.**

Related papers:

- J.N. + P. Penel: Regularity criteria for weak solutions to the Navier–Stokes equations based on spectral projections of vorticity. *Comptes Rendus Mathématique 2012.*
- J.N. + P. Penel: Regularity of a weak solution to the Navier–Stokes equation via one component of a spectral projection of vorticity. *Preprint Feb. 2012, submitted.*